BL-Functions and Free BL-Algebra

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Abstract

Fuzzy logics are designed to support logical inferences on vague or uncertain premises, and they are useful in several theoretical and applicative areas of computer science. A central paradigm in mathematical fuzzy logic, popularized by Hájek [Háj98], is based on the idea of weakening Boolean logic starting from a suitable generalization of Boolean conjunction, namely, a class of $[0,1]$-valued binary functions known as (continuous) triangular norms. Any continuous triangular norm gives raise to a propositional logic, and Hájek's Basic fuzzy logic (for short, Basic logic) is the intersecting common fragment of all these logics. Despite the intensive research efforts devoted to Basic logic in the last decade, this logic still resists to a complete understanding, as it appears from the lack of a satisfactory proof theory [MOG].

The algebraic counterpart of Basic logic is given by a very natural subvariety of residuated bounded lattices, namely commutative, divisible and prelinear residuated lattices, or BL-algebras. A representation result of Aglianó and Montagna [AM03] establishes that the variety generated by all the n-generated BL-algebras is singly generated by the BL-chain $(n+1)[0,1]$, given by the ordinal sum of $n+1$ copies of the generic MV-chain $[0,1]$. As a consequence, validity problems in Basic logic turn out to have the same computational complexity of their Boolean counterparts [BHMV02, BM08]. This fact provides further motivation for the investigation of Basic logic in the computer science setting.

The aforementioned result of Aglianó and Montagna is the starting point of this thesis. By universal algebra, it gives an *implicit* functional representation of the free n-generated BL-algebra: the free n-generated BL-algebra is isomorphic to the clone of n-ary term operations of $(n+$

$1)[0,1]$, with the basic operations defined pointwise. Hence, to provide an *explicit* functional representation of the free n-generated BL-algebra, it is sufficient to describe exactly the subset of n-ary functions over the domain of $(n+1)[0,1]$ that contains all projections and is closed under the basic operations of $(n+1)[0,1]$: we call these functions, *n-ary BL-functions*. By algebraic logic, the Lindenbaum-Tarski algebra of the n-variate fragment of Basic logic is isomorphic to the free BL-algebra over n generators, thus n-ary BL-functions coincide with the truthfunctions of the n-variate fragment of Basic logic.

The main contribution of this thesis is the explicit representation of the free n-generated BL-algebra in terms of n-ary BL-functions. Our result accounts as the BL-algebraic counterpart of Mundici's constructive version of the McNaughton theorem for MV-algebras [Mun94], and improves the previous knowledge on the subject, that was limited to the case of one generator settled by Montagna [Mon00] and Aguzzoli and Gerla [AG05].

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1 Introduction

We present the object of study of this thesis, a propositional *fuzzy* logic, called *Basic fuzzy logic* (for short, Basic logic). The introduction is divided into three parts. In Section 1.1, we describe a natural motivation for investigating fuzzy logics: the phenomenon of vagueness. We adopt the general mathematical framework introduced by Hájek, based on the notion of *(continuous) triangular norm* [Háj98]. In this framework, Basic logic has a prominent importance, being the logic of all (continuous) triangular norms and their residua. Section 1.2 is devoted to the outline of the overall structure of this thesis, focusing on its main contribution. Section 1.3 collects the terminology and notation used throughout the thesis.

1.1 Basic Logic

In this section, we discuss a natural motivation for investigating fuzzy logics, the phenomenon of vagueness, and we introduce Hájek's mathematical framework for the study of fuzzy logics [Háj98]. In this framework, Basic logic is a fundamental object.

1.1.1 Vague Notions

We discuss a logical approach to the phenomenon of vagueness, as it appears in natural languages and reasonings, starting with an experiment.

We are asked to axiomatize the informal notion of *heap (of grains of sand)*. Let us start by examining our idea of heap. The first intuition we can isolate is that a collection of zero grains does not form a heap, but in contrast, for a sufficiently large number $N \geq 1$, a collection of N grains forms a heap. The second, coarse, intuition is that removing a single grain from a heap does not make any difference, that is, if a collection of i grains forms a heap, then a collection of $i-1$ grains forms a heap.

We settle a symbolic notation to write the above intuitions, using a language over the propositional variables $X_0, \ldots, X_N$, and the logical connectives of implication, $\rightarrow$, and negation, $\neg$. It is intended that X_i is the symbolic counterpart of the natural statement,

$$\text{“A collection of } i \text{ grains forms a heap.”}, \tag{1.1}$$

the implication, $A \rightarrow B$, is the symbolic counterpart of the idea that *if* the statement A holds, *then* the statement B holds, and the negation, $\neg A$, is the formal counterpart of the idea that the statement A does *not* hold. Now we can present the above intuitions symbolically:

$$\neg X_0 \tag{H1}$$

$$X_N \tag{H2}$$

$$X_1 \rightarrow X_0 \tag{H3.0}$$

$$\vdots$$

$$X_N \rightarrow X_{N-1} \tag{H3.$N-1$}$$

Let us pretend that the above list of sentences captures the intuition we have about heaps, and let us adopt the previous list as our *theory* of heaps.

As we shall see below, the interpretation of propositional variables, $X_0, \ldots, X_N$, and logical connectives, $\rightarrow$ and $\neg$, is not uniquely determined, rather, there are alternative and competitive interpretations. However, independently of the chosen interpretation, we admit certain manipulations of statements inside natural reasonings, which can be tentatively described in terms of *resources*, as follows. Imagine to record a natural reasoning as a sequence of statements, written in a symbolic notation. Along the sequence, the occurrence of the expression A captures the intuition that the resource A is available; the occurrence of the expression $A \rightarrow B$ captures the intuition that, upon availability of the resource A, also the resource B becomes available; the occurrence of the expression $\neg A$ captures the intuition that the resource A is unavailable.

As reasoning practitioners, we are confident that at the ith step of a reasoning inside a certain theory (for instance, the theory specified above), we are legitimate to infer an axiom A of the theory, because axioms are available resources; in symbols,

$$\begin{array}{lll} \vdots & \vdots & \vdots \\ i & A & \text{axiom} \\ \vdots & \vdots & \vdots \end{array}$$

Similarly, we are confident that within a reasoning, if A has been inferred at the ith step, and $A \to B$ has been inferred at the jth step, with $i, j < k$, then we can safely infer B at the kth step, in symbols (if e.g. $i < j$),

$$\begin{array}{lll} \vdots & \vdots & \vdots \\ i & A & \ldots \\ \vdots & \vdots & \vdots \\ j & A \to B & \ldots \\ \vdots & \vdots & \vdots \\ k & B & i, j \\ \vdots & \vdots & \vdots \end{array}$$

Now consider the reasoning of $2N + 2$ steps, inside the theory of heaps given by axioms (H1)-(H3.$N-1$), displayed below in the adopted symbolic notation:

$$\begin{array}{lll} 1 & X_N & \text{(H2)} \\ 2 & X_N \to X_{N-1} & \text{(H3.}N-1) \\ 3 & X_{N-1} & 1, 2 \\ 4 & X_{N-1} \to X_{N-2} & \text{(H3.}N-2) \\ 5 & X_{N-2} & 3, 4 \\ \vdots & \vdots & \vdots \\ 2N-1 & X_1 & 2N-3, 2N-2 \\ 2N & X_1 \to X_0 & \text{(H3.0)} \\ 2N+1 & X_0 & 2N-1, 2N \\ 2N+2 & \neg X_0 & \text{(H1)} \end{array}$$

In our *prelogical* setting, based on the notion of resource, the statements X_0 and $\neg X_0$, occurring in lines $2N+1$ and $2N+2$, form a critical pair, because they witness simultaneously the availability and the unavailability of the resource X_0.

However, both our initial assumptions and our reasoning schema seem to be substantially defensible, thus we want to somehow admit the sequence above. Our next task is then to identify a *logical* setting, that is an interpretation of propositions, implication, and negation that supports the above reasoning. As we already mentioned, interpretations are not unique, and are competitive: in a given situation, an obvious criterion for preferring a certain interpretation over another is the ability of the former interpretation to capture a natural reasoning in the given situation, in contrast with the inability of the latter to achieve the same goal.

Now consider how Boolean logic behaves in the malicious scenario we settled. Interpret the propositional variables $X_1, \dots, X_N$ as Boolean variables, that is, taking values in $\{0, 1\}$. This is the formal counterpart of the intuition that a natural statement of the form (1.1) either holds with truthvalue 1, or else holds with truthvalue 0, that is, it is either absolutely true, or else absolutely false. Suppose also to interpret the negation, $\neg$, over the familiar Boolean negation, that is, $\neg a$ equals 0 if and only if a equals 1, and to interpret the implication, $\to$, over the familiar Boolean implication, that is, $a \to b$ equals 0 if and only if a equals 1 and $b = 0$. But then, there not exists an assignment of the variables $X_0, \dots, X_N$ in $\{0, 1\}$ making all the axioms (H1), (H2), (H3.0), ..., (H3.$N-1$) true. [1] Hence, the theory has no Boolean models, in contrast with our tangible experience of heaps.

There are three possibilities to fix the problem. The first possibility is to avoid axiom (H1), but in this case the unique model of the theory says that all the X_i's are true, in particular says that X_0 is true, and this

[1]Suppose, for contradiction, that the assignment $\mathbf{a}$ makes all the axioms true. Then, each line of the reasoning contains a statement that is true under $\mathbf{a}$. Indeed, each line is either an axiom, which is true under $\mathbf{a}$ by hypothesis, or the conclusion B of an inference having premises A and $A \to B$ which are true under $\mathbf{a}$ by construction (note that by our definition of $\to$, if both A and $A \to B$ true under $\mathbf{a}$, then B is true under $\mathbf{a}$). This fact is traditionally known as the *sorite's paradox*.

conflicts with our intuition. The second possibility is to avoid axiom (H2), but in this case the unique model of the theory says that all the X_i's are false, in particular says that X_N is false, and again this conflicts with our intuition (we chosen N large enough to verify the statement X_N). The third possibility is to fix a threshold i between 0 and $N-1$ and then negate axiom (H3.i),

$$\begin{aligned} &\vdots \\ X_i &\to X_{i-1} && \text{(H3.}i-1\text{)} \\ \neg(X_{i+1} &\to X_i) && \text{(H3.}i'\text{)} \\ X_{i+2} &\to X_{i+1} && \text{(H3.}i+1\text{)} \\ &\vdots \end{aligned}$$

In this case, the unique model of the theory says that $X_0, \ldots, X_i$ are false, and that $X_{i+1}, \ldots, X_N$ are true, that is, the $(i+1)$th grain makes the difference; but again, this conflicts with our intuition.

Hence, Boolean logic does not support the reasoning scenario we settled, since our theory either has no Boolean models, or else has unsatisfactory Boolean models.

The problem seems to be that the *bivalence* of the underlying logic conflicts with the intrinsic *vagueness* of the axiomatized notion. If Boolean logic was the only admissible logic, then we would be forced to conclude that the phenomenon of vagueness does not admit a logical treatment. In this diagnosis, however, the uniqueness of Boolean logic acts as an assumption. An alternative strategy, avoiding this serious assumption, is to admit the failure of Boolean logic in this context, and to look for a *conservative* repair of the situation, that is, a solution to the problem inside a generalization of Boolean logic. [2]

Let us backtrack and refine our initial intuition regarding the notion of heap. We are confortable in recognizing that X_0 is false (that is, X_0 holds with truthvalue 0), and that X_N is true (that is, X_N holds with truthvalue 1). However, it is reasonable that if X_{i+1} holds with a certain truthvalue, a_{i+1}, then X_i holds with a truthvalue, a_i, that is smaller than a_{i+1}. Moreover, it is reasonable to admit that the truthvalues of the

[2]The solution we adopt follows [Háj98, Example 3.3.21].

X_i's decrease uniformly, as i decreases from N to 0. Hence, adopting this refined intuition, we are interested in a logical framework where we can formalize a theory that is virtually identical, at the linguistic level, to our initial theory, and having as a model an assignments of the propositional variables $X_0, \ldots, X_N$ over the truthvalues,

$$D = \{0, 1/N, 2/N, \ldots, 1\},$$

such that the truthvalue of X_i is exactly i/N. It is intended that D is ordered by $0 < 1/N < \cdots < 1$, meaning that X_j is less true than X_k for every $0 \leq j < k \leq N$. Thus, we abjured the bivalence of Boolean logic in favor of a *many-valued* setting.

Together with bivalence, the other distinctive property of Boolean logic is *truthfunctionality*, which we do not seem to be forced to abjure. Therefore, since the propositional variables range over D, we choose a pair of functions over D as interpretations for implication and negation; namely, we interpret implication over the binary function $a \to b = \min(1, b + 1 - a)$ over D (this operation is traditionally known as Łukasiewicz implication), and negation over the unary function $\neg a = 1 - a$ over D (that is, Łukasiewicz negation). Although it appears arbitrary, the choice of the previous interpretations is defensible under several respects, as we shall see in the next section; for instance, the restrictions of the adopted interpretations of $\to$ and $\neg$ to $\{0, 1\}$ act as Boolean implication and negation.

In Boolean logic, a common practice is that of enriching the language with propositional constants C_0 and C_1, standing respectively for the conventionally true and false statements, say, "$0 = 0$." and "$0 \neq 0$.". Mimicking this practice in our generalized setting, we enrich the language with the propositional constants $C_0, \ldots, C_{N-1}, C_N$, where C_i stands for the statement that conventionally holds with truthvalue i/N, for $i = 0, \ldots, N$. Now by our interpretations of the connectives, $C_i \to A$ holds with truthvalue 1 if and only if A holds with truthvalue $\geq i$, and $A \to C_i$ holds with truthvalue 1 if and only if A holds with truthvalue $\leq i$. With this machinery at hand, let us slightly revise our initial theory by putting,

$$\neg X_0 \tag{H1}$$

$$X_N \tag{H2}$$

$$C_{N-1} \to (X_1 \to X_0) \tag{H3.0$'$}$$

$$(X_1 \to X_0) \to C_{N-1} \tag{H3.0$''$}$$

$$\vdots$$

$$C_{N-1} \to (X_N \to X_{N-1}) \tag{H3.$N-1'$}$$

$$(X_N \to X_{N-1}) \to C_{N-1} \tag{H3.$N-1''$}$$

where, for $i = 0, \ldots, N$, axioms $C_{N-1} \to (X_{i+1} \to X_i)$ and $(X_{i+1} \to X_i) \to C_{N-1}$ say that $X_{i+1} \to X_i$ holds with a truthvalue equal to $(N-1)/N$. In words, we are assuming that the statement $X_{i+1} \to X_i$ is almost true, but not absolutely true, fitting the fine intuition that removing a grain from a heap makes a difference, even though just a small one.

It is not difficult to check that an assignment $\mathbf{a}$ of $X_0, \ldots, X_N$ in D models the revised theory if and only if X_0 holds with truthvalue 0 under $\mathbf{a}$, X_N holds with truthvalue 1 under $\mathbf{a}$, and, for all $0 < i < N$, X_i holds with truthvalue i/N under $\mathbf{a}$. [3]

As regards to the recovery of our natural reasoning on the heap, it is possible to prove [4] that, in the adopted logical framework, the revised theory allows to infer all the lower bounds $C_i \to X_i$ and upper bounds $X_i \to C_i$ for $0 \leq i \leq N$, that is, it allows to infer that the statement X_i holds with truthvalue i/N for $0 \leq i \leq N$. It is also possible to prove that the theory allows to infer $C_N \to \neg X_0$, but the pair $X_0 \to C_0$ and $C_N \to \neg X_0$ is not critical in this setting, because it simply says that X_0 is absolutely false, and $\neg X_0$ is absolutely true.

In summary, we found a generalization of the Boolean logic framework, where propositional variables take values in a linearly ordered

[3]If $\mathbf{a}$ models the revised theory, in particular X_N holds with truthvalue 1 under $\mathbf{a}$, and $X_N \to X_{N-1}$ holds with truthvalue $(N-1)/N$ under $\mathbf{a}$. Then, by definition, the truthvalue of X_{N-1} under $\mathbf{a}$ is equal to the truthvalue of X_N under $\mathbf{a}$ minus $1/N$, that is $(N-1)/N$. Iterating, we get that, for all $0 \leq i < N$, X_i holds with truthvalue i/N under $\mathbf{a}$. The converse is immediate to check, applying the definitions.

[4]Technically, we rely on the completeness theorem of Rational Pavelka logic [Háj98, Theorem 3.3.5].

set D, such that $\{0, 1\} \subseteq D \subseteq [0, 1]$, and implication and negation are functions over D. Adopting this framework, we have been able to write a theory that isolates the expected model and allows for the expected inferences.

Not surprisingly, due to the pervasive presence of vagueness in natural phenomena, fuzzy logics candidate as a suitable kernel of applications such as fuzzy control systems [GH99]. In addition to a robust treatment of vagueness, fuzzy logics may provide a rigorous foundation to the logical treatment of uncertain informations, the prominent applications here being Rényi-Ulam game with lies and error correcting codes [CDM99], the investigation of the probability of fuzzy events [Mon06, KM07, AGM], and satisfiability problems such as MAX-SAT [Mun99, LHdG08].

In the next section, we shall formalize a general mathematical framework, capturing a large family of fuzzy logics, including the logic depicted in this section.

1.1.2 Triangular Norms

In this section, we present a mathematical framework for the study of fuzzy propositional logics, based on the notion of *(continuous) triangular norm*. This framework has been popularized by Hájek [Háj98]. We refer the reader to Section 1.3 for terminology and notation (language, interpretation, calculus, etc.).

Let L be a language, built upon the propositional variables X_1, X_2, $\dots$, the propositional constant $\perp$, and the logical connectives $\odot$ and $\to$. An additional abstraction effort, directed by the introductory discussion on the problem of vagueness, leads us to a family of interpretations of L, satisfying the following four requirements:

Fuzziness: The propositional variables, $X_1, X_2, \dots$, take values over the real unit interval $[0, 1]$, equipped with the usual ordering, $\leq$. Intuitively, 0 and 1 act as Boolean falsity and truth respectively, the values strictly between 0 and 1 act as truth degrees, and the ordering implements the idea that truth degrees are pairwise comparable.

Truthfunctionality: the symbol $\bot$, called *falsum*, is interpreted over 0, and the symbols $\odot$ and $\to$, called *fuzzy conjunction* and *fuzzy implication*, are interpreted over fixed binary operations $\odot^{[0,1]}$ and $\to^{[0,1]}$ over $[0,1]$.

Connectives: Let $a, a_1, a_2 \in [0,1]$. The operation $\odot^{[0,1]}$ is such that $1 \odot^{[0,1]} a = a$ and $0 \odot^{[0,1]} a = 0$, so that the restriction of $\odot^{[0,1]}$ to $\{0,1\}^2$ behave like Boolean conjunction. Moreover, $\odot^{[0,1]}$ is associative, commutative, isotone in both arguments, and continuous.

The operation $\to^{[0,1]}$ is such that $a_1 \to^{[0,1]} a_2 = 1$ if and only if $a_1 \leq a_2$, so that the restriction of $\to^{[0,1]}$ to $\{0,1\}^2$ behave like Boolean implication. Moreover, $\to^{[0,1]}$ is antitone in the first argument and isotone in the second argument.

Inference: The *fuzzy modus ponens* rule allows for the inference of s from the fuzzy conjunction of r and $r \to s$, that is, $r \odot (r \to s)$, for every $r, s \in L$. The operations $\odot^{[0,1]}$ and $\to^{[0,1]}$ are such that the fuzzy modus ponens is *sound* and *powerful*, in the following sense.

Let a_1, a_2, and a_3 denote respectively the values of r, s, and $r \to s$ in $[0,1]$. On the one hand, a_3 must satisfy,

$$a_1 \odot^{[0,1]} a_3 \leq a_2,$$

to preserve the soundness of the inference: we want to exclude the case where the conclusion of the inference has a truth degree strictly lower than the truth degree of the fuzzy conjunction of the premises. On the other hand, a_3 must be the maximal value that preserves soundness, to realize a powerful inference: we want to infer s from premises r and $r \to s$ with a value, a_2, as large as possible; so the value, a_3, of $r \to s$ is chosen to attain the largest possible lower bound on a_2 (as a boundary case, for instance, we avoid the choice $a_3 = 0$, yielding the trivial lower bound $0 \leq a_2$).

The first, second, and third requirement are justified by the solution of the problem of vagueness proposed in the previous section: we want a logical framework that generalizes Boolean logic abjuring bivalence, but maintaining truthfunctionality. The fourth requirement is necessary to implement natural inferences within our framework.

In an important and fruitful insight, Hájek observed that *(continuous) triangular norms*, or *(continuous) t-norms*, and their *residua*, provide suitable interpretations for fuzzy conjunction and implication, that is, interpretations that satisfy the aforementioned requirements. A continuous t-norm, $*$, is a continuous binary function on $[0,1]$ that is associative, commutative, isotone ($a_1 \leq a_2$ implies $a_1 * a_3 \leq a_2 * a_3$, $a_1, a_2, a_3 \in [0,1]$) and has 1 as unit ($a_1 * 1 = a_1$). Given a continuous t-norm $*$, the corresponding *residuum* is the binary operation $\rightarrow_*$ over $[0,1]$ uniquely determined by the residuation equivalence ($a_1, a_2, a_3 \in [0,1]$),

$$a_1 * a_3 \leq a_2 \text{ if and only if } a_3 \leq a_1 \rightarrow_* a_2,$$

which turns out to be given by,

$$a_1 \rightarrow_* a_2 = \max\{a_3 \in [0,1] \mid a_1 * a_3 \leq a_2\}.$$

For every t-norm, $*$, the corresponding *t-algebra*,

$$[0,1]_* = ([0,1], *, \rightarrow_*, 0),$$

is the algebra over the signature $(\odot, \rightarrow, \bot)$ of type $(2,2,0)$, having $\odot$ realized by the t-norm $*$, $\rightarrow$ realized by its residuum $\rightarrow_*$, and $\bot$ realized by 0.

It is immediate to check that the interpretation of the language L into $[0,1]_*$ satisfies the requirements listed above. This fact motivates the introduction of the fuzzy propositional logic *based* on the t-norm $*$ as the set of formulae,

$$\{s \in L \mid [0,1]_* \models s = \top\}.$$

In this setting, the most general fuzzy propositional logic is the logic of *all* continuous t-norms and their residua, that is,

$$\bigcap_* \{s \in L \mid [0,1]_* \models s = \top\}, \tag{1.2}$$

where $*$ ranges over continuous t-norms. In [CEGT00], Cignoli, Esteva, Godo, and Torrens proved that the following logical calculus is complete with respect to all continuous t-norms and their residua. [5]

Definition 1 (Basic Logic Calculus, $\vdash_{BL}$)**.** *The* Basic logic calculus*, in symbols* $\vdash_{BL}$*, is defined by the modus ponens inference rule and the axioms* ($r, s, t \in L$)*:*

(BL1) $(r \to s) \to ((s \to t) \to (r \to t))$

(BL2) $(r \odot s) \to r$

(BL3) $(r \odot (r \to s)) \to (s \odot (s \to r))$

(BL4) $(r \to (s \to t)) \to ((r \odot s) \to t)$

(BL5) $((r \odot s) \to t) \to (r \to (s \to t))$

(BL6) $((r \to s) \to t) \to (((s \to r) \to t) \to t)$

(BL6) $\bot \to r$

It turns out that a formula s is provable from formulae $r_1, \ldots, r_i$ in the Basic logic calculus if and only if, for every t-algebra $[0,1]_*$ and every assignment $\mathbf{a}$, if $r_1, \ldots, r_i$ are equal to 1 under $\mathbf{a}$ in $[0,1]_*$, then s is equal to 1 under $\mathbf{a}$ in $[0,1]_*$. Formally, noticing that for every finite set of terms $r_1, \ldots, r_i, s \in L$, there exists $n \geq 1$ such that $r_1, \ldots, r_i, s \in L_n$, and recalling that $\top$ abbreviates $\bot \to \bot$, we have the following completeness theorem.

Theorem 2 (Cignoli et al.; Aglianó and Montagna)**.** *Let* $r_1, \ldots, r_i, s$ *be terms in* L_n*. Then,* $r_1, \ldots, r_i \vdash_{BL} s$ *if and only if, for every t-norm* $*$ *and every* $\mathbf{a} \in [0,1]^n$*,*

$$r_1^{[0,1]_*}(\mathbf{a}) = \cdots = r_i^{[0,1]_*}(\mathbf{a}) = \top^{[0,1]_*} \textit{ implies } s^{[0,1]_*}(\mathbf{a}) = \top^{[0,1]_*}.$$

In particular, $\vdash_{BL} s$ *if and only if, for every t-norm* $*$ *and every* $\mathbf{a} \in [0,1]^n$*,*

$$s^{[0,1]_*}(\mathbf{a}) = \top^{[0,1]_*}.$$

[5] As conjectured by Hájek [Háj98, Remark 2.3.23]. We mention that Hájek's version of the calculus included the commutativity of $\odot$ as an axiom [Háj98, Definition 2.2.4]. This axiom has been proved redundant by Cintula [Cin05].

Thus, Basic logic is the logic of all continuous t-norms and their residua. [6] This result has a major benefit and a major weakness, which we now consider in turn.

The benefit is that the completeness of Basic logic with respect to all continuous t-norms and their residua furnishes a satisfactory philosophical motivation for Basic logic. Indeed, Basic logic abstracts a number of properties of Boolean logic that are natural, or essential, and, upon requiring exclusively the satisfaction of these properties, extends Boolean logic from the domain $\{0, 1\}$ to the domain $[0, 1]$. [7] Hence, as far as the distilled properties are regarded as *the* essential features of Boolean logic, Basic logic is *the* natural extension of Boolean logic from $\{0, 1\}$ to $[0, 1]$. From this viewpoint, Basic logic is easier to motivate as a fuzzy logic than a logic based on a fixed t-norm, because in the latter case there is a commitment on a specific t-norm that lacks in the former, and this specific asks for additional motivations such as, for instance, a domain of application. [8] On the other hand, if some feature of Basic logic is considered as groundless or misleading, the same objection automatically applies to any of its concrete realizations.

The weakness is that Theorem 2 does not suggest an algorithm for solving the most natural logical problems related to Basic logic, such as the problem of deciding Basic logic tautologies or (finite) consequences, or the problem of representing Basic logic truthfunctions.

In the rest of this section we insist on the decision issue; we shall devote the central part of the thesis to the investigation of the representation issue.

Let us formalize the problem of deciding tautologies and conse-

[6] In particular, Basic logic forms a common fragment of Łukasiewicz [Łuk20], Gödel [Göd32], and Product logics [HGE96], which are fundamental in the sense of the Mostert-Shields theorem [MS57]. In this introduction, we are attempting to articulate a direct justification of Basic logic as an autonomous object, a defensible generalization of Boolean logic, instead of as a common fragment of the most important fuzzy logics. We refer the reader to [Háj98, Got01] for alternative authoritative introductions to Basic logic.

[7] A possible objection to this affirmation is that the continuity of t-norms is not necessary, because left-continuity suffices to attain a residuum. However, continuity may be asked as a natural requirement for a conjunction operation over $[0, 1]^2$.

[8] For instance, Łukasiewicz logic can be presented as the logic of Rényi-Ulam games with lies [Mun93].

quences in Basic logic. The *(finite) consequence* problem of Basic logic is the problem of deciding if, given a finite set of formulae $\{r_1, \ldots, r_i\}$, and a formula s, there exists a proof in Basic logic of s using $r_1, \ldots, r_i$ as additional axioms, that is, whether the consequence relation,

$$r_1, \ldots, r_i \vdash_{BL} s, \tag{1.3}$$

holds or not. Say that $r_1, \ldots, r_i, s$ contain variables among $X_1, \ldots, X_n$. By Theorem 2, this problem is equivalent to decide the following statement: for every t-norm $*$ and every assignment $\mathbf{a} \in [0,1]^n$ of the variables, if $r_1^{[0,1]_*}(\mathbf{a}) = \cdots = r_i^{[0,1]_*}(\mathbf{a}) = 1$, then it must also hold that $s^{[0,1]_*}(\mathbf{a}) = 1$. The *tautology* problem of Basic logic reduces to the special case of deciding whether or not a given a formula s is a consequence of the empty set in Basic logic. Say that s contains variables among $X_1, \ldots, X_n$. By Theorem 2, this problem is equivalent to decide the following statement: for every t-norm $*$, and every assignment $\mathbf{a} \in [0,1]^n$ of the variables, $s^{[0,1]_*}(\mathbf{a}) = 1$. [9] But deciding the previous statements asks for an exhaustive nested search over two infinite spaces, since both the interpretations of the logical signature, and the assignments of the propositional variables, are infinitely many (in fact, uncountably many).

Hence, in the terms above, Theorem 2 does not provide a decision procedure for Basic logic. A stronger completeness result is needed, ideally, the completeness of Basic logic with respect to a *single* and *manageable* t-algebra. This algebra has been eventually described thanks to the effort of several resarchers [AM03, BHMV02, AFM07]. In this thesis, we freely follow the presentation of Aglianó and Montagna [AM03].

Definition 3. *Let* $n \geq 1$*. The algebra* $[0, n+1] = ([0, n+1], \odot, \rightarrow, \bot)$ *is the*

[9]In contrast with the Boolean case, an instance of the consequence problem does not reduce to an instance of the tautology problem, due to the lack of the deduction theorem in its classical form. In fact, in general Basic logic does not satisfy: $r_1, \ldots, r_i \vdash_{BL} s$ if and only if $r_1, \ldots, r_{i-1} \vdash_{BL} r \rightarrow s$, but only: $r \vdash_{BL} s$ if and only if there exists $n \geq 0$ such that $r_1, \ldots, r_{i-1} \vdash_{BL} r_i^n \rightarrow s$ [Háj98].

algebra of type $(2,2,0)$*, defined as follows* ($a_1, a_2 \in [0, n+1]$)*:*

$$a_1 \odot a_2 = \begin{cases} \min(a_1, a_2) & \textit{if } \lfloor a_1 \rfloor \neq \lfloor a_2 \rfloor \\ \max(\lfloor a_1 \rfloor, a_1 + a_2 - \lfloor a_1 \rfloor - 1) & \textit{otherwise} \end{cases}$$

$$a_1 \to a_2 = \begin{cases} a_2 & \textit{if } \lfloor a_2 \rfloor < \lfloor a_1 \rfloor \\ a_2 + \lfloor a_1 \rfloor + 1 - a_1 & \textit{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor \textit{ and } a_2 < a_1 \\ n+1 & \textit{otherwise} \end{cases}$$

$$\bot = 0$$

For $\circ \in \{\odot, \to, \bot\}$*, we let* $\circ^{[0,n+1]}$ *denote the realization of* $\circ$ *in* $[0, n+1]$*.* [10]

The following result of Aglianó and Montagna refines the completeness theorem of Cignoli, Esteva, Godo and Torrens.

Theorem 4 (Aglianó and Montagna)**.** *Let* $r_1, \ldots, r_i, s$ *be terms in* L_n*. Then,* $r_1, \ldots, r_i \vdash_{BL} s$ *if and only if, for every* $\mathbf{a} \in [0, n+1]^n$*,*

$$r_1^{[0,n+1]}(\mathbf{a}) = \cdots = r_i^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]} \textit{ implies } s^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]}.$$

In particular, $\vdash_{BL} s$ *if and only if, for every* $\mathbf{a} \in [0, n+1]^n$*,*

$$s^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]}.$$

To the extent of the decision issue of Basic logic, the previous result is a dramatic improvement of Theorem 2. Indeed, in terms of Theorem 4, Basic logic proves s from $r_1, \ldots, r_i$ if and only if, for every $\mathbf{a} \in [0, n+1]^n$, $r_1^{[0,n+1]}(\mathbf{a}) = \cdots = r_i^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]}$ implies $s^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]}$. The crux is that the decision of the right side of the previous equivalence is in coNP: intuitively, the algebra $[0, n+1]$ allows for a finite reduction of the infinite space of the assignments $\mathbf{a} \in [0, n+1]^n$.

The case where $i = 0$, that is, the problem whether s is a Basic logic tautology or not, has been proved to be in coNP by Baaz et al. in [BHMV02]. In the vein of [CFM04, MPT03], Bova and Montagna presented in [BM08] an improvement of the seminal algorithm of Baaz

[10] The fact that the domain is $[0, n+1]$ instead of $[0, 1]$ is immaterial, modulo a resizing. We adopt this technical trick to streamline notations and computations.

et al. in [BHMV02]. The algorithm takes in input a Basic logic finite consequence (thus implementing the general case $i \geq 0$), and reduces the instance, of size l, to a collection of exponentially many systems of linear equality and inequality constraints (each having size polynomially bounded in l), such that the instance is valid if and only if all of the systems are unsatisfiable. A careful organization of the reductions allows to show that checking at most 2^{3l} systems suffices to decide the instance. This improves the upper bound implicit in the algorithm of [BHMV02], that involves a number $\geq l!$ of witnesses. This nice bound is attained applying techniques inspired by proof theory, despite the algorithm is not still satisfactory as a proof system for Basic logic. [11]

As for lower bounds, the problem of deciding Basic logic consequences is coNP-hard, by the following reduction from the coNP-complete problem of Łukasiewicz tautology [Mun87]. Let $s \in L_n$. It is easy to check that s is a tautology of Łukasiewicz logic if and only if $\neg\neg s$ is a tautology of Basic logic, if and only if Basic logic proves $\neg\neg s$ from the empty set. Thus, as a consequence of Theorem 4, not only deciding Basic logic has the same computational complexity of deciding Boolean logic, but there is also a decision algorithm that matches the worst-case upper bounds of Boolean logic decision algorithms.

Before formalizing and attacking the *representation* issue of Basic logic, let us summarize the motivations we presented in favor of the investigation of Basic logic. Along the lines of [Háj98], we introduced Basic logic as a natural generalization of Boolean logic based upon minimal assumptions, possibly with the exception of conjunction continuity. Then, we presented a strong completeness result of Aglianó and Montagna [AM03], which has as a corollary that Basic logic finite consequences, and hence Basic logic tautologies, have the same computational complexity of the Boolean counterparts, that is, are coNP-complete problems. As a strengthen of the coNP upper bound of Baaz et al. in [BHMV02], we mentioned an algorithm of Bova and Montagna [BM08], that matches the worst case running time of the familiar

[11] Adopting the terminology and the notation in [BM08], Basic logic proves s from $r_1, \ldots, r_i$ if and only if all the leaves of the reduction tree having the root labeled by, $\overline{\top \preccurlyeq r_1 \odot \cdots \odot r_i} \mid \top \preccurlyeq s$, are axioms. In particular, Basic logic proves s if and only if all the leaves of the reduction tree having the root labeled by, $\top \preccurlyeq s$, are axioms.

truthtable algorithm for Boolean logic. This bound is optimal, in the general case, if coNP$\neq$NP [CR73]. In our opinion, this nucleus of logical and computational facts is sufficient to legitimate Basic logic as an autonomous object of study. [12]

1.2 Outline and Contribution

In the previous section we introduced fuzzy logics as a generalization of Boolean logic, aimed to support logical inferences on vague or uncertain premises. We observed that these logics arose as practical mathematical tools for tackling problems in several theoretical and applicative areas of computer science, for instance fuzzy control, error correcting codes, fuzzy probability, and maximum satisfiability. We also depicted Hájek's paradigm for mathematical fuzzy logics, relying upon the idea of generalizing Boolean logic starting from a suitable generalization of Boolean conjunction, given by $[0,1]$-valued functions known as (continuous) triangular norms. In this setting, Basic logic is a fundamental object, being the intersecting common fragment of all triangular norms based logics. Eventually we discussed the importance of the completeness theorem of Basic logic with respect to the semantics $[0, n+1]$ as regards to the decision issue of Basic logic.

Chapter 2 constitutes the central part of this thesis, and it is devoted to an explicit functional representation of Basic logic.

In Section 2.1 we discuss how the completeness of the n-variate fragment of Basic logic with respect to $[0, n+1]$ furnishes an *implicit* representation of the truthfunctions of the n-variate fragment of Basic logic, or equivalently of the elements of the Lindenbaum-Tarski algebra of the n-variate fragment of Basic logic, for every $n \geq 1$. The main contribution of this thesis is an *explicit* description of these truthfunctions, which we call *BL-functions*, for every $n \geq 1$. This very natural and elementary logical problem has been solved only in the case $n = 1$, by Montagna [Mon00]. Even the case $n = 2$ eluded intensive research efforts [Mon01].

[12] The computational cost of renouncing to the right-continuity of fuzzy conjunction is not clear. Indeed, it is still unknown if the logic of all left-continuous t-norms and their residua, called MTL [EG01], is in coNP.

In Section 2.2 we mention the completeness of Łukasiewicz logic with respect to the standard MV-chain $[0, 1]$, and the functional representation of the truthfunctions of Łukasiewicz logic in terms of McNaughton functions. The framework in which we shall characterize BL-functions inherits several beautiful ideas and technical tools from the case of Łukasiewicz logic and McNaughton functions.

In Section 2.3 we study in full generality the class of BL-functions: these functions act in the BL-algebraic setting as McNaughton functions act in the MV-algebraic setting. In Section 2.3.1 we prepare a conceptual and terminological framework suitable for the study of BL-functions. In Section 2.3.2 we recast the case of unary BL-functions into our framework. This case was already known, thanks to the work of Montagna [Mon00]. In Section 2.3.3, to elicit intuitions in view of the general case, we study in detail binary BL-functions. The step from the unary to the binary case furnishes the heuristic for generalizing the construction to the case of $n \geq 1$ variables, that is treated in Section 2.3.4.

In Section 2.4 we capitalize our explicit description of n-ary BL-functions in the universal algebraic setting. Since the variety of BL-algebras forms the equivalent algebraic semantics of Basic logic, by universal algebra, the free n-generated BL-algebra is isomorphic to the Lindenbaum-Tarski algebra of the n-variate fragment of Basic logic. Hence our main result accounts as an explicit and constructive functional representation of the free n-generated BL-algebra, for $n \geq 1$.

In Chapter 3 we summarize our results in geometrical terms, and discuss three possible developments of the present work, namely the combinatorial representation of locally finite subvarieties of BL-algebras, the identification of tight finite countermodels to BL-quasiequations, and the construction of deductive interpolants in Basic logic.

1.3 Terminology and Notation

In this section, we introduce the terminology and notation used throughout the thesis.

For every $n \geq 1$, we let $[n] = \{1, \ldots, n\}$. For any $a \in \mathbb{R}$, we let $\lfloor a \rfloor$ denote the integer part of a, stipulating that $\lfloor +\infty \rfloor = +\infty$. Let $n \geq 1$ be fixed. For any $r \in \mathbb{R}$, we let $\mathbf{r}$ be the vector $(r, \ldots, r) \in \mathbb{R}^n$. In particular, $\mathbf{1} = (1, \ldots, 1) \in [0,1]^n$, and $\mathbf{n}+\mathbf{1} = (n+1, \ldots, n+1) \in [0, n+1]^n$. For $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{a}' = (a'_1, \ldots, a'_n) \in \mathbb{R}^n$, we let $\mathbf{a}-\mathbf{a}' = (a_1-a'_1, \ldots, a_n-a'_n) \in \mathbb{R}^n$. Let f be an n-ary function from B to C, and let $A \subseteq B$. Then, $f|_A$ denotes the restriction of the function f to A. Let g be a function from A to C. We write $f|_A = g$ if $f(a) = g(a)$ for every $a \in A$.

Definition 5 (Language)**.** *The set $X = \{X_i \mid i \in \mathbb{N}\}$ is a set of symbols, called* variables*. The set L is the smallest set of strings over the alphabet $X \cup \{\odot, \to, \bot,), (\}$ such that: $X \subseteq L$ and $\bot \in L$; $s, t \in L$ implies $(s \odot t), (s \to t) \in L$. The set L is called* language*, and the strings in L are referred to either as* terms *or as* formulae*.*

*Let $n \geq 1$. We let L_n denote the n-*variate fragment *of the language L, that is the subset of L containing exactly the strings built upon variables $X_1, \ldots, X_n$. Let $I \subseteq [n]$. We let L_I denote the subset of L containing exactly the strings built upon variables X_i for $i \in I$.*

We let L^+ denote the subset of terms in L not containing occurrences of $\bot$, and similarly we let L^+_n and L^+_I be the subsets of terms in L_n and L_I respectively not containing occurrences of $\bot$.

Note that, for every $t \in L$, there exists $n \geq 1$ such that $t \in L_n$. For every term $t \in L$, we write t^m for the compound term $t \odot \cdots \odot t$, with t occurring m times.

Definition 6 (Substitution)**.** *Let $t \in L_n$ and let $I = \{i_1, \ldots, i_m\} \subseteq [n]$ such that $i_1 < \cdots < i_m$. We let,*

$$t_{\{i_1 \backslash 1, \ldots, i_m \backslash m\}},$$

denote the term in L_m obtained by substituting simultaneously and uniformly in t variable X_{i_1} with variable X_1, …, variable X_{i_m} with variable X_m. Let $t \in L_n$, let $I = \{i_1, \ldots, i_m\} \subseteq [n]$ such that $i_1 < \cdots < i_m$, and let $t_1, \ldots, t_m$ be in L_n. We let,

$$t_{\{i_1 \backslash t_1, \ldots, i_m \backslash t_m\}},$$

denote the term in L_n obtained by substituting simultaneously and uniformly in t variable X_{i_1} with term t_1, …, variable X_{i_m} with term t_m. Let $r, s, t \in L_n$, and let s be a subterm of t. We let,

$$t_{\{s \backslash r\}},$$

denote the term in L_n obtained by substituting simultaneously and uniformly in t the term s with the term r.

Definition 7 (Logical Calculus)**.** *A* (logical) calculus*, in symbols $\vdash$, is defined by a finite collection of axiom schemata together with the modus ponens inference rule. A* proof *of a formula t in the calculus is a tuple $(t_1, \ldots, t_m) \in L^m$ such that $t = t_m$ and for $i = 1, \ldots, m-1$, t_i is either an instance of an axiom schema, or the conclusion of a modus ponens inference rule,*

$$t_j, t_j \to t_k \vdash t_k,$$

where $j, k \in [m]$ and $j, k < i$. We say that $t \in L$ is provable *in a fixed calculus, notation $\vdash t$, if and only if there exists a proof of t in the calculus.*

Definition 8 (Algebraic Semantics, Term Operation)**.** *An* (algebraic) semantics *is an algebra $\mathbf{A}$ having domain A and signature $(\odot, \to, \bot)$ of type $(2, 2, 0)$. We let $\circ^{\mathbf{A}}$ denote the operation realizing the symbol $\circ \in \{\odot, \to, \bot\}$ in $\mathbf{A}$. Let $\mathbf{A}$ be a semantics, and let $t \in L_n$. We let,*

$$t^{\mathbf{A}} : A^n \to A,$$

denote the n-ary operation over A, uniquely determined by the following inductive clauses, for every $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$:

(i) if $t = X_i$, then $t^{\mathbf{A}}(\mathbf{a}) = a_i$;

(ii) if $t = \bot$, then $t^{\mathbf{A}}(\mathbf{a}) = \bot^{\mathbf{A}}$;

(iii) if $t = r \odot s$, then $(r \odot s)^{\mathbf{A}}(\mathbf{a}) = r^{\mathbf{A}}(\mathbf{a}) \odot^{\mathbf{A}} s^{\mathbf{A}}(\mathbf{a})$;

(iv) if $t = r \to s$, then $(r \to s)^{\mathbf{A}}(\mathbf{a}) = r^{\mathbf{A}}(\mathbf{a}) \to^{\mathbf{A}} s^{\mathbf{A}}(\mathbf{a})$.

We call $t^{\mathbf{A}}$ the (term) operation *corresponding to t in $\mathbf{A}$.*

Definition 9 (Validity, Completeness)**.** *Let $\mathbf{A}$ be a semantics, and let $r, s \in L_n$. The equation $r = s$ is* valid *in $\mathbf{A}$, in symbols $\mathbf{A} \models r = s$, if for every $\mathbf{a} \in A^n$, it holds that $r^{\mathbf{A}}(\mathbf{a}) = s^{\mathbf{A}}(\mathbf{a})$. The term r is a* valid *in $\mathbf{A}$, in symbols, $\mathbf{A} \models t = \top$, if the equation $r = \top$ is valid in $\mathbf{A}$. $\mathbf{A}$ is* complete *for a calculus $\vdash$, if, for every $t \in L$,*

$$\vdash t \text{ if and only if } \mathbf{A} \models t = \top.$$

We shall adopt the following abbreviations.

Notation 10 (Abbreviations)**.** *For every* $r, s \in L$ *we write* $r \wedge s$ *instead of* $r \odot (r \to s)$*,* $r \vee s$ *instead of* $((r \to s) \to s) \wedge ((s \to r) \to r)$*,* $\neg r$ *instead of* $r \to \bot$*, and* $\top$ *instead of* $\neg\bot$*.*

2 BL-Functions and Free BL-Algebra

In this chapter, we present the main contribution of this thesis. In light of the motivations furnished in the introduction, the problem we are going to deal with is very natural. Roughly speaking, we want to describe explicitly the class of functions that stands to Basic logic truthfunctions as Boolean functions stand to Boolean logic truthfunctions.

More precisely, let $n \geq 1$. Say that terms $r, s \in L_n$ are *provably equivalent (in Basic logic)* if Basic logic proves both $r \to s$ and $s \to r$, and write,

$$[t] = \{t' \in L_n \mid t' \text{ provably equivalent to } t\},$$

for every $t \in L_n$. The relation of equiprovability is an equivalence relation over L_n. Let the algebra $\mathbf{A}_n = (A_n, \odot, \to, \bot)$ of type $(2, 2, 0)$ be defined as follows. The domain A_n contains exactly the classes of provably equivalent terms in L_n,

$$A_n = \{[t] \mid t \in L_n\}.$$

The constant $\bot$ is realized by $[\bot]$, the operation $\odot$ is realized by $[r] \odot [s] = [r \odot s]$, and the operation $\to$ realized by $[r] \to [s] = [r \to s]$. [1] The algebra $\mathbf{A}_n$ is the *Lindenbaum-Tarski algebra* of the n-variate fragment of Basic logic. Every element $[t] \in A_n$ represents a distinct *truthfunction* of the n-variate fragment of Basic logic: indeed, by the completeness result in Theorem 4, Basic logic proves $r \to s$ and $s \to r$ if and only if

[1] It is possible to prove that the definitions of the operations $\odot$ and $\to$ over pairs of classes of equiprovable terms are independent of the choice of the representatives [Háj98, Lemma 2.2.16].

$r^{[0,n+1]}$ and $s^{[0,n+1]}$ coincide as functions from $[0, n+1]^n$ to $[0, n+1]$. [2] A natural question is that of describing explicitly the class of functions that contains exactly the truthfunctions of the n-variate fragment of Basic logic. We call these functions, *BL-functions*, and we call the problem of describing explicitly BL-functions, the *representation problem of Basic logic*.

As far as the representation problem of Boolean logic is concerned, the familiar *functional* completeness property guarantees that for every n-variate Boolean function f, there exists a Boolean term over n variables computing f. Conversely, every Boolean term over n variables computes an n-variate Boolean function, and two terms are equiprovable in Boolean logic if and only if they compute the same function by *semantic* completeness. Thus, there is a bijection between n-variate Boolean functions, and Boolean terms of n variables modulo equiprovability. This is not the case in the Basic logic setting, where we evaluate variables over $[0, n+1]$. Whereas it still holds that a Basic logic term over n variables computes an n-variate function over $[0, n+1]$, the converse does not hold: the set of terms is countable, but the set of functions is uncountable. It turns out that the task of isolating those n-variate functions over $[0, n+1]$ that are computed by terms in L_n is not trivial.

The representation problem of the n-variate fragment of Basic logic is equivalent to the problem of providing an explicit functional representation of the free BL-algebra over n-generators. To be precise, it is well known that the equivalent semantics of Basic logic, in the sense of Blok and Pigozzi [BP89], is a very natural subvariety of residuated lattices, defined as follows.

Definition 11 (BL-algebra)**.** *A* commutative bounded divisible residuated lattice *is an algebra* $(A, \vee, \wedge, \odot, \to, \top, \bot)$ *of type* $(2, 2, 2, 2, 0, 0)$ *such that:*

(i) $(A, \odot, \top)$ *is a commutative monoid;*

[2] Basic logic proves $r \to s$ and $s \to r$ if and only if, $(r \to s)^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]}$ and $(s \to r)^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]}$ for every $\mathbf{a} \in [0, n+1]^n$, if and only if $r^{[0,n+1]}(\mathbf{a}) \leq s^{[0,n+1]}(\mathbf{a})$ and $s^{[0,n+1]}(\mathbf{a}) \leq r^{[0,n+1]}(\mathbf{a})$ for every $\mathbf{a} \in [0, n+1]^n$, if and only if $r^{[0,n+1]} = s^{[0,n+1]}$.

(ii) $(A, \vee, \wedge, \top, \bot)$ *is a bounded lattice ($x \leq y$ if and only if $x \wedge y = x$);*

(iii) residuation *holds, that is, for all $a_1, a_2, a_3 \in A$:*

$$a_1 \odot a_3 \leq a_2 \text{ if and only if } a_3 \leq a_1 \to a_2;^{3} \tag{2.1}$$

(iv) divisibility *holds, that is, for all $a_1, a_2 \in A$:*

$$a_1 \wedge a_2 = a_1 \odot (a_1 \to a_2). \tag{2.2}$$

A BL-algebra *is a* prelinear *commutative bounded divisible residuated lattice, that is, for all $a_1, a_2 \in A$,*

$$(a_1 \to a_2) \vee (a_2 \to a_1) = \top. \tag{2.3}$$

The completeness result of Theorem 4 has the following algebraic counterpart.

Definition 12. *Let $n \geq 1$. The algebra,*

$$[0, n+1]' = ([0, n+1], \vee, \wedge, \odot, \to, \top, \bot),$$

is the algebra of type $(2, 2, 2, 2, 0, 0)$, where $\circ \in \{\odot, \to, \bot\}$ is realized by the operation $\circ^{[0,n+1]}$ of Definition 3, and $\circ \in \{\vee, \wedge, \top\}$ is realized by the following operations ($a_1, a_2 \in [0, n+1]$):

$$\begin{aligned} a_1 \wedge a_2 &= a_1 \odot (a_1 \to a_2) \\ a_1 \vee a_2 &= ((a_1 \to a_2) \to a_2) \wedge ((a_2 \to a_1) \to a_1) \\ \top &= \bot \to \bot \end{aligned}$$

For $\circ \in \{\vee, \wedge, \odot, \to, \top, \bot\}$, we let $\circ^{[0,n+1]'}$ denote the realization of $\circ$ in $[0, n+1]'$.

Observing that $\top^{[0,n+1]'} = n+1$ and, for every $a_1, a_2 \in [0, n+1]$, $a_1 \wedge^{[0,n+1]'} a_2 = \min(a_1, a_2)$, and $a_1 \vee^{[0,n+1]'} a_2 = \max(a_1, a_2)$, it is easy to check that the algebra $[0, n+1]'$ is a BL-algebra. Now, Theorem 4 is equivalent to the following statement.

[3] The residuation equivalence (2.1) can be written equivalently by equations [Háj98, Lemma 2.3.10]. Hence, BL-algebras form a variety.

Theorem 13 (Aglianó and Montagna)**.** *Let $n \geq 1$. The algebra $[0, n+1]'$ generates as a quasivariety the variety generated by the class of all n-generated BL-algebras.*

By universal algebraic facts [MMT81], the free n-generated BL-algebra is isomorphic to the Lindenbaum-Tarski algebra of the n-variate fragment of Basic logic, or equivalently to the algebra of truthfunctions of the n-variate fragment of Basic logic, with pointwise defined operations. Formally,

Corollary 14. *Let $n \geq 1$. The free n-generated BL-algebra is isomorphic to the algebra,*

$$(B_n, \vee, \wedge, \odot, \rightarrow, \top, \bot),$$

of type $(2, 2, 2, 2, 0, 0)$, where $B_n = \{t^{[0,n+1]} \mid t \in L_n\} \subseteq [0, n+1]^{[0,n+1]^n}$, $\bot$ and $\top$ are realized by $\bot^{[0,n+1]'}$ and $\top^{[0,n+1]'}$ respectively, and each $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$ is realized by the binary operation $\circ^{[0,n+1]'}$ defined pointwise.

In Section 2.1 we collect the previous work done on the representation issue of Basic logic. Section 2.2 is devoted to the functional representation of Łukasiewicz logic in terms of McNaughton functions. In Section 2.3, we shall define an explicit class of functions, the class of n-ary BL-functions F_n, that coincides with the class B_n of n-ary functions over $[0, n+1]$ computed by terms in L_n. Finally, in Section 2.4, we shall obtain a representation of the free n-generated BL-algebra, in terms of such explicit class of functions.

2.1 Previous Work

In this section, we formalize the representation issue of Basic logic, distinguishing between implicit and explicit functional representations, and then we summarize the previous work done on this subject.

In light of our introductory discussion, given a term $t \in L_n$, we call the n-ary function $t^{[0,n+1]}$ over $[0, n+1]$ the *(Basic logic) truthfunction* of t. For each $n \geq 1$, we let the set,

$$B_n = \{t^{[0,n+1]} \mid t \in L_n\} \subseteq [0, n+1]^{[0,n+1]^n}, \tag{2.4}$$

denote the truthfunctions of the n-variate fragment of Basic logic. It is immediate to observe (say, for cardinality reasons) that,

$$B_n \subset [0, n+1]^{[0,n+1]^n};$$

precisely, by the inductive definition (Definition 8) of the map,

$$t \mapsto t^{[0,n+1]},$$

sending each $t \in L_n$ to a function $t^{[0,n+1]} \colon [0, n+1]^n \to [0, n+1]$, the set B_n is attained as the smallest set of n-ary functions over $[0, n+1]$ satisfying the following inductive clauses:

Basis Clause: B_n contains the n-ary constant function 0, and the n-ary projection functions $x_1, \dots, x_n$. [4] Indeed, $0 = \bot^{[0,n+1]} \in B_n$, and $x_i = X_i^{[0,n+1]} \in B_n$ for $i = 1, \dots, n$.

Inductive Clause: B_n is closed under pointwise application of the binary operations $\odot^{[0,n+1]}$ and $\to^{[0,n+1]}$ of Definition 3, that is: if $f, g \in B_n$, then $(f \odot g), (f \to g) \in B_n$, where, for every $\mathbf{a} \in [0, n+1]^n$:

$$(f \odot g)(\mathbf{a}) = f(\mathbf{a}) \odot^{[0,n+1]} g(\mathbf{a}),$$
$$(f \to g)(\mathbf{a}) = f(\mathbf{a}) \to^{[0,n+1]} g(\mathbf{a}).$$

Indeed, if $f, g \in B_n$, then $f = r^{[0,n+1]}$ and $g = s^{[0,n+1]}$ for some $r, s \in L_n$. Therefore, $(r \odot s), (r \to s) \in L_n$, and $(r \odot s)^{[0,n+1]}, (r \to s)^{[0,n+1]} \in B_n$. But, for every $\mathbf{a} \in [0, n+1]^n$,

$$\begin{aligned}(r \odot s)^{[0,n+1]}(\mathbf{a}) &= r^{[0,n+1]}(\mathbf{a}) \odot^{[0,n+1]} s^{[0,n+1]}(\mathbf{a}) \\ &= f(\mathbf{a}) \odot^{[0,n+1]} g(\mathbf{a}) \\ &= (f \odot g)(\mathbf{a}),\end{aligned}$$

and,

$$\begin{aligned}(r \to s)^{[0,n+1]}(\mathbf{a}) &= r^{[0,n+1]}(\mathbf{a}) \to^{[0,n+1]} s^{[0,n+1]}(\mathbf{a}) \\ &= f(\mathbf{a}) \to^{[0,n+1]} g(\mathbf{a}) \\ &= (f \to g)(\mathbf{a}),\end{aligned}$$

[4] Let $u, n \geq 1$ be fixed. The n-ary *constant* function $0 \colon [0, u]^n \to [0, u]$ is such that $0(\mathbf{a}) = 0$ for every $\mathbf{a} \in [0, u]^n$. Similarly, the n-ary constant function $u \colon [0, u]^n \to [0, u]$ is such that $0(\mathbf{a}) = u$ for every $\mathbf{a} \in [0, u]^n$. For $i = 1, \dots, n$, the n-ary *projection* function $x_i \colon [0, u]^n \to [0, u]$ is such that $x_i(\mathbf{a}) = a_i$ for every $\mathbf{a} = (a_1, \dots, a_n) \in [0, u]^n$.

so that $(f \odot g), (f \to g) \in B_n$.

The inductive definition above furnishes an *implicit* description of the class B_n, in the sense that functions in B_n are characterized as the functions computed by a certain class of algebraic circuits; precisely, the algebraic circuits of n inputs over $[0, n+1]$, implementing terms $t \in L_n$ over the basis $\{\odot^{[0,n+1]}, \to^{[0,n+1]}, \perp^{[0,n+1]}\}$.

The aim of this work is to provide an *explicit* characterization of the functions in B_n, adopting a notion of explicitness inspired by the familiar definition of McNaughton functions (Definition 18).

Definition 15 (Explicitness)**.** *Let $n \geq 0$, let $f : [0, n+1]^n \to [0, n+1]$, and let $\mathbf{b} = (b_1, \ldots, b_n) \in [0, n+1]^n$. An explicit description of $f(\mathbf{b})$ is an expression of the form,*

$$f(\mathbf{b}) = p(\mathbf{b}), \tag{2.5}$$

where p is an n-variate linear polynomial with integer coefficients over the real numbers. We say that f is described explicitly if there exists a finite collection of n-variate linear polynomials with integer coefficients over the real numbers that describe $f(\mathbf{b})$ explicitly for every $\mathbf{b} \in [0, n+1]^n$. We say that a class of functions has an explicit *description if it contains only explicitly described functions.*

An explicit description of a function f improves an implicit description for at least two reasons. The first is that if f has an explicit description, then f has exactly one explicit description, whereas if f has an implicit description, then it is easy to realize that f has infinitely many implicit descriptions. The second is that, in contrast with implicit descriptions, explicit descriptions give a geometrical intuition on the described functions, that are fruitful for several logical and algebraic applications: this geometrical insight and its applications are discussed in the conclusion of this thesis (Chapter 3).

The problem of describing explicitly the class B_n splits into the following two subproblems:

Subproblem 1: Describe explicitly a class F_n of n-ary functions over $[0, n+1]$.

Subproblem 2: Prove that $B_n \subseteq F_n$ that is, prove that for every term t in L_n, the function $t^{[0,n+1]}$ is in F_n. Conversely, prove that $F_n \subseteq$

B_n that is, prove that for every function f in F_n, there exists a term t in L_n such that $f = t^{[0,n+1]}$; as an additional benefit, given an effective specification of f, provide an effective construction of t.

We will apply the previous schema to the special cases $n = 1$ and $n = 2$ (Section 2.3.2 and Section 2.3.3), and eventually to the general case $n \geq 1$ (Section 2.3.4).

Despite its very natural and elementary statement, the problem of describing explicitly the class B_n has been solved only for the case $n = 1$, by Montagna [Mon00]. An effective construction of terms $t \in L_1$ computing functions $f \in B_1$ has been provided by Aguzzoli and Gerla [AG05]. However, even the case $n = 2$ eluded intensive research efforts [Mon01].

In the next section, we preliminarily study the representation issue of Łukasiewicz logic.

2.2 McNaughton Functions and Free MV-algebra

In this section, we study the case of Łukasiewicz logic. The completeness theorem of Chang, together the representation theorem of McNaughton, guarantees that the truthfunctions of the n-variate fragment of Łukasiewicz logic coincide with n-ary McNaughton functions [Cha58, McN51, Mun94].

The logical calculus of *Łukasiewicz logic*, $\vdash_L$, is defined by adding the following axiom to the axioms of Basic logic in Definition 1 ($t \in L$):

(L1) $\neg\neg t \to t$,

An MV-algebra $\mathbf{A} = (A, \vee, \wedge, \odot, \to, \top, \bot)$ is a BL-algebra satisfying *involutiveness* , that is,

$$a = \neg\neg a, \tag{2.6}$$

for all $a \in A$, where $\neg a$ is for $a \to \bot$. The variety of MV-algebras forms the algebraic semantics of Łukasiewicz logic, that is, for every $t \in L$,

$$\vdash_L t \text{ if and only if } \mathbf{A} \models t = \top,$$

for every MV-algebra $\mathbf{A}$. As a strengthening of this completeness result, we recall the celebrated completeness theorem of Chang [Cha58].

Definition 16. *The algebra* $[0,1] = ([0,1], \odot, \to, \bot)$ *is the algebra of type* $(2,2,2,2,0,0)$ *defined as follows* ($a_1, a_2 \in [0,1]$)*:*

$$a_1 \odot a_2 = \max(0, a_1 + a_2 - 1)$$
$$a_1 \to a_2 = \min(1, a_2 + 1 - a_1)$$
$$\bot = 0$$

For $\circ \in \{\odot, \to, \bot\}$*, we let* $\circ^{[0,1]}$ *denote the realization of* $\circ$ *in* $[0,1]$*.*

Theorem 17 (Chang)**.** *Let* $t \in L$*. Then,*

$$\vdash_{\mathsf{L}} t \textit{ if and only if } [0,1] \models t = \top.$$

In light of the previous completeness result, given a term $t \in L_n$, we call the n-ary function $t^{[0,1]}$ over $[0,1]$ the *Łukasiewicz logic truthfunction* of t. For each $n \geq 1$, we let the set,

$$Ł_n = \{t^{[0,1]} \mid t \in L_n\} \subseteq [0,1]^{[0,1]^n},$$

denote the truthfunctions of the n-variate fragment of Łukasiewicz logic.

As above, it is immediate to observe that $Ł_n \subset [0, n+1]^{[0,n+1]^n}$, precisely, $Ł_n$ is the smallest set of n-ary functions over $[0,1]$ that contains the n-ary constant 0, the n-ary projections $x_1, \ldots, x_n$, and is closed under pointwise application of the binary operations $\odot^{[0,1]}$ and $\to^{[0,1]}$.

It is known that the implicit class $Ł_n$ coincides with the explicit class of n-ary McNaughton functions. We rephrase this important result in terms of the solution schema presented in Section 2.1. The first step of our schema consists in guessing an explicit class of n-ary functions over $[0,1]$, for every $n \geq 1$.

Definition 18 (McNaughton Function)**.** *Let* $n \geq 1$*. A continuous* n*-ary function* $f : [0,1]^n \to [0,1]$ *is an* n*-ary* McNaughton function *if and only if there exists a finite collection* C_f *of* n*-variate linear polynomials with integer coefficients over the real numbers such that, for every* $\mathbf{a} \in [0,1]^n$*,* $f(\mathbf{a}) = p(\mathbf{a})$ *for some* $p \in C_f$*. We let* M_n *denote the set of* n*-ary McNaughton functions.*

Note that the n-ary constant 0, and the n-ary constant 1, are n-ary McNaughton functions.

Since the notion of explicitness introduced in Definition 15 abstracts upon the definition of McNaughton functions,

Fact 19. *M_n has an explicit description, for every $n \geq 1$.*

The explicit class M_n settles the first step of our solution schema. The second step asks for a proof that the class M_n of n-ary McNaughton functions coincides with the class $Ł_n$ of n-ary Łukasiewicz truthfunctions, thus replacing the implicit, circuital, description furnished by the latter by the explicit, geometric, description furnished by the former. As it is well know,

Theorem 20 (McNaughton)**.** *The truthfunctions of the n-variate fragment of Łukasiewicz logic coincide with the n-ary McNaughton functions, that is, $M_n = Ł_n$.*

In fact, Theorem 20 gives a functional representation of the free n-generated MV-algebra, in terms of n-ary McNaughton functions. Formally,

Definition 21. *The algebra $[0,1]' = ([0,1], \vee, \wedge, \odot, \rightarrow, \top, \bot)$ is the algebra of type $(2,2,2,2,0,0)$, where $\circ \in \{\odot, \rightarrow, \bot\}$ is realized by the operation $\circ^{[0,1]}$ of Definition 16, and $\circ \in \{\vee, \wedge, \top\}$ is realized by the following defined operations ($a_1, a_2 \in [0,1]$):*

$$\begin{aligned} a_1 \wedge a_2 &= a_1 \odot (a_1 \rightarrow a_2) \\ a_1 \vee a_2 &= ((a_1 \rightarrow a_2) \rightarrow a_2) \wedge ((a_2 \rightarrow a_1) \rightarrow a_1) \\ \top &= \bot \rightarrow \bot \end{aligned}$$

For $\circ \in \{\vee, \wedge, \odot, \rightarrow, \top, \bot\}$, we let $\circ^{[0,1]'}$ denote the realization of $\circ$ in $[0,1]'$.

Fact 22. *The algebra $[0,1]'$ is an MV-algebra.*

The completeness statement of Theorem 17 has the following, equivalent, universal algebraic rephrasing.

Theorem 23. *The algebra $[0,1]'$ generates the variety of MV-algebras.*

Corollary 24. *Let $n \geq 1$. The free n-generated MV-algebra is isomorphic to the algebra $(Ł_n, \vee, \wedge, \odot, \rightarrow, \top, \bot)$ of type $(2,2,2,2,0,0)$, where $\bot$ and $\top$ are realized by $\bot^{[0,1]}$ and $\top^{[0,1]}$ respectively, and each $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$ is realized by the binary operation $\circ^{[0,1]'}$ defined pointwise.*

For instance, if f and g are in $Ł_n$, say $f = r^{[0,1]}$ and $g = s^{[0,1]}$ for $r, s \in L_n$, then $(f \vee g)$ is in $Ł_n$, where $(f \vee g)$ is defined by,

$$\begin{aligned}(f \vee g)(\mathbf{a}) &= f(\mathbf{a}) \vee^{[0,1]'} g(\mathbf{a}) \\ &= (f(\mathbf{a}) \rightarrow^{[0,1]'} g(\mathbf{a})) \rightarrow^{[0,1]'} g(\mathbf{a}) \\ &= (f(\mathbf{a}) \rightarrow^{[0,1]} g(\mathbf{a})) \rightarrow^{[0,1]} g(\mathbf{a}) \\ &= (r^{[0,1]}(\mathbf{a}) \rightarrow^{[0,1]} s^{[0,1]}(\mathbf{a})) \rightarrow^{[0,1]} s^{[0,1]}(\mathbf{a}) \\ &= (r \rightarrow s)^{[0,1]}(\mathbf{a}) \rightarrow^{[0,1]} s^{[0,1]}(\mathbf{a}) \\ &= ((r \rightarrow s) \rightarrow s)^{[0,1]}(\mathbf{a}) \\ &= (r \vee s)^{[0,1]}(\mathbf{a})\end{aligned}$$

for every $\mathbf{a} \in [0,1]^n$.

Now, since M_n has an explicit description, and $M_n = Ł_n$, we obtain a functional representation of the free n-generated MV-algebra.

Theorem 25 (Functional Representation)**.** *Let $n \geq 1$. The free n-generated MV-algebra is isomorphic to the algebra $(M_n, \vee, \wedge, \odot, \rightarrow, \top, \bot)$, having type $(2,2,2,2,0,0)$, where $\bot$ and $\top$ are realized by the n-ary constant 0 and constant 1 functions respectively, and each $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$ is realized by the binary operation $\circ^{[0,1]'}$ defined pointwise.*

For instance, if f and g are in M_n, then $(f \vee g)$ is in M_n, where $(f \vee g)$ is defined by,

$$\begin{aligned}(f \vee g)(\mathbf{a}) &= f(\mathbf{a}) \vee^{[0,1]'} g(\mathbf{a}) \\ &= \max(f(\mathbf{a}), g(\mathbf{a}))\end{aligned}$$

for every $\mathbf{a} \in [0,1]^n$.

As an additional benefit, in [Mun94], Mundici described an effective construction that, given any n-ary McNaughton function f in input, returns in output a term $t \in L_n$ such that $t^{[0,1]} = f$. To describe such construction, an effective encoding of the input function f is necessary. The following notion is crucial.

Definition 26 (Unimodular Triangulation)**.** *A set $S \subseteq [0,1]^n$ is an n-dimensional* simplex *if it is the convex hull of a set of $n+1$ affinely independent points $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ in $[0,1]^n$, called the* vertices *of S. Let $\mathbf{v} \in [0,1]^n$ be* *a rational point, w.l.o.g.* $\mathbf{v} = (a_1/d, \ldots, a_n/d)$ *for uniquely determined relatively prime integers* $a_1, \ldots, a_n, d$ *with* $d \geq 1$. *We say that* $(a_1, \ldots, a_n, d)$ *are the* homogeneous coordinates *of* $\mathbf{v}$. *An* n*-dimensional simplex* S *in* $[0,1]^n$ *with rational vertices listed by* $(\mathbf{v}_1, \ldots, \mathbf{v}_{n+1})$ *can be displayed as the integer square matrix* M_S *having the homogeneous coordinates of* $\mathbf{v}_i$ *as* i*th row. An* n*-dimensional simplex* S *with rational vertices is* unimodular *if* $|\det(M_S)| = 1$. *The convex hull of any subset of* $(k+1)$ *vertices of* S *is a* k*-dimensional simplex, called a* face *of* S *(the empty set is a face of any simplex).*

A unimodular triangulation *of* $[0,1]^n$ *is a finite set* U *of* n*-dimensional unimodular simplexes, such that each face of each simplex of* U *belongs to* U*, any two simplexes of* U *intersect in a common face, and* $[0,1]^n$ *is the union of all simplexes in* U*.*

Let f *be an* n*-ary McNaughton function, and let* U *be a unimodular triangulation of* $[0,1]^n$*. We say that* U linearizes f *if for every simplex* $S \in U$*, there is* $p \in C_f$ *such that* $f(\mathbf{a}) = p(\mathbf{a})$ *for every* $\mathbf{a} \in S$*. Let* U *and* U' *be unimodular triangulations. Then, we say that* U' *is a* refinement *of* U *if, for every* $S \in U$*, there exist* $S_1, \ldots, S_m \in U'$ *such that* $\bigcup_{j \in [m]} S_j = S$*.*

The following statements follow directly from Theorem 20.

Fact 27. *The following statements hold.*

(i) *Let* f *be an* n*-ary McNaughton function. Then, there exists a unimodular triangulation of* $[0,1]^n$ *that linearizes* f*.*

(ii) *Let* f *and* f' *be* n*-ary McNaughton functions, let* U *and* U' *be unimodular triangulations linearizing* f *and* f' *respectively, and let* $g = f \circ f'$ *for* $\circ \in \{\odot, \to\}$*. Then, there exists a unimodular triangulation* V *that refines both* U *and* U' *and linearizes the* n*-ary McNaughton function* g*.*

By Definition 26 and Fact 27(i), any McNaughton function f can be effectively encoded by a finite set of pairs (S, p) such that $S \in U$ for some unimodular triangulation U linearizing f, and $p \in C_f$ is such that $f(\mathbf{b}) = p(\mathbf{b})$ for every $\mathbf{b} \in S$.

Theorem 28 (Mundici)**.** *Let* f *be an* n*-ary McNaughton function. Then, there is an algorithm that receives in input an effective encoding of* f *and returns in output a term* $t \in L_n$ *such that* $t^{[0,1]} = f$*.*

We shall use the following notion.

Definition 29 (1-Reproducing)**.** *Let $n \geq 1$. If $t \in L_n$ is such that $t^{[0,1]}(\mathbf{1}) = 1$, we say that t is* 1-reproducing*. Let $I = \{i_1, \ldots, i_m\} \subset [n]$ be such that $i_1 < \cdots < i_m$. If $t \in L_I$ is such that $t_{\{i_1 \backslash 1, \ldots, i_m \backslash m\}} \in L_m$ is 1-reproducing, we say that t is* 1-reproducing.

Note that if t is in L_n^+, then t is 1-reproducing, but the converse does not hold. However,

Corollary 30. *Let $s \in L_n$ be a 1-reproducing term. Then, there is an algorithm that receives in input s and returns in output a term $t \in L_n^+$ such that $t^{[0,1]} = s^{[0,1]}$.*

Proof. It is easy to check that the following equations are valid in the MV-algebra $[0, 1]$:

(i) $r \odot \bot = \bot$, $\bot \odot r = \bot$, $\bot \to r = r \to r$, and $r \to \bot = \neg r$;

(ii) $(r \odot \neg s) = \neg(r \to s)$ and $(\neg r \odot s) = \neg(s \to r)$;

(iii) $(r \to \neg s) = \neg(r \odot s)$ and $(\neg r \to s) = (r \to (r \odot s)) \to s$;

(iv) $\neg\neg r = r$.

Now, applying the equations above, compute a term t starting from s by subsequent substitutions, as follows. First, remove the symbol $\bot$, possibly introducing the symbol $\neg$, applying the equations in (i). Then, either remove or move outwards the symbol $\neg$ applying the equations in (ii)-(iii). Eventually, remove pairs of the symbol $\neg$ from the prefix applying the equation in (iv). Clearly, $s^{[0,1]} = t^{[0,1]}$. Moreover, $t \in L_n^+$. For otherwise, $t = \neg r$ for some $r \in L_n^+$. But then, $r^{[0,1]}(\mathbf{1}) = 1$, and $(\neg r)^{[0,1]}(\mathbf{1}) = 0$, contradiction since $(\neg r)^{[0,1]}(\mathbf{1}) = t^{[0,1]}(\mathbf{1}) = s^{[0,1]}(\mathbf{1}) = 1$ by hypothesis. □

As an application of the previous corollary, given a term $t \in L_n$ such that t is not 1-reproducing, it is possible to compute a term $s = \neg s'$ with $s' \in L_n^+$ such that $t^{[0,1]} = s^{[0,1]}$.

In the next section, we introduce and study the main object of this thesis, that is, the class of BL-functions.

2.3 BL-Functions

The goal of this section is to describe explicitly a class of n-ary functions over $[0, n+1]$, called *n-ary BL-functions* ($n \geq 1$), and to show that these functions coincide with the truthfunctions of the n-variate fragment of Basic logic (Section 2.3.1 and Section 2.3.4). Preliminarily, we study the case $n = 1$ (Section 2.3.2) and the case $n = 2$ (Section 2.3.3). As we mentioned in Section 2.1, the case $n = 1$ has already been solved by Montagna [Mon00] and Aguzzoli and Gerla [AG05].

2.3.1 Definition

This section is devoted to the description of a suitable framework for defining and investigating BL-functions. We refer the reader to Section 3.1.1 for an alternative, equivalent, definition of BL-functions in geometrical terms.

Definition 31 (Parameter, Neighborhood). *Let $n \geq 1$ and let $\mathbf{a} \in [0,1]^n$. We call the set,*

$$\mathrm{par}(\mathbf{a}) = \{i \in [n] \mid a_i = 1\},$$

the set of parameters *of* $\mathbf{a}$*, and we call the set,*

$$\mathrm{neigh}(\mathbf{a}) = \{\mathbf{c} \in [0,1]^n \mid c_i = 1 \textit{ if } i \in \mathrm{par}(\mathbf{a}) \textit{ and } 0 \leq c_i < 1 \textit{ otherwise}\}$$

the neighborhood *of* $\mathbf{a}$*. See Figure 2.1(a) and (c), Figure 2.2(a) and (c), and Figure 2.3(a) and (c).*

Note that $\mathrm{par}(\mathbf{a}) = [n]$ if and only if $\mathbf{a} = \mathbf{1}$.

Definition 32 (Realm). *Let $n \geq 1$. The map* controller *from $[0, n+1]^n$ to $[0,1]^n$ is defined as follows, for every $\mathbf{b} = (b_1, \ldots, b_n) \in [0, n+1]^n$:*

$$\mathrm{controller}(\mathbf{b}) = \begin{cases} \mathbf{1} & \textit{if } \mathbf{b} = \mathbf{n+1} \\ \mathbf{b} - \mathbf{j} & \textit{if } \lfloor b_1 \rfloor = \cdots = \lfloor b_n \rfloor = j \\ (a_1, \ldots, a_n) & \textit{otherwise} \end{cases}$$

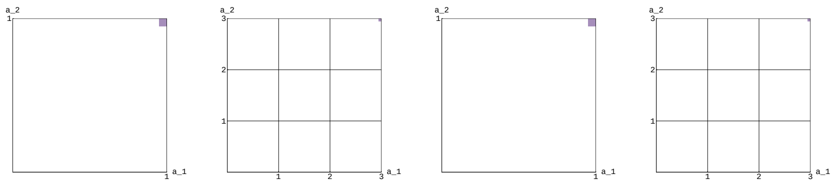
where, for $i = 1, \ldots, n$,

$$a_i = \begin{cases} b_i - \lfloor b_i \rfloor & \textit{if } \lfloor b_i \rfloor = \min\{\lfloor b_k \rfloor \mid k \in [n]\} \\ 1 & \textit{otherwise} \end{cases}$$

We say that $\mathbf{a}$ controls $\mathbf{b}$ *if* $\mathrm{controller}(\mathbf{b}) = \mathbf{a}$. *Let* $A \subseteq [0,1]^n$. *The set,*

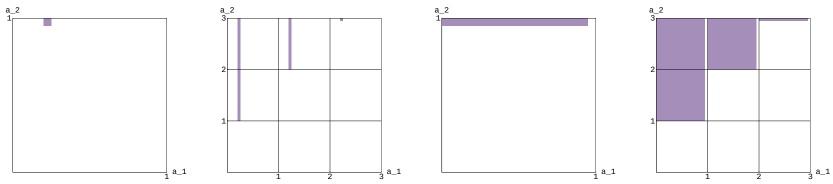
$$\mathrm{realm}(A) = \{\mathbf{b} \in [0, n+1]^n \mid \textit{there exists } \mathbf{a} \in A \textit{ that controls } \mathbf{b}\},$$

is called the realm *of* A. *See Figure 2.1(b) and (d), Figure 2.2(b) and (d), and Figure 2.3(b) and (d).*



(a) (b) (c) (d)

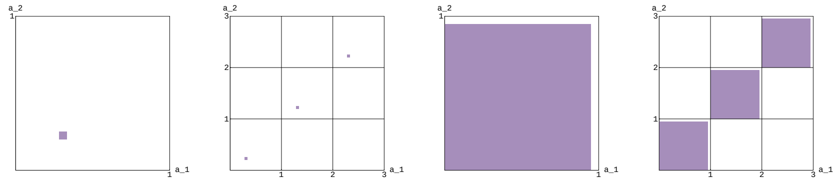
Figure 2.1: Sampling Definitions 31 and 32 with $n = 2$. Figure (a) highlights the point $\mathbf{a} = \mathbf{1} \in [0,1]^2$. Here, $\mathrm{par}(\mathbf{a}) = \{1, 2\}$. Figures (b), (c), and (d) respectively highlight $\mathrm{realm}(\{\mathbf{a}\}) = \{\mathbf{3}\}$, $\mathrm{neigh}(\{\mathbf{a}\}) = \{\mathbf{1}\}$, and $\mathrm{realm}(\mathrm{neigh}(\{\mathbf{a}\})) = \{\mathbf{3}\}$.



(a) (b) (c) (d)

Figure 2.2: Sampling Definitions 31 and 32 with $n = 2$. Figure (a) highlights the points $\mathbf{a} = (a_1, a_2) = (1/4, 1) \in [0,1]^2$. Here, $\mathrm{par}(\mathbf{a}) = \{2\}$. Figure (b) highlights $\mathrm{realm}(\{\mathbf{a}\}) = \{(a_1, b_2) \mid 1 \leq b_2 \leq 3\} \cup \{(a_1 + 1, b_2) \mid 2 \leq b_2 \leq 3\} \cup \{(a_1 + 2, 3)\} \subseteq [0,3]^2$. Figure (c) highlights $\mathrm{neigh}(\{\mathbf{a}\}) = \{(c_1, 1) \mid 0 \leq c_1 < 1\} \subseteq [0,1]^2$. Figure (d) highlights $\mathrm{realm}(\mathrm{neigh}(\{\mathbf{a}\})) = \{(b_1, b_2) \mid 0 \leq b_1 < 1 \leq b_2 \leq 3\} \cup \{(b_1, b_2) \mid 1 \leq b_1 < 2 \leq b_2 \leq 3\} \cup \{(b_1, 3) \mid 2 \leq b_1 < 3\} \subseteq [0,3]^2$.

In contrast with the case of n-ary McNaughton functions and unimodular triangulations of $[0,1]^n$, we deal with a class of discontinuous n-ary functions over $[0, n+1]^n$, so that, to provide a blockwise description of these functions, we need a partition of $[0, n+1]^n$ into disjoint blocks. As a preliminary step, given a unimodular triangulation U of $[0,1]^n$, we determine a finite collection of rational points in $[0,1]^n$, such



(a) (b) (c) (d)

Figure 2.3: Sampling Definitions 31 and 32 with $n = 2$. Figure (a) highlights the point $\mathbf{a} = (a_1, a_2) = (1/3, 1/4) \in [0,1]^2$. Here, $\mathrm{par}(\mathbf{a}) = \emptyset$. Figure (b) highlights $\mathrm{realm}(\{\mathbf{a}\}) = \{(a_1, a_2), (a_1 + 1, a_2 + 1), (a_1 + 2, a_2 + 2)\} \subseteq [0,3]^2$. Figure (c) highlights $\mathrm{neigh}(\{\mathbf{a}\}) = \{(c_1, c_2) \mid 0 = \lfloor c_1 \rfloor = \lfloor c_2 \rfloor < 1\} \subseteq [0,1]^2$. Figure (d) highlights $\mathrm{realm}(\mathrm{neigh}(\{\mathbf{a}\})) = \{(b_1, b_2) \mid 0 \leq \lfloor b_1 \rfloor = \lfloor b_2 \rfloor < 3\} \subseteq [0,3]^2$.

that each point represents the relative interior of a fixed simplex in U, as follows.

Definition 33 (Quasipartition, Parent)**.** *Let $n \geq 1$. Let U be any unimodular triangulation of $[0,1]^n$. The set,*

$$\tilde{U} \subseteq \mathbb{Q}^n \cap [0,1]^n,$$

is a quasipartition *of $[0,1]^n$ if the following holds: $\mathbf{u} \in \tilde{U}$ if and only if $\mathbf{u}$ is the mediant of the vertices of some simplex S in U. $\mathbf{u}$ is called the* delegate *of S. The points lying in the relative interior of S are called* siblings *of $\mathbf{u}$, and $\mathbf{u}$ is their* parent*. In symbols,*

$$\mathrm{sibl}(\mathbf{u}) = \{\mathbf{a} \in [0,1]^n \mid \mathbf{a} \in \mathrm{relint}\ S, \mathbf{u} \textit{ delegate of } S\}.$$

Note that $\tilde{U}$ encodes a genuine partition of $[0,1]^n$, that is, the blocks in,

$$\{\mathrm{sibl}(\mathbf{u}) \subseteq [0,1]^n \mid \mathbf{u} \in \tilde{U}\},$$

are disjoint and their union is equal to $[0,1]^n$. Note also that the blocks in,

$$\{\mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \subseteq [0,n+1]^n \mid \mathbf{u} \in \tilde{U}\},$$

form a genuine partition of $[0,n+1]^n$, that is, they are disjoint and their union is equal to $[0,n+1]^n$. This partitioning strategy is crucial in the description of n-ary BL-functions. See Figure 2.4.

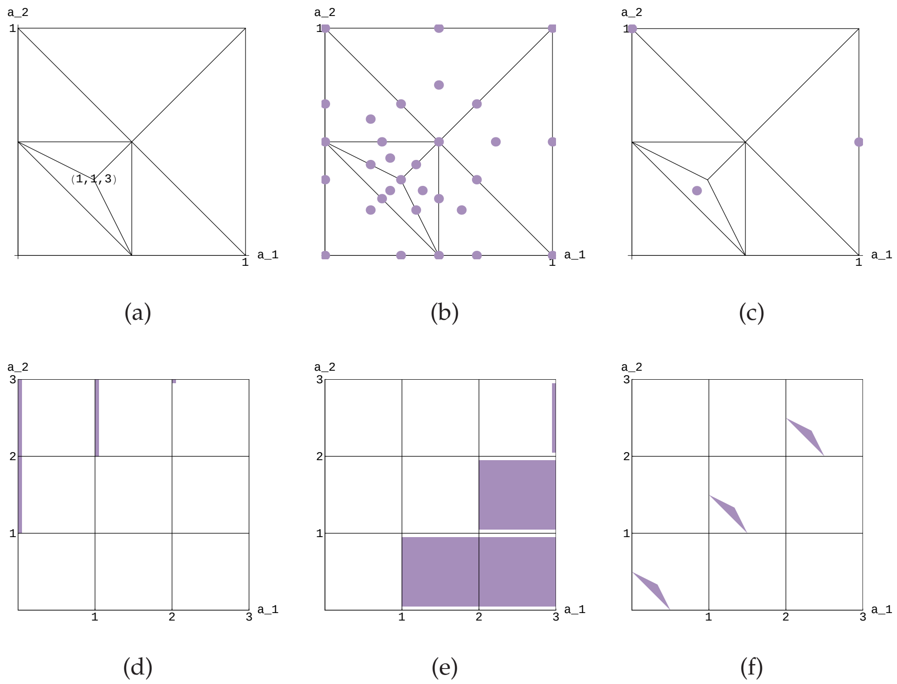


Figure 2.4: Sampling Definition 33 with $n = 2$. Figures (a) and (b) depict respectively a unimodular triangulation U of $[0,1]^2$, and the corresponding quasipartition $\tilde{U}$. Figure (c) highlights three delegates in $\tilde{U}$, namely $\mathbf{u} = (0,1) \in \tilde{U}$, $\mathbf{u}' = (1,1/2) \in \tilde{U}$, and $\mathbf{u}'' = (2/7,2/7) \in \tilde{U}$. Figures (d), (e), and (f) highlight respectively $\mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \subseteq [0,3]^2$, $\mathrm{realm}(\mathrm{sibl}(\mathbf{u}')) \subseteq [0,3]^2$, and $\mathrm{realm}(\mathrm{sibl}(\mathbf{u}'')) \subseteq [0,3]^2$.

Definition 34 (System). *Let $n \geq 1$. Let $r \in L_n$ be such that $r \in L_n^+$ if r is 1-reproducing, and $r = \neg r'$ with $r' \in L_n^+$ otherwise. Let U be a unimodular triangulation linearizing $r^{[0,1]}$. A* system *$\tilde{r}$ for r (with underlying quasipartition $\tilde{U}$) is a map,*

$$\tilde{r} \colon \tilde{U} \to L_n,$$

such that, for every $\mathbf{u} \in \tilde{U}$:

(i) if $\mathbf{u} = \mathbf{1}$, then, if r is 1-reproducing, $\tilde{r}(\mathbf{1}) \in L_n^+$;

(ii) otherwise, if $\mathbf{u} \notin [0,1)^n$ and $r^{[0,1]}(\mathbf{u}) = 1$, then $\tilde{r}(\mathbf{u}) \in L_{\mathrm{par}(\mathbf{u})}$ and $\tilde{r}(\mathbf{u})$ is 1-reproducing;

(iii) otherwise, $\tilde{r}(\mathbf{u}) = r$.

We say that $\tilde{r}(\mathbf{u})$ is the module *of the system* responsible *over* $\mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$.

Note that, if r is not 1-reproducing, then $\tilde{r}(\mathbf{1}) = r$.

Definition 35 (Implementation)**.** *Let $n \geq 1$ and let $r \in L_n$. Let $\tilde{r}$ be a system for r, over the quasipartition $\tilde{U}$, and let $\mathbf{u} \in \tilde{U}$. Let $\mathbf{b} \in [0, n+1]^n$ such that $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$. Then, we say that $\tilde{r}$* implements $f(\mathbf{b})$ *if the following holds:*

(i) If $\mathbf{b} \notin [1, n+1]^n$ and $\tilde{r}(\mathbf{u}) = t$, then:

$$f(\mathbf{b}) = t^{[0,n+1]}(\mathbf{b}).$$

(ii) If $\mathbf{b} \in [1, n+1]^n$ and $\tilde{r}(\mathbf{1}) = s$, the following holds. If r is not 1-reproducing, then,

$$f(\mathbf{b}) = r^{[0,n+1]}(\mathbf{b});$$

otherwise, $s \in L_n^+$ and there exists an auxiliary *system $\tilde{s}$ for s such that $\tilde{s}(\mathbf{u}) = t$ and,*

$$f(\mathbf{b}) = t^{[0,n+1]}(\mathbf{b}).$$

Let $f : [0, n+1]^n \to [0, n+1]$. We say that $\tilde{r}$ implements f *if $\tilde{r}$ implements $f(\mathbf{b})$ for every $\mathbf{b} \in [0, n+1]^n$.*

We are now in the position to define the class of n-ary BL-functions.

Definition 36 (n-ary BL-functions, F_n)**.** *Let $n \geq 1$. An n-ary function $f : [0, n+1]^n \to [0, n+1]$ is an* n-ary BL-function *if and only if there exists a system $\tilde{r}$ over a term $r \in L_n$ such that $\tilde{r}$ implements f. We let F_n denote the class of n-ary BL-functions.*

Our goal is to prove that, for every $n \geq 1$, the class F_n is explicit in terms of Definition 15, and coincides with the class B_n in (2.4). The proof is by induction on n. In the next section, we settle the base case, $n = 1$.

2.3.2 Unary Case

We want to show that F_1 has an explicit description in terms of Definition 15, and coincides with B_1. This case has already been solved by Montagna [Mon00] and Aguzzoli and Gerla [AG05].

We instatiate Definition 3 with $n = 1$.

Definition 37. *The algebra* $[0,2] = ([0,2], \odot, \to, \bot)$ *is the algebra of type* $(2,2,0)$, *defined as follows* ($a_1, a_2 \in [0,2]$)*:*

$$a_1 \odot a_2 = \begin{cases} \min(a_1, a_2) & \text{if } \lfloor a_1 \rfloor \neq \lfloor a_2 \rfloor \\ a_1 \odot^{[0,1]} a_2 & \text{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = 0 \\ (a_1 - 1 \odot^{[0,1]} a_2 - 1) + 1 & \text{otherwise} \end{cases}$$

$$a_1 \to a_2 = \begin{cases} 2 & \text{if } a_1 \leq a_2 \\ a_2 & \text{if } \lfloor a_2 \rfloor < \lfloor a_1 \rfloor \\ a_1 \to^{[0,1]} a_2 & \text{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = 0 \\ (a_1 - 1 \to^{[0,1]} a_2 - 1) + 1 & \text{otherwise} \end{cases}$$

$$\bot = 0$$

We enrich the signature with the defined operations $\vee$, $\wedge$, $\neg$, *and* $\top$ *of arity* 2, 2, 1, *and* 0 *respectively* ($a_1, a_2 \in [0,2]$)*:*

$$\begin{aligned} \neg a_1 &= a_1 \to \bot \\ a_1 \wedge a_2 &= a_1 \odot (a_1 \to a_2) \\ a_1 \vee a_2 &= ((a_1 \to a_2) \to a_2) \wedge ((a_2 \to a_1) \to a_1) \\ \top &= \neg \bot \end{aligned}$$

For $\circ \in \{\vee, \wedge, \odot, \to, \neg, \top, \bot\}$, *we let* $\circ^{[0,2]}$ *denote the realization of the symbol* $\circ$ *in the algebra* $[0,2]$.

Fact 38. *For every* $a_1, a_2 \in [0,2]$, $a_1 \wedge^{[0,2]} a_2 = \min(a_1, a_2)$, $a_1 \vee^{[0,2]} a_2 = \max(a_1, a_2)$, $\top^{[0,2]} = 2$, *and,*

$$\neg^{[0,2]} a_1 = \begin{cases} 2 & \text{if } a_1 = 0 \\ 1 - a_1 & \text{if } 0 < a_1 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that, by Notation 10, for every $r, s \in L_1$ and every $\mathbf{a} \in [0,2]$, $(\neg r)^{[0,2]}(\mathbf{a}) = \neg^{[0,2]} r^{[0,2]}(\mathbf{a})$, $(r \wedge s)^{[0,2]}(\mathbf{a}) = r^{[0,2]}(\mathbf{a}) \wedge^{[0,2]} s^{[0,2]}(\mathbf{a})$, $(r \vee s)^{[0,2]}(\mathbf{a}) = r^{[0,2]}(\mathbf{a}) \vee^{[0,2]} s^{[0,2]}(\mathbf{a})$, and $\top^{[0,2]}(\mathbf{a}) = \top^{[0,2]}$.

We mentioned that the truthfunctions of the 1-variate fragment of Basic logic, B_1, coincide with the smallest set of unary functions over $[0,2]$ that contains the projection function x_1, the constant function 0, and is closed under pointwise application of the operations $\odot^{[0,2]}$ and $\to^{[0,2]}$. The problem is to provide an explicit description of B_1: to this aim, we first guess an explicit class of unary functions over $[0,2]$, and we then prove that the guessed class coincides with B_1.

Definition 39 (Unary BL-functions, F_1)**.** *A unary function f over $[0,2]$ is a* unary BL-function *if and only if there exists a system $\tilde{r}$ over a term $r \in L_1$ that implements f. We let F_1 denote the class of unary BL-functions.*

We claim that the class of unary BL-functions is explicit, in the sense that if $\tilde{r}$ implements f, then $\tilde{r}$ provides an explicit description of f in terms of Definition 15. A formal proof requires preliminarily the following lemma (compare also Example 47 and Example 48).

Lemma 40 (Lifting)**.** *Let $t \in L_1$ be such that, if t is 1-reproducing, then $t \in L_1^+$, and $t = \neg t'$ with $t' \in L_1^+$ otherwise. Then, $t^{[0,2]}$ has an explicit description in terms of Definition 15.*

Proof. Let $\mathbf{b} = (b_1) \in [0,2]$ and let $\mathbf{a} \in [0,1]$ be such that $\mathbf{b} \in \mathrm{realm}(\{\mathbf{a}\})$. We distinguish two cases.

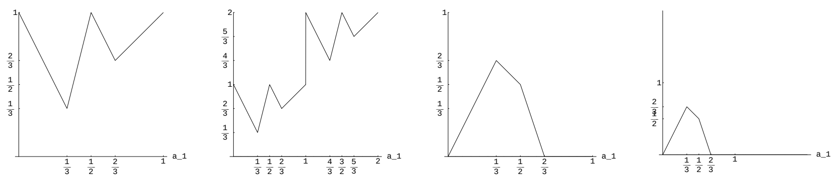
If t is 1-reproducing, so that $t \in L_1^+$, by induction on t, applying Definition 37, we have:

$$t^{[0,2]}(\mathbf{b}) = \begin{cases} t^{[0,1]}(\mathbf{b}) & \text{if } \lfloor b_1 \rfloor = 0 \text{ and } t^{[0,1]}(\mathbf{a}) < 1 \\ t^{[0,1]}(\mathbf{b} - \mathbf{1}) + 1 & \text{if } \lfloor b_1 \rfloor = 1 \text{ and } t^{[0,1]}(\mathbf{a}) < 1 \\ 2 & \text{otherwise} \end{cases} \tag{2.7}$$

If t is not 1-reproducing, so that $t = \neg t'$ with $t' \in L_1^+$, by Fact 38 and applying the previous case to t', we have:

$$t^{[0,2]}(\mathbf{b}) = \begin{cases} t^{[0,1]}(\mathbf{b}) & \text{if } \lfloor b_1 \rfloor = 0 \text{ and } 0 < t'^{[0,1]}(\mathbf{a}) < 1 \\ 2 & \text{if } \lfloor b_1 \rfloor = 0 \text{ and } t'^{[0,1]}(\mathbf{a}) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

In both cases, since $t^{[0,1]}$ has an explicit description by Fact 19, we have that $t^{[0,2]}(\mathbf{b})$ has an explicit description. Since this holds for any $\mathbf{b} \in [0,2]$, we conclude that $t^{[0,2]}$ has an explicit description, precisely, the description given by equations (2.7)-(2.8). $\square$



(a) $r^{[0,1]}$. (b) $r^{[0,2]}$. (c) $s^{[0,1]}$. (d) $s^{[0,2]}$.

Figure 2.5: A sample of Lemma 40 enlightens the *lifting* phenomenon. In plots (a)-(b), the term r is 1-reproducing, thus in L_1^+. In plots (c)-(d), the term $s \in L_1$ is not 1-reproducing.

Figure 2.5 enlightens the *lifting* phenomenon.

Proposition 41 (Explicitness)**.** *F_1 has an explicit description in terms of Definition 15.*

Proof. Let $f \in F_1$. Let $\tilde{r}$ be a system implementing f, over the quasipartition $\tilde{U}$. We distinguish two cases. First suppose that r is 1-reproducing. By Definition 34, if r is 1-reproducing, then $\tilde{r}(\mathbf{u}) \in L_1^+$ for every $\mathbf{u} \in \tilde{U}$. Let $\mathbf{b} \in [0,2]$. Thus, by Definition 35, we have that $f(\mathbf{b}) = t^{[0,2]}(\mathbf{b})$ with $t \in L_1^+$. Since, by Lemma 40, $t^{[0,2]}(\mathbf{b})$ has an explicit description, we conclude that $f(\mathbf{b})$ has an explicit description. Thus, f has an explicit description. Next suppose that r is not 1-reproducing, so that $r = \neg r'$ with $r' \in L_1^+$. By Definition 34, if r is not 1-reproducing, then $f(\mathbf{b}) = r^{[0,2]}(\mathbf{b})$ for every $\mathbf{b} \in [0,2]$. By Lemma 40, $r^{[0,2]}$ has an explicit description, hence, f has an explicit description. $\square$

The explictly described class F_1 settles the first step of our solution schema. The second step consists in proving that the class F_1 of unary BL-functions coincides with the truthfunctions of the 1-variate fragment of Basic logic,

$$B_1 = \{t^{[0,2]} \mid t \in L_1\} \subseteq [0,2]^{[0,2]}.$$

We prove the inclusion $B_1 \subseteq F_1$.

Lemma 42 (Closure)**.** $t^{[0,2]} \in F_1$ *for every* $t \in L_1$*.*

Proof. The proof is by induction on t. We show that there exists a function $f \in F_1$ such that $t^{[0,2]} = f$.

For the base case, let $t = X_1$. Take $r, s = X_1$ and fix a unimodular triangulation U of $[0,1]$ linearizing $X_1^{[0,1]} : [0,1] \to [0,1]$. Then, $\tilde{r} = \tilde{s} = \{(\mathbf{u}, X_1) \mid \mathbf{u} \in \tilde{U}\}$ forms a system for r and s. By definition, $\tilde{r}$ implements x_1, thus $x_1 \in F_1$. But, by definition, $X_1^{[0,2]}(\mathbf{a}) = a_1$ for every $\mathbf{a} = (a_1) \in [0,2]$, that is $X_1^{[0,2]}$ is the projection function x_1 over $[0,2]$. Now, let $t = \perp$. Take $r = \perp$ and fix a unimodular triangulation U linearizing $\perp^{[0,1]}$. Then, $\tilde{r} = \{(\mathbf{u}, \perp) \mid \mathbf{u} \in \tilde{U}\}$ forms a system for r. By definition, $\tilde{r}$ implements 0, thus $0 \in F_1$. But, by definition, $\perp^{[0,2]}(\mathbf{a}) = 0$ for every $\mathbf{a} \in [0,2]$, that is $\perp^{[0,2]}$ is the constant function 0 over $[0,2]$. The base case is proved.

For the inductive step, let $t = t_1 \circ t_2$ for $\circ \in \{\odot, \to\}$. By the induction hypothesis, $t_1^{[0,2]}, t_2^{[0,2]} \in F_1$. Below, we construct a system $\tilde{r}$ (over a certain quasipartition $\tilde{U}$) such that that $\tilde{r}$ implements the function $t_1^{[0,2]} \circ^{[0,2]} t_2^{[0,2]}$, so that $t_1^{[0,2]} \circ^{[0,2]} t_2^{[0,2]} \in F_1$. But, by definition, $t^{[0,2]} = t_1^{[0,2]} \circ^{[0,2]} t_2^{[0,2]}$, thus proving that $t^{[0,2]} \in F_1$.

The system $\tilde{r}$ is defined as follows. Since $t_1^{[0,2]}, t_2^{[0,2]} \in F_1$, there exist systems $\tilde{r_1}$ (say over quasipartition $\tilde{U_1}$) and $\tilde{r_2}$ (say over quasipartition $\tilde{U_2}$) implementing $t_1^{[0,2]}$ and $t_2^{[0,2]}$. On the basis of $\tilde{r_1}$ and $\tilde{r_2}$, we define $\tilde{r}$, as follows. There are eight cases ($\circ$ is equal to $\odot$ or to $\to$; both r_1 and r_2 are 1-reproducing, only r_1 is 1-reproducing, only r_2 is 1-reproducing, neither r_1 nor r_2 are 1-reproducing). We examine the case where $\circ$ is equal to $\odot$ and both r_1 and r_2 are 1-reproducing in L_1^+ (the other cases are similar, compare also Lemma 65). In this case, $\tilde{r_1}(\mathbf{1}) = s_1$ with s_1 1-reproducing in L_1^+, and $\tilde{r_2}(\mathbf{1}) = s_2$ with s_2 1-reproducing in L_1^+. Let $\tilde{s_1}$ (say over $\tilde{V_1}$) and $\tilde{s_2}$ (say over $\tilde{V_2}$) be the systems for s_1 and s_2 respectively (these systems exist by induction hypothesis). By Fact 27(ii), let U be a unimodular triangulation refining both U_1 and U_2 and linearizing $r_1 \odot r_2$, and let V be a unimodular triangulation refining both V_1 and V_2 and linearizing $s_1 \odot s_2$. We let $r = r_1 \odot r_2$ and $s = s_1 \odot s_2$ (note that both r and s are 1-reproducing in L_1^+). Now we define $\tilde{r}$ and $\tilde{s}$. First we put $\tilde{r}(\mathbf{1}) = s$. Next observe that, for every $\mathbf{1} \neq \mathbf{u} \in \tilde{U}$, there exists exactly one pair $(\mathbf{u}_1, \mathbf{u}_2) \in \tilde{U_1} \times \tilde{U_2}$ such that $\mathbf{u} \in \mathrm{sibl}(\mathbf{u}_1) \cap \mathrm{sibl}(\mathbf{u}_2)$. In this case we put $\tilde{r}(\mathbf{u}) = \tilde{r_1}(\mathbf{u}_1) \odot \tilde{r_2}(\mathbf{u}_2)$. Similarly, for every $\mathbf{v} \in \tilde{V}$, there exists exactly one pair $(\mathbf{v}_1, \mathbf{v}_2) \in \tilde{V_1} \times \tilde{V_2}$ such that $\mathbf{v} \in \mathrm{sibl}(\mathbf{v}_1) \cap \mathrm{sibl}(\mathbf{v}_2)$. In this case we put $\tilde{s}(\mathbf{v}) = \tilde{s_1}(\mathbf{v}_1) \odot \tilde{s_2}(\mathbf{v}_2)$. But now, by construction, $\tilde{r}$ implements $t_1^{[0,2]} \odot^{[0,2]} t_2^{[0,2]} = t^{[0,2]} = f$. The inductive step is proved. $\square$

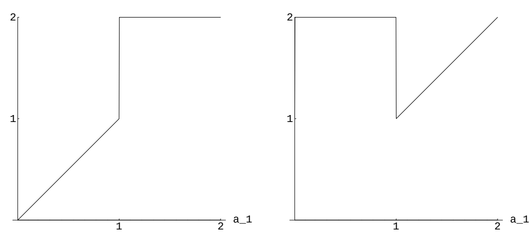
We now deal with the inclusion $F_1 \subseteq B_1$. We exploit the following *isolation* mechanism.

Definition 43 (Isolation). *Let $D \in \{[0,1),[1,2]\}$. Say that the term $t \in L_1$* isolates *the term $s \in L_1$ over D if $t^{[0,2]}(\mathbf{b}) = s^{[0,2]}(\mathbf{b})$ if $\mathbf{b} \in D$, and $t^{[0,2]}(\mathbf{b}) = \top^{[0,2]}$ if $\mathbf{b} \notin D$.*

The first observation is that it is easy to isolate X_1, either over $[0,1)$, or over $[1,2]$. Formally,

Lemma 44 (Variable Isolation). *There exist terms $v_1, w_1 \in L_1$ such that v_1 isolates X_1 over $[0,1)$, and w_1 isolates X_1 over $[1,2]$.*

Proof. Let $v_1 = \neg\neg X_1$ and $w_1 = \neg\neg X_1 \to X_1$. By Definition 37 and Fact 38, v_1 and w_1 satisfy the claim. Compare also Figure 2.6. □



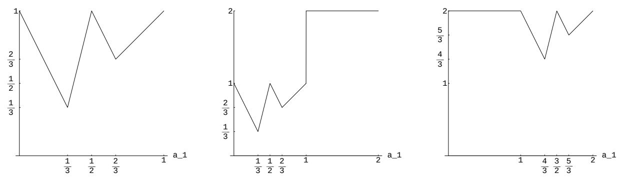
(a) $v_1^{[0,2]}$. (b) $w_1^{[0,2]}$.

Figure 2.6: The isolation of variable X_1.

The second observation is that, relying on the availability of v_1 and w_1, a direct inspection of the definitions of $\odot^{[0,2]}$ and $\to^{[0,2]}$ is sufficient to realize that, given a term $t \in L_1^+$, it is easy to isolate t over $[0,1)$ or over $[1,2]$. Formally,

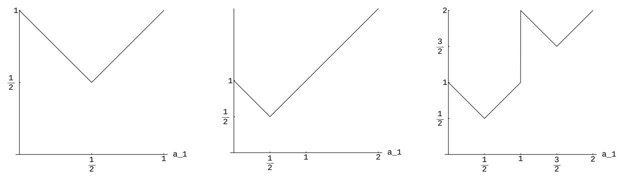
Lemma 45 (Term Isolation). *Let t be a term in L_1 such that, if t is 1-reproducing, then $t \in L_1^+$, otherwise $t = \neg t'$ with $t' \in L_1^+$. Then, there exist terms $\hat{t}$ and $\check{t}$ in L_1 such that, $\hat{t}$ isolates t over $[0,1)$, and $\check{t}$ isolates t over $[1,2]$.*

Proof. We distinguish two cases. If $t \in L_1$ is 1-reproducing ($t \in L_1^+$), put $\hat{t} = t_{\{1\backslash v_1\}}$ and $\check{t} = t_{\{1\backslash w_1\}}$, where v_1 and w_1 are as in Lemma 44. A routine induction on t, with appeal to Lemma 44 and Definition 37, shows that $\hat{t}$ and $\check{t}$ satisfy the claim. Compare also Figure 2.7 and Figure 2.8. Otherwise, if $t \in L_1$ is not 1-reproducing ($t = \neg t'$ with $t' \in L_1^+$), then put $\hat{t} = \check{t} = \neg t'_{\{1\backslash v_1\}}$, where v_1 is as in Lemma 44. Applying the first case to t' and the definition of $\neg^{[0,2]}$ in Fact 50, we have that $\hat{t}$ isolates $\neg t'$, that is t, over $[0, 1)$, and $\check{t}$ isolates $\neg t'$, that is $\perp$, over $[1, 2]$. □



(a) $t^{[0,1]}$. (b) $\hat{t}^{[0,2]}$. (c) $\check{t}^{[0,2]}$.

Figure 2.7: Sampling Lemma 45 with $t \in L_1^+$ 1-reproducing depicted in (a). By Lemma 44, $\hat{t}^{[0,2]}$ and $\check{t}^{[0,2]}$ are as in plots (b) and (c) respectively.



(a) $r^{[0,1]} = s^{[0,1]}$. (b) $r^{[0,2]}$. (c) $s^{[0,2]}$.

Figure 2.8: Let $t \in L_1$ be a 1-reproducing term. In this case, the syntactic assumption $t \in L_1^+$ in Lemma 44, supported by Corollary 30, is necessary, for otherwise the defined terms $\hat{t}^{[0,2]}$ and $\check{t}^{[0,2]}$ do not satisfy the statement. For instance, letting $r = (X_1 \vee \neg X_1) \in L_1$ and $s = (X_1 \to (X_1 \odot X_1)) \in L_1^+$, we have that $r^{[0,1]} = s^{[0,1]}$ and, in particular, $r^{[0,1]}(\mathbf{1}) = s^{[0,1]}(\mathbf{1}) = 1$, so that both r and s are 1-reproducing by Definition 29. However, $\hat{s}^{[0,2]}$ isolates s over $[0, 1)$ and $\check{s}^{[0,2]}$ isolates s over $[1, 2]$ as depicted in (c), but $\check{r}^{[0,2]}$ does not isolate r over $[1, 2]$ as depicted in (b).

Relying on the availability of the term isolation mechanism described above, it is easy to construct a term $t \in L_1$ that computes a function $f \in F_1$, given by its implementing system. Formally,

Lemma 46 (Normal Form). *For every function $f \in F_1$, there exists a term $t \in L_1$ such that $f = t^{[0,2]}$.*

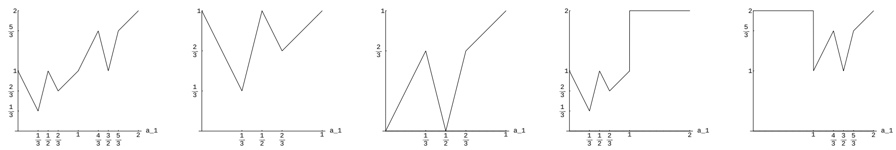
Proof. Let $f \in F_1$ be implemented by the system $\tilde{r}$ (with quasipartition $\tilde{U}$) for some $r \in L_1$, and let $\tilde{r}(\mathbf{1}) = s$. We distinguish two cases.

If r is 1-reproducing, then, by Definition 34, $\tilde{r}(\mathbf{1}) = s$ with both r and s in L_1^+. By Definition 35, for every $\mathbf{b} \in [0, 1)$, $f(\mathbf{b}) = r^{[0,2]}(\mathbf{b})$, and for every $\mathbf{b} \in [1, 2]$, $f(\mathbf{b}) = s^{[0,2]}(\mathbf{b})$. But then, putting,

$$t = \hat{r} \wedge \check{s}, \tag{2.9}$$

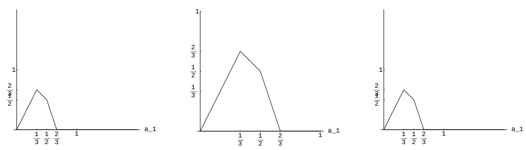
where $\hat{r}$ and $\check{s}$ are given by applying Lemma 45 to r and s respectively, settles the claim, that is $f = t^{[0,2]}$. Compare also Figure 2.9.

If r is not 1-reproducing, then putting $t = \hat{r} \wedge \check{s} = \hat{r}$ settles the claim, that is $f = t^{[0,2]}$. Compare also Figure 2.10. □



(a) f. (b) $r^{[0,1]}$. (c) $s^{[0,1]}$. (d) $\hat{r}^{[0,2]}$. (e) $\check{s}^{[0,2]}$

Figure 2.9: Sampling Lemma 46 with $f \in F_1$ depicted in (a). f is implemented by the system $\tilde{r}$ for r, such that $\tilde{r}(\mathbf{1}) = s$, and the auxiliary system $\tilde{s}$ for s. r and s are the 1-reproducing terms in L_1^+ depicted in (b) and (c) respectively. $\tilde{r}(\mathbf{u}) = r$ and $\tilde{s}(\mathbf{u}) = s$ for every $\mathbf{u} \in \tilde{U}$, where $\tilde{U}$ is the quasipartition of Example 47. By Lemma 40 and Lemma 44, $\hat{r}^{[0,2]}$ and $\check{s}^{[0,2]}$ are as in (d) and (e) respectively. By direct inspection, $f = (\hat{r} \wedge \check{s})^{[0,2]} = t^{[0,2]}$.



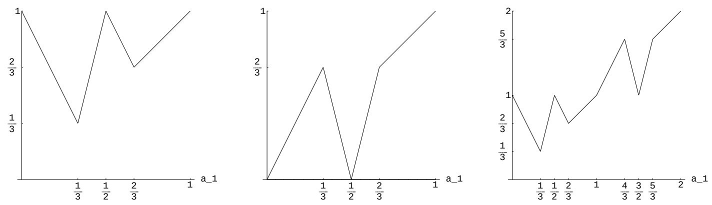
(a) f. (b) $r^{[0,1]}$. (c) $t^{[0,2]}$.

Figure 2.10: Sampling Lemma 46 with $f \in F_1$ depicted in (a). f is implemented by the system $\tilde{r}$ such that $\tilde{r}(\mathbf{u}) = r$ for every $\mathbf{u} \in \tilde{U}$, where $r \in L_1$ is the not 1-reproducing term depicted in (b), and $\tilde{U}$ is the quasipartition of Example 47. By Lemma 40, $r^{[0,2]} = t^{[0,2]}$ is depicted in (c). By direct inspection, $f = t^{[0,2]}$.

Thus, by Lemma 42 and Lemma 46, we conclude that $B_1 = F_1$. This settles the case $n = 1$. Notice that, as an additional benefit, in the proof of the inclusion $F_1 \subseteq B_1$, we also described a construction of a term $t \in L_1$ that computes the given $f \in F_1$.

We conclude this section by means of two examples.

Example 47. *Let $r \in L_1$ be such that $r^{[0,1]}$ is as in Figure 2.11, so that r is* 1*-reproducing in L_1^+. $r^{[0,1]}$ has four linear components p_1, p_2, p_3, and p_4, lin-*



(a) $r^{[0,1]}$. (b) $s^{[0,1]}$. (c) f.

Figure 2.11: $\tilde{r}$ implement f.

earized by the unimodular triangulation U with simplexes $S_1 = \mathrm{conv}\{(0),(1/3)\}$, $S_2 = \mathrm{conv}\{(1/3),(1/2)\}$, $S_3 = \mathrm{conv}\{(1/2),(2/3)\}$, $S_4 = \mathrm{conv}\{(2/3),(1)\}$. *$r^{[0,1]}$ coincides with p_i over S_i, for $i = 1,\ldots,4$. The corresponding quasipartition is,*

$$\tilde{U} = \{(0),(1/4),(1/3),(2/5),(1/2),(3/5),(2/3),(3/4),(1)\}.$$

The points (0), $(1/3)$, $(1/2)$, $(2/3)$, (1) have theirselves as parents in $\tilde{U}$, and the points in $\mathrm{relint}\ \mathrm{conv}\{(0),(1/3)\}$, $\mathrm{relint}\ \mathrm{conv}\{(1/3),(1/2)\}$, $\mathrm{relint}\ \mathrm{conv}\{(1/2),(2/3)\}$, *and* $\mathrm{relint}\ \mathrm{conv}\{(2/3),(1)\}$, *have respectively $(1/4)$, $(2/5)$, $(3/5)$, and $(3/4)$ as parents in $\tilde{U}$. The map,*

$$\tilde{r}(\mathbf{u}) = \begin{cases} r & \textit{if } \mathbf{u} \in \tilde{U} \setminus \{\mathbf{1}\} \\ s & \textit{otherwise} \end{cases}$$

forms a system for r, with $s \in L_1^+$ 1*-reproducing such that $s^{[0,1]}$ is as in Figure 2.11. $s^{[0,1]}$ has four components q_1, q_2, q_3, and q_4. U and $\tilde{U}$ are as above, and $s^{[0,1]}$ coincides with q_i over S_i, for $i = 1,\ldots,4$. The map,*

$$\tilde{s}(\mathbf{u}) = \begin{cases} s & \textit{if } \mathbf{u} \in \tilde{U} \setminus \{\mathbf{1}\} \\ t & \textit{otherwise} \end{cases}$$

where $t \in L_1$ is 1*-reproducing (possibly, $t = s$), forms a system for s.*

Applying Definition 35 and Lemma 40, it is easy to see that $\tilde{r}$ implements the unary function f over $[0,2]$ in Figure 2.11. Indeed, if $\mathbf{b} \in [0,1)$,

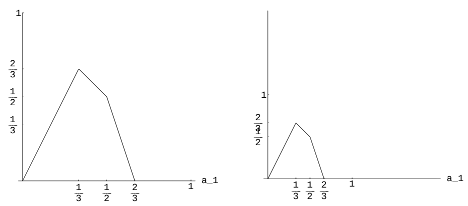
$$f(\mathbf{b}) = \begin{cases} 2 = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = 0 \\ p_1(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{0, \frac{1}{3}\} \\ p_1(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{1}{3} \\ p_2(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{1}{3}, \frac{1}{2}\} \\ 2 = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{1}{2} \\ p_3(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{1}{2}, \frac{2}{3}\} \\ p_3(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{2}{3} \\ p_4(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{2}{3}, 1\} \end{cases}$$

and if $\mathbf{b} \in [1,2]$,

$$f(\mathbf{b}) = \begin{cases} q_1(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = 1 \\ q_1(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{1, \frac{4}{3}\} \\ q_1(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{4}{3} \\ q_2(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{4}{3}, \frac{3}{2}\} \\ q_2(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{3}{2} \\ q_3(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{3}{2}, \frac{5}{3}\} \\ q_3(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{5}{3} \\ q_4(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,1]}(\mathbf{b}-\mathbf{1}) + 1 = s^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{5}{3}, 2\} \\ 2 = t^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = 2 \end{cases}$$

Thus, $\tilde{r}$ implements f. Note that the implementation describes f explicitly.

Example 48. *As a second example of implementation, let $r \in L_1$ be such that $r^{[0,1]}$ is as in Figure 2.12, so that r is not 1-reproducing. $r^{[0,1]}$ has four linear*



(a) $r^{[0,1]}$. (b) f.

Figure 2.12: $\tilde{r}$ implements f.

components p_1, p_2, p_3, and p_4, linearized by the unimodular triangulation U

above, with the quasipartition $\tilde{U}$ above. The map $\tilde{r}$ that sends every $\mathbf{u} \in \tilde{U}$ to r forms a system for r. Note that $\tilde{r}(\mathbf{1}) = r$. Applying Definition 35 and Lemma 40, it is easy to see that $\tilde{r}$ implements the unary function f over $[0,2]$ in Figure 2.12. Indeed, for every $\mathbf{b} \in [0,2]$,

$$f(\mathbf{b}) = \begin{cases} 2 = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = 0 \\ p_1(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{0, \frac{1}{3}\} \\ p_1(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{1}{3} \\ p_2(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{1}{3}, \frac{1}{2}\} \\ 2 = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{1}{2} \\ p_3(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{1}{2}, \frac{2}{3}\} \\ p_3(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} = \frac{2}{3} \\ p_4(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in \text{relint conv}\{\frac{2}{3}, 1\} \\ 0 = r^{[0,2]}(\mathbf{b}) & \textit{if } \mathbf{b} \in [1,2] \end{cases}$$

Thus, $\tilde{r}$ implements f. Note that the implementation describes f explicitly.

In the next section, we study the case $n = 2$.

2.3.3 Binary Case

In this section, we prove that the class of binary BL-functions F_2 coincides with the class of binary term functions B_2.

We instatiate Definition 3 with $n = 2$. Compare also Figure 2.13.

Definition 49. *The algebra $[0,3] = ([0,3], \odot, \rightarrow, \bot)$ is an algebra of type $(2,2,0)$ such that $\bot = 0$ and, for every $a_1, a_2 \in [0,3]$:*

$$a_1 \odot a_2 = \begin{cases} \min(a_1, a_2) & \textit{if } \lfloor a_1 \rfloor \neq \lfloor a_2 \rfloor \\ a_1 \odot^{[0,1]} a_2 & \textit{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = 0 \\ (a_1 - 1 \odot^{[0,1]} a_2 - 1) + 1 & \textit{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = 1 \\ (a_1 - 2 \odot^{[0,1]} a_2 - 2) + 2 & \textit{otherwise} \end{cases}$$

$$a_1 \rightarrow a_2 = \begin{cases} 3 & \textit{if } a_1 \leq a_2 \\ a_2 & \textit{if } \lfloor a_2 \rfloor < \lfloor a_1 \rfloor \\ a_1 \rightarrow^{[0,1]} a_2 & \textit{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = 0 \\ (a_1 - 1 \rightarrow^{[0,1]} a_2 - 1) + 1 & \textit{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = 1 \\ (a_1 - 2 \rightarrow^{[0,1]} a_2 - 2) + 2 & \textit{otherwise} \end{cases}$$

We enrich the signature with the operations $\vee$, $\wedge$, $\neg$, *and* $\top$, *of arity* 2, 2, 1, *and* 0 *respectively, defined as in Definition 37. For* $\circ \in \{\vee, \wedge, \odot, \to, \neg, \top, \bot\}$, *we let* $\circ^{[0,3]}$ *denote the realization of the symbol* $\circ$ *in the algebra* $[0,3]$.

Fact 50. *For every* $a_1, a_2 \in [0,3]$, $a_1 \wedge^{[0,3]} a_2 = \min(a_1, a_2)$, $a_1 \vee^{[0,3]} a_2 = \max(a_1, a_2)$, $\top^{[0,3]} = 3$, *and,*

$$\neg^{[0,3]} a_1 = \begin{cases} 3 & \textit{if } a_1 = 0 \\ 1 - a_1 & \textit{if } 0 < a_1 < 1 \\ 0 & \textit{otherwise} \end{cases}$$

Note that, by Notation 10, for every $r, s \in L_2$ and every $\mathbf{a} \in [0,3]^2$, $(\neg r)^{[0,3]}(\mathbf{a}) = \neg^{[0,3]} r^{[0,3]}(\mathbf{a})$, $(r \wedge s)^{[0,3]}(\mathbf{a}) = r^{[0,3]}(\mathbf{a}) \wedge^{[0,3]} s^{[0,3]}(\mathbf{a})$, $(r \vee s)^{[0,3]}(\mathbf{a}) = r^{[0,3]}(\mathbf{a}) \vee^{[0,3]} s^{[0,3]}(\mathbf{a})$, and $\top^{[0,3]}(\mathbf{a}) = \top^{[0,3]}$.

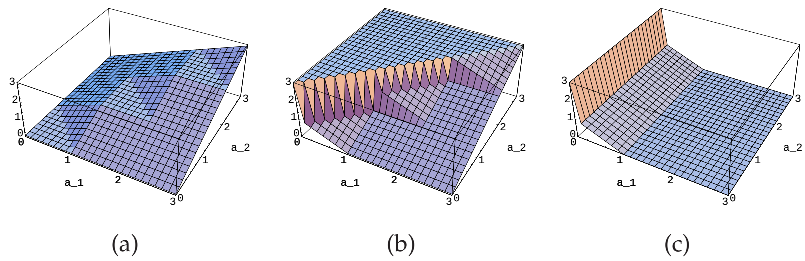


Figure 2.13: Let $X_1 \odot X_2, X_1 \to X_2, \neg X_1 \in L_2$. Plot (a) shows $(X_1 \odot X_2)^{[0,3]}$, plot (b) shows $(X_1 \to X_2)^{[0,3]}$, and plot (c) shows $(\neg X_1)^{[0,3]}$. Note that, by Definition 8, $(\neg X_1)^{[0,3]}$ is a binary function over $[0,3]$; it is essentially unary on its first coordinate.

We mentioned that the truthfunctions of the 2-variate fragment of Basic logic, B_2, coincide with the smallest set of binary functions over $[0,3]$ that contains the projections x_1 and x_2, the constant 0, and is closed under pointwise application of the operations $\odot^{[0,3]}$ and $\to^{[0,3]}$.

The problem is to provide an explicit description of B_2. The first step in our solution schema consists in guessing an explicit class of binary functions over $[0,3]$.

Definition 51 (Binary BL-functions, F_2)**.** *A binary function* f *over* $[0,3]$ *is a* binary BL-function *if and only if there exists a system* $\tilde{r}$ *over a term* $r \in L_2$ *that implements* f*. We let* F_2 *denote the class of binary BL-functions.*

We claim that the class of binary BL-functions is explicit, in the sense that if $\tilde{r}$ implements f, then $\tilde{r}$ actually furnishes an explicit description of f (compare also Example 61 and Example 62). To prove the claim, we preliminarily prove two technical lemmas.

The first lemma addresses a phenomenon which we call *extension*. This phenomenon arises only if $n \geq 2$. The underlying intuition is that if a 1-reproducing term $t \in L_2$ contains only one variable, say the variable X_1, then the binary function $t^{[0,3]} \colon [0,3]^2 \to [0,3]$ can be described in terms of the unary function $t^{[0,2]} \colon [0,2] \to [0,2]$. But then, $t^{[0,3]}$ is described explicitly: indeed, $t \in L_1$, so that $t^{[0,2]} \in B_1 \subseteq F_1$ by Lemma 42, and therefore $t^{[0,2]}$ has an explicit description by Proposition 41. The formal details follow.

Definition 52 (Projection)**.** *Let* $I = \{i\} \subset \{1,2\}$ *and let* $\mathbf{b} = (b_1, b_2) \in [0,3]^2$*. Let* $\mathbf{c} = (c_1) \in [0,2]$ *be the unique element in* $[0,2]$ *such that* $c_1 - \lfloor c_1 \rfloor = b_i - \lfloor b_i \rfloor$ *and:* $\lfloor c_1 \rfloor = 0$*, if* $\lfloor b_i \rfloor = 0$*;* $\lfloor c_1 \rfloor = 2$*, if* $\lfloor b_i \rfloor = 3$*;* $\lfloor c_1 \rfloor = 1$*, otherwise. We call* $\mathbf{c}$ *the* I-projection *of* $\mathbf{b}$ *over* $[0,2]$.

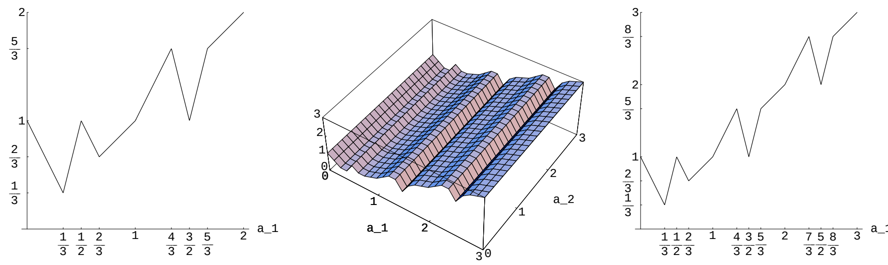
Lemma 53 (Extension)**.** *Let* $I = \{i\} \subset \{1,2\}$*, and let* $t \in L_I$ *be a* 1-*reproducing term. Then,* $t^{[0,3]}$ *has an explicit description in terms of Definition 15.*

Proof. Let $\mathbf{b} = (b_1, b_2) \in [0,3]^2$, and let $\mathbf{c} = (c_1) \in [0,2]$ be the I-projection of $\mathbf{b}$ over $[0,2]$. Let $\bar{t} = t_{\{i \backslash 1\}} \in L_1$. By induction on t, applying Definition 49, we get,

$$t^{[0,3]}(\mathbf{b}) = \begin{cases} 0 & \text{if } \bar{t}^{[0,2]}(\mathbf{c}) = 0 \\ 3 & \text{if } \bar{t}^{[0,2]}(\mathbf{c}) = 2 \\ \bar{t}^{[0,2]}(\mathbf{c}) - \lfloor \bar{t}^{[0,2]}(\mathbf{c}) \rfloor + \lfloor b_i \rfloor & \text{otherwise} \end{cases} \tag{2.10}$$

By Lemma 42, we know that $\bar{t}^{[0,2]}$ is equal to some $f \in F_1$, and f has an explicit description by Proposition 41. So, $\bar{t}^{[0,2]}$ has an explicit description. Therefore, $t^{[0,3]}$ has an explicit description, precisely, the description given by equation (2.10). See also Figure 2.14. □

The second lemma generalizes the *lifting* phenomenon, already encountered in the unary case, to the binary case.



(a) (b) (c)

Figure 2.14: Sampling Lemma 53 with $I = \{1\}$, so that t is a 1-reproducing term in L_1. In this case, $\bar{t} = t_{\{i\backslash 1\}} = t$. Let $\bar{t}^{[0,2]}$ be depicted in (a). By Lemma 53, $t^{[0,3]}$ is described explicitly in terms of equation (2.10), that is, $t^{[0,3]}$ is as in (b). Plot (c) shows $t^{[0,3]}((b_1, 0))$ for $0 \leq b_1 \leq 3$. A comparison of (a) and (c) enlightens the *extension* phenomenon.

Lemma 54 (Lifting)**.** *Let $t \in L_2$ be such that $t \in L_2^+$ if t is 1-reproducing, and $t = \neg t'$ with $t' \in L_2^+$ otherwise. Then, $t^{[0,3]}(\mathbf{b})$ has an explicit description in terms of Definition 15.*

Proof. Let $\tilde{U}$ be a quasipartition for t, let $\mathbf{b} = (b_1, b_2) \in [0,3]^2$, and let $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ for $\mathbf{u}$ in $\tilde{U}$. We examine two cases.

First suppose that t is 1-reproducing ($t \in L_2^+$). By induction on t, applying Definition 49, we get,

$$t^{[0,3]}(\mathbf{b}) = \begin{cases} t^{[0,1]}(\mathbf{b} - \mathbf{j}) + j & \text{if } j = \lfloor b_1 \rfloor = \lfloor b_2 \rfloor \text{ and } t^{[0,1]}(\mathbf{u}) < 1 \\ 3 & \text{if } \lfloor b_1 \rfloor = \lfloor b_2 \rfloor \text{ and } t^{[0,1]}(\mathbf{u}) = 1 \\ t^{[0,1]}(\mathrm{controller}(\mathbf{b})) + j & \text{if } t^{[0,1]}(\mathbf{u}) < 1 \text{ and } j = \min\{\lfloor b_1 \rfloor, \lfloor b_2 \rfloor\} \\ \bar{t}^{[0,3]}(\mathbf{b}) & \text{otherwise,} \end{cases} \tag{2.11}$$

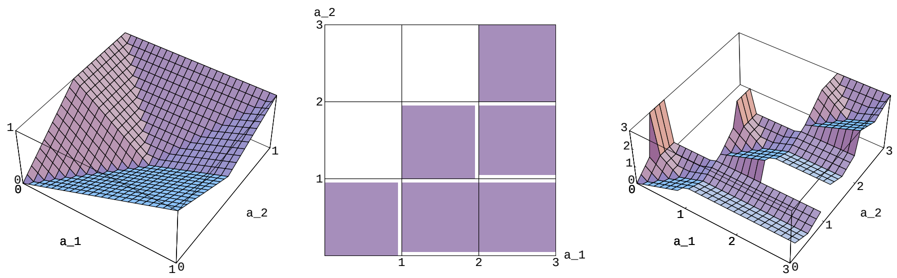
where $\bar{t}$ is a 1-reproducing term in $L_{\mathrm{par}(\mathbf{u})}$. Noticing that, if $\lfloor b_1 \rfloor \neq \lfloor b_2 \rfloor$, then $\emptyset \subset \mathrm{par}(\mathbf{u}) \subset \{1, 2\}$, we apply Lemma 53 to $\bar{t}$, and we have that $\bar{t}^{[0,3]}$ has an explicit description. Moreover, by Fact 19, $t^{[0,1]}$ has an explicit description. Hence, $t^{[0,3]}$ has an explicit description, precisely, the description given by equation (2.11). Compare also Figure 2.15.

Now suppose that t is not 1-reproducing ($t = \neg t'$ with $t' \in L_2^+$). By

Fact 50, and applying the previous case to t', we get,

$$t^{[0,3]}(\mathbf{b}) = \begin{cases} t^{[0,1]}(\mathbf{b}) & \text{if } \mathbf{b} \in [0,1)^2 \text{ and } t^{[0,1]}(\mathbf{u}) < 1 \\ 3 & \text{if } \mathbf{b} \in [0,1)^2 \text{ and } t^{[0,1]}(\mathbf{u}) = 1 \\ 0 & \text{if } \mathbf{b} \in [1,3]^2 \\ t^{[0,1]}(\mathrm{controller}(\mathbf{b})) & \text{if } t^{[0,1]}(\mathbf{u}) < 1 \\ 3 & \text{otherwise,} \end{cases} \tag{2.12}$$

Since $t^{[0,1]}$ has an explicit description by Fact 19, so that $t^{[0,3]}$ has an explicit description, precisely, the description given by equation (2.12). Compare also Figure 2.16. □



(a) (b) (c)

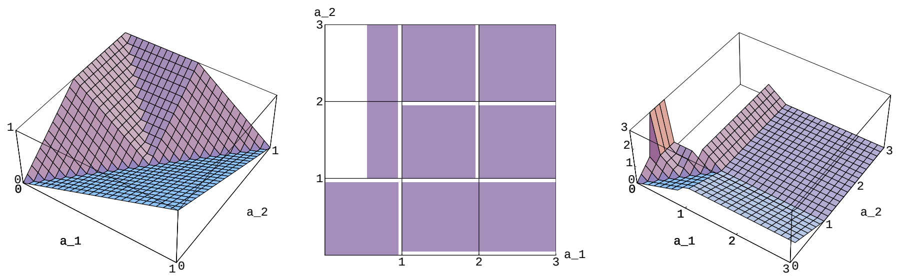
Figure 2.15: Sampling the first case of Lemma 54 with the 1-reproducing term $t \in L_2^+$ depicted in (a). The underlying unimodular triangulation U and quasipartition $\tilde{U}$ are sketched in Figure 2.25. The region highlighted in (b) coincides with the set of $\mathbf{b} \in [0,3]^2$ such that $\mathbf{u} \in [0,1)^2$ or $t^{[0,1]}(\mathbf{u}) < 1$. The restriction of $t^{[0,3]}$ to the highlighted region is then described in terms of the first, second and third clause of equation (2.11), as it is depicted in (c). A comparison of (a) and (c) enlightens the *lifting* phenomenon.

The previous lemmas on lifting and extension allow to conclude that F_2 has an explicit description.

Proposition 55 (Explicitness)**.** *F_2 has an explicit description in terms of Definition 15.*

Proof. Let $f \in F_2$, and let $\tilde{r}$ be the system (over quasipartition $\tilde{U}$) implementing f, with $r \in L_2$. We distinguish two cases.

If r is 1-reproducing, then $\tilde{r}(\mathbf{1}) = s$ with s is 1-reproducing in L_2, and there is an auxiliary system $\tilde{s}$ for s, say over quasipartition $\tilde{V}$. Both



(a) (b) (c)

Figure 2.16: Sampling the second case of Lemma 54 with the not 1-reproducing term $t \in L_2$ depicted in (a). The underlying unimodular triangulation U and quasipartition $\tilde{U}$ are sketched in Figure 2.25. The region highlighted in (b) coincides with the set $[1,3]^2$ and the set of $\mathbf{b} \notin [1,3]^2$ such that $\mathbf{u} \in [0,1)^2$ or $t^{[0,1]}(\mathbf{u}) < 1$. The restriction of $t^{[0,3]}$ to the highlighted region is then described in terms of the first, second, third, and fourth clause of equation (2.12), as it is depicted in (c).

r and s are in L_2^+. Let $\mathbf{b} \in [0,3]^2$. First suppose that $\mathbf{b} \notin [1,3]^2$. Let $\mathbf{u}$ in $\tilde{U}$ be such that $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, and let $\tilde{r}(\mathbf{u}) = t$. Note that $|\mathrm{par}(\mathbf{u})| = 1$. By Definition 34, either $t \in L_{\mathrm{par}(\mathbf{u})}$ or $t = r$; and, by Definition 35, $f(\mathbf{b}) = t^{[0,3]}(\mathbf{b})$. But then, by Lemma 53 and Lemma 54, we have that $t^{[0,3]}(\mathbf{b})$ is described explicitly, hence $f(\mathbf{b})$ is described explicitly. Next suppose that $\mathbf{b} \in [1,3]^2$. Let $\mathbf{v}$ in $\tilde{V}$ be such that $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{v}))$, and let $\tilde{s}(\mathbf{v}) = t'$. Noticing that $|\mathrm{par}(\mathbf{v})| = 1$ and reasoning as above, we have that $t'^{[0,3]}(\mathbf{b})$ is described explicitly, hence $f(\mathbf{b})$ is described explicitly. Thus we conclude that $\tilde{r}$ describes explicitly f, so that in this case, f has an explicit description. If r is not 1-reproducing, an entirely similar argument shows that $\tilde{r}$ describes explicitly f, so that in this case also, f has an explicit description.

Hence, every $f \in F_2$ has an explicit description, so that F_2 has an explicit description. $\square$

The explicit class F_2 of binary functions over $[0,3]$ settles the first step of our solution schema, in the case $n = 2$. The second step consists in proving that the class F_2 of binary BL-functions coincides with the truthfunctions of the 2-variate fragment of Basic logic,

$$B_2 = \{t^{[0,3]} \mid t \in L_2\} \subseteq [0,3]^{[0,3]^2}.$$

We prove the inclusion $B_2 \subseteq F_2$.

Lemma 56 (Closure). *$t^{[0,3]} \in F_2$ for every $t \in L_2$.*

Proof. The proof is by induction on t. We refer the reader to the proof of the general case $n \geq 1$ in Lemma 65. $\square$

We prove the inclusion $F_2 \subseteq B_2$. To this aim we preliminarily generalize the *isolation* mechanism to the case $n = 2$.

Definition 57 (Isolation). *Let $D \subseteq [0,3]^2$. Say that the term $t \in L_2$* isolates *the term $s \in L_2$ over D if $t^{[0,3]}(\mathbf{b}) = s^{[0,3]}(\mathbf{b})$ if $\mathbf{b} \in D$, and otherwise $t^{[0,3]}(\mathbf{b}) = \top^{[0,3]}$.*

Let $\tilde{r}$ be a system, over the quasipartition $\tilde{U}$, and let $\mathbf{u} \in \tilde{U}$. The final objective is that of isolating the term $\tilde{r}(\mathbf{u})$ over $\mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$. We proceed in two steps. In the first step, we isolate *certain* variables over *certain* regions of $[0,3]^2$ (compare Lemma 58 below). In the second step, on the basis of the variable isolation mechanism, we isolate *certain* terms over *certain* regions of $[0,3]^2$ (compare Lemma 59 below).

Lemma 58 (Variable Isolation). *Let $i \in \{1,2\}$.*

(i) Let $\mathbf{a} \in [0,1]^2$ be such that $i \in \mathrm{par}(\mathbf{a})$. There exists terms $r_{(i,\mathbf{a})}, s_{(i,\mathbf{a})} \in L_2$ such that, $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ isolate X_i over $D_{(i,\mathbf{a})}$ and $E_{(i,\mathbf{a})}$ respectively, where,

$$D_{(i,\mathbf{a})} = \{\mathbf{b} \notin [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))\},$$
$$E_{(i,\mathbf{a})} = \{\mathbf{b} \in [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))\}.$$

(ii) Let $\tilde{U}$ be a quasipartition of $[0,1]^2$ and let $\mathbf{u} \in \tilde{U}$ be such that $i \in \mathrm{par}(\mathbf{u})$. There exists terms $r_{(i,\mathbf{u})}, s_{(i,\mathbf{u})} \in L_2$ such that, $r_{(i,\mathbf{u})}$ and $s_{(i,\mathbf{u})}$ isolate X_i over $D_{(i,\mathbf{u})}$ and $E_{(i,\mathbf{u})}$ respectively, where,

$$D_{(i,\mathbf{u})} = \{\mathbf{b} \notin [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\},$$
$$E_{(i,\mathbf{u})} = \{\mathbf{b} \in [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\}.$$

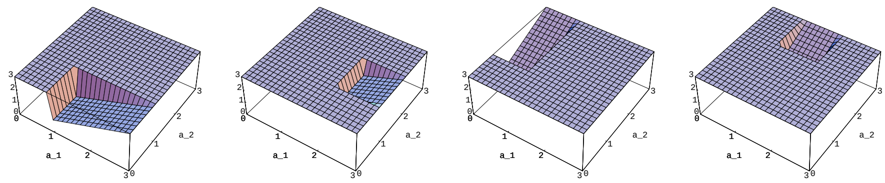
(iii) There exist terms $v_i, w_i \in L_2$ such that, v_i and w_i isolate X_i over J_i and K_i respectively, where,

$$J_i = \{\mathbf{b} \notin [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a})), i \notin \mathrm{par}(\mathbf{a})\},$$
$$K_i = \{\mathbf{b} \in [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a})), i \notin \mathrm{par}(\mathbf{a})\}.$$

Proof. (i) If $\mathbf{a}$ is equal to $\mathbf{1}$, then $r_{(i,\mathbf{a})} = s_{(i,\mathbf{a})} = \top$ settles the claim, simply noticing that $\mathrm{realm}(\mathrm{neigh}(\mathbf{a})) = \mathbf{n}+\mathbf{1}$. Otherwise, put,

$$
\begin{aligned}
r_{(1,\mathbf{a})} &= (\neg\neg X_2) \vee (\neg\neg X_1 \to X_1)\\
s_{(1,\mathbf{a})} &= ((X_1 \to X_2) \to X_2) \to ((\neg\neg X_1 \to X_1) \vee (\neg\neg X_2 \to X_2))\\
r_{(2,\mathbf{a})} &= (\neg\neg X_1) \vee (\neg\neg X_2 \to X_2)\\
s_{(2,\mathbf{a})} &= ((X_2 \to X_1) \to X_1) \to ((\neg\neg X_2 \to X_2) \vee (\neg\neg X_1 \to X_1)),
\end{aligned}
$$

But then, by Definition 49 and Fact 50, for $i = 1, 2$, $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ isolate X_i over $D_{(i,\mathbf{a})}$ and $E_{(i,\mathbf{a})}$ respectively. See Figure 2.17.



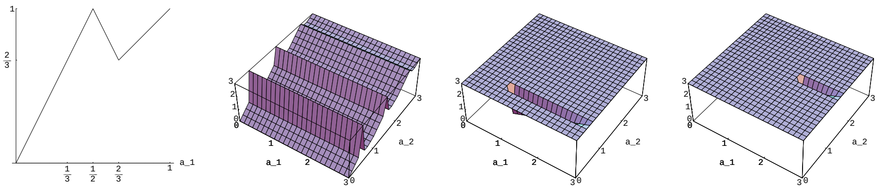
(a) $r^{[0,3]}_{(1,\mathbf{a})}$. (b) $s^{[0,3]}_{(1,\mathbf{a})}$. (c) $r^{[0,3]}_{(2,\mathbf{a})}$. (d) $s^{[0,3]}_{(2,\mathbf{a})}$.

Figure 2.17: Lemma 58(i).

(ii) If $\mathbf{u}$ is equal to $\mathbf{1}$, then $r_{(i,\mathbf{u})} = s_{(i,\mathbf{u})} = \top$ settles the claim, simply noticing that $\mathrm{realm}(\mathrm{sibl}(\mathbf{u})) = \mathbf{n}+\mathbf{1}$. Otherwise, $\mathrm{par}(\mathbf{u}) = \{i\}$. Suppose w.l.o.g. that $i = 1$ (the case $i = 2$ is similar). There are two cases.

As a first case, suppose that $\mathrm{sibl}(\mathbf{u})$ contains exactly a rational vertex in the unimodular triangulation U. Say, w.l.o.g. that $\mathrm{sibl}(\mathbf{u}) = \{\mathbf{u}_{28}\}$ in Figure 2.25(b). Note that $\mathbf{u}_{28} = (u_{28,1}, u_{28,2}) = (1, 1/2)$. Fix $t \in L_1^+$ such that $t^{[0,1]}(\mathbf{c}) = 1$ if and only if $\mathbf{c} = (u_{28,2}) = 1/2$ or $\mathbf{c} = \mathbf{1} = 1$. Such t exists by Theorem 28 and Corollary 30. Let $r_{(1,\mathbf{u})} = t_{\{1\backslash 2\}} \to r_{(1,\mathbf{a})}$ and $s_{(1,\mathbf{u})} = t_{\{1\backslash 2\}} \to s_{(1,\mathbf{a})}$, where $r_{(1,\mathbf{a})}$ and $s_{(1,\mathbf{a})}$ are as in part (i), say with $\mathbf{a} = \mathbf{u}$. By Lemma 53 and Definition 49, we have that $r_{(1,\mathbf{u})}$ and $s_{(1,\mathbf{u})}$ isolate X_1 over $D_{(1,\mathbf{a})}$ and $E_{(1,\mathbf{a})}$ respectively. See Figure 2.18.

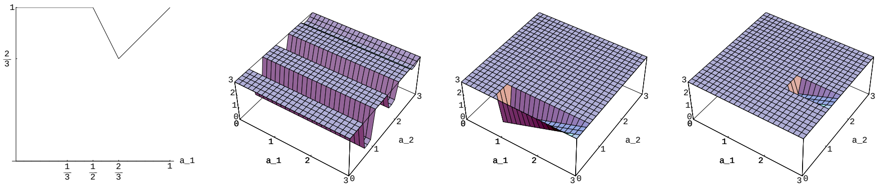
As a second case, suppose that $\mathrm{sibl}(\mathbf{u})$ is an open line segment having as endpoints a pair of rational vertices in the unimodular triangulation U. Say, w.l.o.g. that $\mathrm{sibl}(\mathbf{u}) = \mathrm{relint\,conv}\{\mathbf{u}_{28}, \mathbf{u}_{30}\}$ in Figure 2.25(b). Note that $\mathbf{u}_{28} = (u_{28,1}, u_{28,2}) = (1, 1/2)$, and $\mathbf{u}_{30} = (u_{30,1}, u_{30,2}) = (1, 0)$. Fix $t \in L_1$ such that $t^{[0,1]}(\mathbf{c}) = 1$ if and only if $\mathbf{c} \in \mathrm{conv}\{u_{28,2}, u_{30,2}\} = \mathrm{conv}\{0, 1/2\}$ or $\mathbf{c} = \mathbf{1} = 1$. Such t exists by Theorem 28. Let $r_{(1,\mathbf{u}_{28})}$, $r_{(1,\mathbf{u}_{30})}$, $s_{(1,\mathbf{u}_{28})}$, and $s_{(1,\mathbf{u}_{30})}$ be settled as in the previous case. Let $r_{(1,\mathbf{u})} =$



(a) $t^{[0,1]}$. (b) $t^{[0,3]}_{\{1\backslash 2\}}$ (c) $r^{[0,3]}_{(1,\mathbf{u})}$. (d) $s^{[0,3]}_{(1,\mathbf{u})}$.

Figure 2.18: Lemma 58(ii), Case 1: $\mathrm{sibl}(\mathbf{u}) = \{\mathbf{u}_{28}\}$, $\mathbf{u}_{28} = (1, 1/2)$.

$(r_{(1,\mathbf{u}_{28})} \wedge r_{(1,\mathbf{u}_{30})}) \to (t_{\{1\backslash 2\}} \to r_{(1,\mathbf{a})})$ and $s_{(1,\mathbf{u})} = (s_{(1,\mathbf{u}_{28})} \wedge s_{(1,\mathbf{u}_{30})}) \to (t_{\{1\backslash 2\}} \to s_{(1,\mathbf{a})})$, where $r_{(1,\mathbf{a})}$ and $s_{(1,\mathbf{a})}$ are as in part (i), say with $\mathbf{a} = \mathbf{u}$. By Lemma 53, Definition 49, and the previous case, we have that $r_{(1,\mathbf{u})}$ and $s_{(1,\mathbf{u})}$ isolate X_1 over $D_{(1,\mathbf{a})}$ and $E_{(1,\mathbf{a})}$ respectively. See Figure 2.19.



(a) $t^{[0,1]}$. (b) $t^{[0,3]}_{\{1\backslash 2\}}$ (c) $r^{[0,3]}_{(1,\mathbf{u})}$. (d) $s^{[0,3]}_{(1,\mathbf{u})}$.

Figure 2.19: Lemma 58(ii), Case 2: $\mathrm{sibl}(\mathbf{u}) = \mathrm{relint}\ \mathrm{conv}\{\mathbf{u}_{28}, \mathbf{u}_{30}\}$, $\mathbf{u}_{28} = (u_{28,1}, u_{28,2}) = (1, 1/2)$, and $\mathbf{u}_{30} = (u_{30,1}, u_{30,2}) = (1, 0)$.

(iii) For $i = 1, 2$, put,

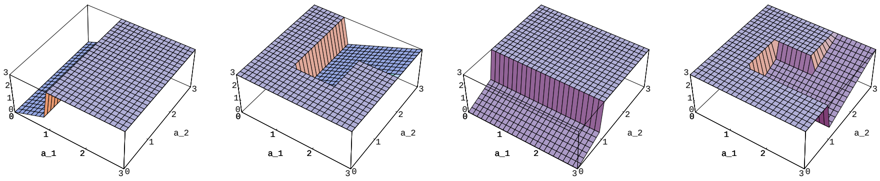
$$v_i = \neg\neg X_i \tag{2.13}$$

$$w_i = \left(\bigwedge_{\mathbf{a} \in A} (r_{(i,\mathbf{a})} \wedge s_{(i,\mathbf{a})}) \right) \to (\neg\neg X_i \to X_i), \tag{2.14}$$

where $A = \{\mathbf{a} \in [0,1]^2 \mid i \in \mathrm{par}(\mathbf{a})\}$, and $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ are as in part (i). But then, by part (i), Definition 49 and Fact 50, we have that v_i and w_i isolate X_i over J_i and K_i respectively. See Figure 2.20. $\square$

We now extend the term isolation mechanism to the case $n = 2$.

Lemma 59 (Term Isolation)**.** *The following statements hold.*



(a) $v_1^{[0,1]}$. (b) $w_1^{[0,1]}$. (c) $v_2^{[0,1]}$. (d) $w_2^{[0,1]}$.

Figure 2.20: Lemma 58(iii).

(i) Let $t \in L_2^+$ and let $\tilde{U}$ be a quasipartition for t. Then, there exist terms $\hat{t}$ and $\check{t}$ in L_2 such that, $\hat{t}$ isolates t over:

$$\{\mathbf{b} \notin [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^2 \textit{ or } t^{[0,1]}(\mathbf{u}) < 1\},$$

and $\check{t}$ isolates t over:

$$\{\mathbf{b} \in [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^2 \textit{ or } t^{[0,1]}(\mathbf{u}) < 1\}.$$

(ii) Let $t = \neg t'$ with $t' \in L_2^+$ and let $\tilde{U}$ be a quasipartition for t. Then, there exists a term $\hat{t}$ in L_2 such that, $\hat{t}$ isolates t over

$$[1,3]^2 \cup \{\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \mid \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^2 \textit{ or } t^{[0,1]}(\mathbf{u}) < 1\}.$$

(iii) Let $\tilde{U}$ be a quasipartition of $[0,1]^2$, and let $\mathbf{u} \in \tilde{U}$ be such that $|\mathrm{par}(\mathbf{u})| = 1$. Let t be a 1-reproducing term in $L_{\mathrm{par}(\mathbf{u})}$. Then, there exist terms $\dot{t}$ and $\ddot{t}$ in L_2 such that, $\dot{t}$ isolates t over:

$$\{\mathbf{b} \notin [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathbf{u} \in \tilde{U}\},$$

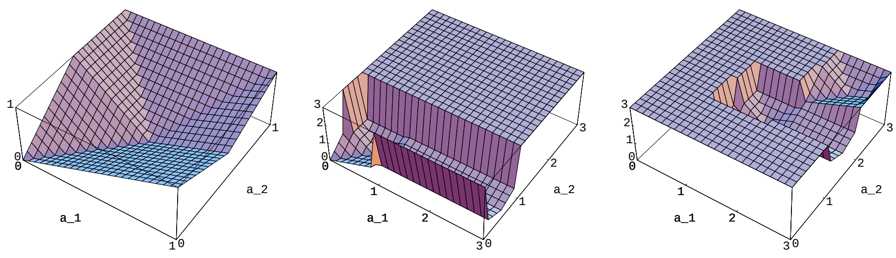
and $\ddot{t}$ isolates t over:

$$\{\mathbf{b} \in [1,3]^2 \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathbf{u} \in \tilde{U}\}.$$

Proof. (i) Put:

$$\hat{t} = t_{\{1 \backslash v_1, 2 \backslash v_2\}},$$
$$\check{t} = t_{\{1 \backslash w_1, 2 \backslash w_2\}},$$

where v_1, v_2, w_1, and w_2 are as in Lemma 58(iii). A routine induction on t, appealing to Lemma 58(iii) and to the definition of $\odot^{[0,3]}$ and $\rightarrow^{[0,3]}$, shows that $\hat{t}$ and $\check{t}$ satisfy the claim. Compare also Figure 2.21.



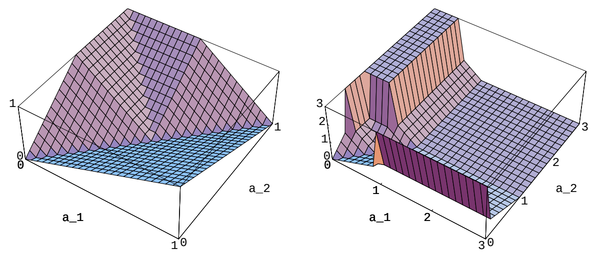
(a) $t^{[0,1]}$. (b) $\hat{t}^{[0,3]}$. (c) $\check{t}^{[0,3]}$.

Figure 2.21: Sampling Lemma 59(i) with the 1-reproducing $t \in L_2^+$ depicted in (a). $\hat{t}$ and $\check{t}$ satisfy the statement of the lemma.

(ii) Put:

$$\hat{t} = \neg t'_{\{1\backslash v_1, 2\backslash v_2\}},$$

where v_1 and v_2 are as in Lemma 58(iii). Now, by part (i) of the present lemma, applied to t', and by the definition of $\neg^{[0,3]}$, we have that $\hat{t}$ satisfies the statement. Compare also Figure 2.22.



(a) $t^{[0,1]}$. (b) $\hat{t}^{[0,3]}$.

Figure 2.22: Sampling Lemma 59(ii) with the not 1-reproducing $t \in L_2$ depicted in (a). $\hat{t}$ satisfies the statement of the lemma.

(iii) Let $\mathrm{par}(\mathbf{u}) = \{i\}$, so that t is 1-reproducing in $L_{\{i\}}$. Put:

$$\dot{t}_{\mathbf{u}} = t_{\{i\backslash r_{(i,\mathbf{u})}\}} \tag{2.15}$$

$$\ddot{t}_{\mathbf{u}} = t_{\{i\backslash s_{(i,\mathbf{u})}\}}, \tag{2.16}$$

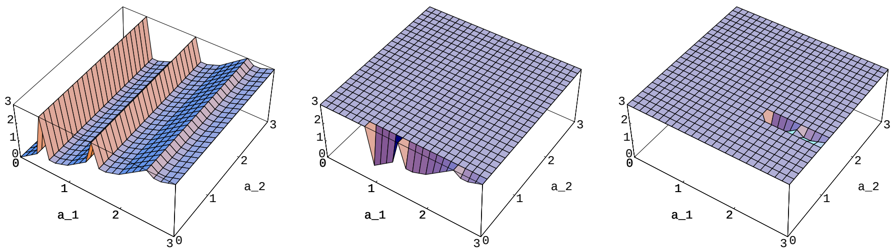
where $r_{(i,\mathbf{u})}$ and $s_{(i,\mathbf{u})}$ are the terms given by Lemma 58(ii). We claim that $\dot{t}_{\mathbf{u}}$ and $\ddot{t}_{\mathbf{u}}$ satisfy the claim. Indeed, consider $\dot{t}_{\mathbf{u}}$ (the case of $\ddot{t}_{\mathbf{u}}$ is analogous). Let $\mathbf{b} \in [0,3]^2 \backslash [1,3]^2$. If $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, by Lemma 58(ii), $r^{[0,3]}_{(i,\mathbf{u})}(\mathbf{b}) = X^{[0,3]}_i(\mathbf{b})$, so that,

$$\dot{t}_{\mathbf{u}}(\mathbf{b}) = (\dot{t}_{\mathbf{u}})_{\{r_{(i,\mathbf{u})}\backslash X_i\}}(\mathbf{b}) = t(\mathbf{b}).$$

Otherwise, suppose $\mathbf{b} \notin \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$. By Lemma 58(ii), $r^{[0,3]}_{(i,\mathbf{u})}(\mathbf{b}) = \top^{[0,3]}$, so that,

$$\dot{t}_{\mathbf{u}}(\mathbf{b}) = (\dot{t}_{\mathbf{u}})_{\{r_{(i,\mathbf{u})}\backslash\top\}}(\mathbf{b}) = t_{\{X_i\backslash\top\}}(\mathbf{b}) = \top^{[0,3]},$$

where the last equality holds because because t is 1-reproducing in $L_{\{i\}}$. Compare also Figure 2.23. $\square$



(a) $t^{[0,3]}$. (b) $\dot{t}^{[0,3]}_{\mathbf{u}}$. (c) $\ddot{t}^{[0,3]}_{\mathbf{u}}$.

Figure 2.23: Sampling Lemma 59(iii) with $\mathbf{u} = \mathbf{u}_{30}$ in the quasipartition $\tilde{U}$ of Figure 2.25, so that $i = 1$ and t is a 1-reproducing term in $L_{\{1\}} = L_1$. Then, $\dot{t}_{\mathbf{u}}$ and $\ddot{t}_{\mathbf{u}}$ satisfy the statement of the lemma.

With the background of the term isolation mechanism, the problem of constructing a term $t \in L_2$ that computes a given function $f \in F_2$ reduces to the following: given a system $\tilde{r}$ that implements f, isolate each term $\tilde{r}(\mathbf{u})$ over the region $\mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, and output the meet of the resulting terms.

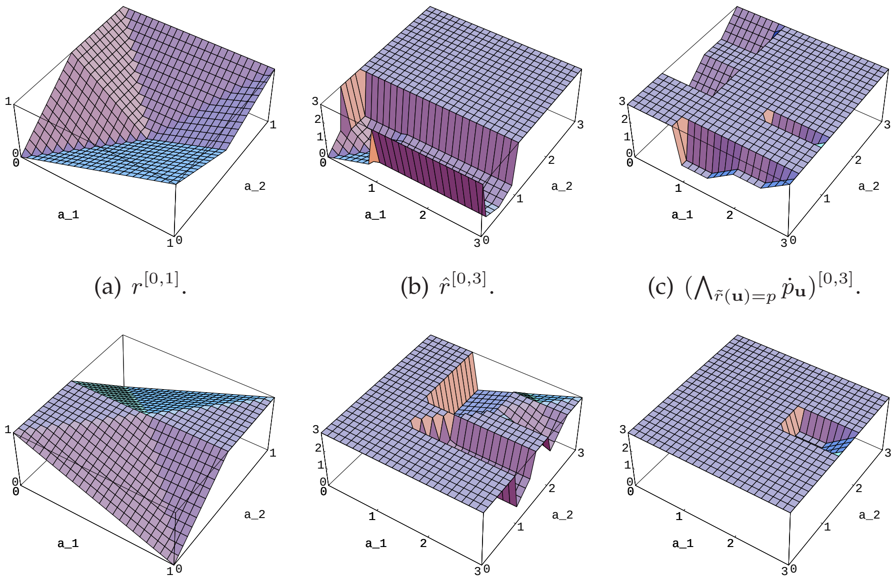
Lemma 60 (Normal Form)**.** *For every function $f \in F_2$, there exists a term $t \in L_2$ such that $f = t^{[0,3]}$.*

Proof. Let $f \in F_2$ be implemented by the system $\tilde{r}$ over the quasipartition $\tilde{U}$, for $r \in L_2$. We distinguish two cases.

First suppose that r is 1-reproducing (so, $r \in L_2^+$). Then, by Definition 34, $\tilde{r}(\mathbf{1}) = s$ with s 1-reproducing in L_2^+, and, by Definition 35, there exists a system $\tilde{s}$, say over quasipartition $\tilde{V}$. Put,

$$t = \left(\hat{r} \wedge \bigwedge_{\tilde{r}(\mathbf{u})=p} \dot{p}_{\mathbf{u}}\right) \wedge \left(\check{s} \wedge \bigwedge_{\tilde{s}(\mathbf{v})=q} \ddot{q}_{\mathbf{v}}\right),$$

where: $\hat{r}$ corresponds to r as by Lemma 59(i); $\check{s}$ corresponds to s as by Lemma 59(i); $\mathbf{u}$ ranges over all the $\mathbf{u}' \in \tilde{U}$ such that $\mathbf{u}' \notin [0,1)^2$ and $r^{[0,1]}(\mathbf{u}') = 1$, and $\dot{p}_{\mathbf{u}}$ corresponds to p and $\mathbf{u}$ as by Lemma 59(iii); $\mathbf{v}$ ranges over all the $\mathbf{v}' \in \tilde{V}$ such that $\mathbf{v}' \notin [0,1)^2$ and $s^{[0,1]}(\mathbf{v}') = 1$, and $\ddot{q}_{\mathbf{v}}$ corresponds to q and $\mathbf{v}$ as by Lemma 59(iii). But then, applying Lemma 59(i) and (iii) and Definition 35, we have that $f = t^{[0,2]}$. See Figure 2.24.



(a) $r^{[0,1]}$. (b) $\hat{r}^{[0,3]}$. (c) $(\bigwedge_{\tilde{r}(\mathbf{u})=p} \dot{p}_{\mathbf{u}})^{[0,3]}$.

(d) $s^{[0,1]}$. (e) $\check{s}^{[0,3]}$. (f) $(\bigwedge_{\tilde{s}(\mathbf{v})=q} \ddot{q}_{\mathbf{v}})^{[0,3]}$.

Figure 2.24: Sampling Lemma 60 with $f \in F_2$ depicted in Figure 2.28.

Next suppose that r is not 1-reproducing. Put,

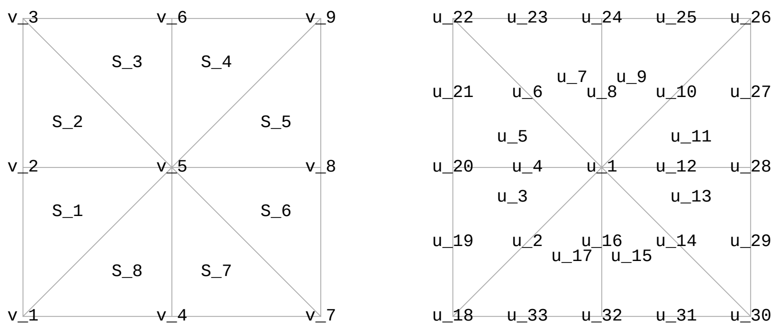
$$t = \hat{r} \wedge \bigwedge_{\tilde{r}(\mathbf{u})=p} \dot{p}_{\mathbf{u}},$$

where: $\hat{r}$ corresponds to r as by Lemma 59(ii); $\mathbf{u}$ ranges over all the $\mathbf{u}' \in \tilde{U}$ such that $\mathbf{u}' \notin [0,1)^2$ and $r^{[0,1]}(\mathbf{u}') = 1$, and $\dot{p}_{\mathbf{u}}$ corresponds to p and $\mathbf{u}$ as by Lemma 59(iii). But then, applying Lemma 59(ii) and (iii) and Definition 35, we have that $f = t^{[0,2]}$. $\square$

Thus, by Lemma 56 and Lemma 60, we conclude that $B_2 = F_2$. This settles the case $n = 2$. Notice that, as an additional benefit, in the proof of the inclusion $F_2 \subseteq B_2$, we also described a construction of a term

$t \in L_2$ that computes the given $f \in F_2$. We conclude this section by means of two examples.

Example 61. *Let $U = \{S_1, \ldots, S_8\}$ be the unimodular triangulation of $[0,1]^2$ sketched in Figure 2.25(a), with quasipartition $\tilde{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_{33}\}$ sketched in Figure 2.25(b). Consider, for instance, the 2-dimensional simplex $S_1 = \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$, with edges $F_1 = \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2\}$, $F_2 = \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_5\}$, and $F_3 = \mathrm{conv}\{\mathbf{v}_2, \mathbf{v}_5\}$, and vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_5$. Points in $\mathrm{relint}\ S_1$ have $\mathbf{u}_3$ as parent, points in $\mathrm{relint}\ F_1$ have $\mathbf{u}_{19}$ as parent, points in $\mathrm{relint}\ F_2$ have $\mathbf{u}_2$ as parent, points in $\mathrm{relint}\ F_3$ have $\mathbf{u}_4$ as parent, and eventually points in $\{\mathbf{v}_1\}$, $\{\mathbf{v}_2\}$, and $\{\mathbf{v}_5\}$ have respectively $\mathbf{u}_{18}$, $\mathbf{u}_{20}$, and $\mathbf{u}_1$ as parents. Let*

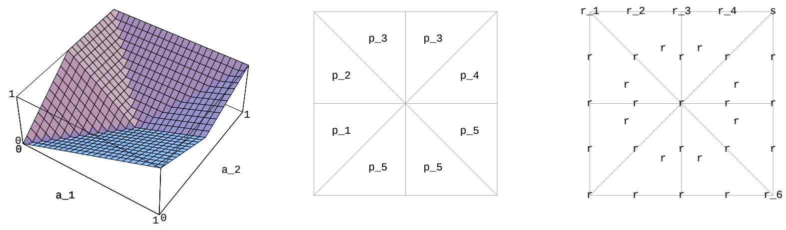


(a) U. (b) $\tilde{U}$.

Figure 2.25: Example 61.

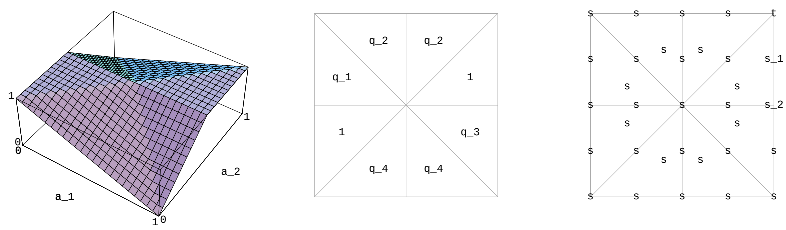
$r, s \in L_2$ be such that $r^{[0,1]}$ and $s^{[0,1]}$ are as in Figure 2.26(a) and 2.27(a) respectively. Both r and s are 1-reproducing, $r^{[0,1]}$ has linear components $p_1, \ldots, p_5$, and $s^{[0,1]}$ has linear components $1, q_1, \ldots, q_4$. Both $r^{[0,1]}$ and $s^{[0,1]}$ are linearized by U, as shown in Figure 2.26(b) and 2.27(b) respectively. Let $r_1 = \cdots = r_4 = X_2 \odot X_2$, $r_5 = X_1 \odot X_1$, and $s_1 = s_2 = X_1 \odot X_1$. Let the map $\tilde{r}$ be specified by the pairs: $(\mathbf{u}_{22}, r_1)$, $(\mathbf{u}_{23}, r_2)$, $(\mathbf{u}_{24}, r_3)$, $(\mathbf{u}_{25}, r_4)$, $(\mathbf{u}_{26}, s)$, $(\mathbf{u}_{30}, r_5)$, $(\mathbf{u}_i, r)$ for all $i \notin \{22, \ldots, 26, 30\}$. Let the map $\tilde{s}$ be specified by the pairs: $(\mathbf{u}_{26}, t)$, $(\mathbf{u}_{27}, s_1)$, $(\mathbf{u}_{28}, s_2)$, $(\mathbf{u}_i, s)$ for all $i \notin \{26, 27, 28\}$. Then, $\tilde{r}$ and $\tilde{s}$ form systems for r and s. These systems are depicted in Figure 2.26(c) and Figure 2.27(c).

We claim that $\tilde{r}$ implements the binary function f over $[0,3]$ depicted in Figure 2.28. It is sufficient to prove that $\tilde{r}$ implements $f(\mathbf{b})$ for every



(a) (b) (c)

Figure 2.26: r from Example 61. (a) $r^{[0,1]}$. (b) The linear decomposition of $r^{[0,1]}$ over U. (c) The system $\tilde{r}$ over $\tilde{U}$.



(a) (b) (c)

Figure 2.27: s from Example 61. (a) $s^{[0,1]}$. (b) The linear decomposition of $s^{[0,1]}$ over U. (c) The system $\tilde{s}$ over $\tilde{U}$.

$\mathbf{b} \in [0,3]^2$, *by applying Definition 35. Let* $\mathbf{b} \in [0,3]^2$. *We distinguish two cases. First suppose that* $\mathbf{b} \notin [1,3]^2$, *and let* $i \in [33]$ *be such that* $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}_i))$. *Notice that* $i \in \{27, 28, 29\}$ *implies* $0 = \lfloor b_2 \rfloor < \lfloor b_1 \rfloor$. *Then,* $\tilde{r}$ *implements* $f(\mathbf{b})$, *indeed, by Lemma 54(i) and Lemma 53 (see Figure 2.29):*

$$f(\mathbf{b}) = \begin{cases} p_1(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{1,2,3,4,18,19,20\} \\ p_2(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{5,6,21\} \\ p_3(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{7,8,9,10\} \\ p_4(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{11,12\} \\ p_4(\mathbf{a}) = r^{[0,1]}(\mathbf{a}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{27,28\} \text{ and } \mathrm{controller}(\mathbf{b}) = \mathbf{a} \\ p_5(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{13,14,15,16,17,31,32,33\} \end{cases}$$

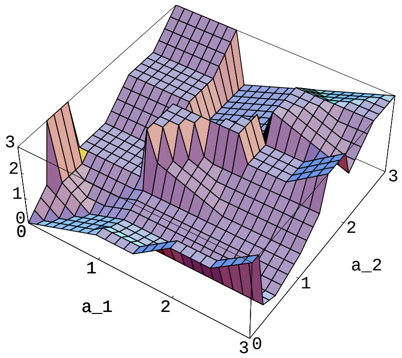


Figure 2.28: The claim of Example 61 is that $\tilde{r}$ implements the function $f : [0,3]^2 \to [0,3]$ above.

$$f(\mathbf{b}) = \begin{cases} p_5(\mathbf{a}) = r^{[0,1]}(\mathbf{a}) = r^{[0,3]}(\mathbf{b}) & \textit{if } i = 29 \textit{ and } \mathrm{controller}(\mathbf{b}) = \mathbf{a} \\ (X_2 \odot X_2)^{[0,3]}(\mathbf{b}) = r_1^{[0,3]}(\mathbf{b}) & \textit{if } i = 22 \\ (X_2 \odot X_2)^{[0,3]}(\mathbf{b}) = r_2^{[0,3]}(\mathbf{b}) & \textit{if } i = 23 \\ (X_2 \odot X_2)^{[0,3]}(\mathbf{b}) = r_3^{[0,3]}(\mathbf{b}) & \textit{if } i = 24 \\ (X_2 \odot X_2)^{[0,3]}(\mathbf{b}) = r_4^{[0,3]}(\mathbf{b}) & \textit{if } i = 25 \\ (X_1 \odot X_1)^{[0,3]}(\mathbf{b}) = r_5^{[0,3]}(\mathbf{b}) & \textit{if } i = 30 \end{cases}$$

Suppose that $\mathbf{b} \in [1,3]^2$*, and let* $i \in [33]$ *be such that* $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}_i))$*. Notice that* $i \in \{22, 23, 24, 25\}$ *implies* $1 = \lfloor b_1 \rfloor < \lfloor b_2 \rfloor$*, and* $i \in \{29, 30\}$ *implies* $1 = \lfloor b_2 \rfloor < \lfloor b_1 \rfloor$*. Then,* $\tilde{r}$ *implements* $f(\mathbf{b})$*, indeed, by Lemma 54 and Lemma 53 (see Figure 2.30):*

$$f(\mathbf{b}) = \begin{cases} 3 = s^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{1,2,3,4,10,11,12,18,19,20\} \\ q_1(\mathbf{b}) = s^{[0,1]}(\mathbf{b}) = s^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{5,6,21\} \\ q_2(\mathbf{b}) = s^{[0,1]}(\mathbf{b}) = s^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{7,8,9,10\} \\ q_2(\mathbf{a}) + 1 = s^{[0,1]}(\mathbf{a}) + 1 = s^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{22,\dots,25\} \textit{ and } \mathrm{controller}(\mathbf{b}) = \mathbf{a} \\ q_3(\mathbf{b}) = s^{[0,1]}(\mathbf{b}) = s^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{13,14\} \\ q_3(\mathbf{a}) + 1 = s^{[0,1]}(\mathbf{a}) + 1 = s^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{29,30\} \textit{ and } \mathrm{controller}(\mathbf{b}) = \mathbf{a} \\ q_4(\mathbf{b}) = s^{[0,1]}(\mathbf{b}) = s^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{15,16,17,31,32,33\} \\ (X_1 \odot X_1)^{[0,3]}(\mathbf{b}) = s_1^{[0,3]}(\mathbf{b}) & \textit{if } i = 27 \\ (X_1 \odot X_1)^{[0,3]}(\mathbf{b}) = s_2^{[0,3]}(\mathbf{b}) & \textit{if } i = 28 \\ 3 = s^{[0,3]}(\mathbf{b}) = t^{[0,3]}(\mathbf{b}) & \textit{if } i = 26 \end{cases}$$

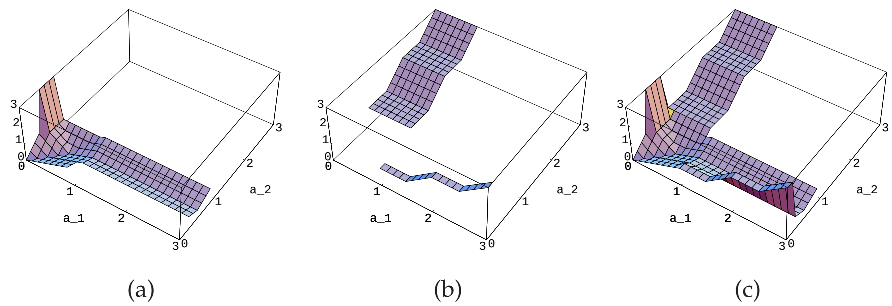


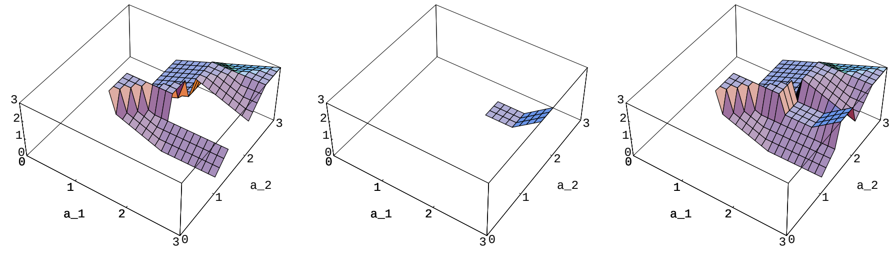
Figure 2.29: Example 61, Case $\mathbf{b} \in [0,3]^2 \setminus [1,3]^2$. Let $\mathbf{u} \in \tilde{U}$, where $\tilde{U}$ is the quasipartition underlying $\tilde{r}$. (a) $\tilde{r}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ such that either $\mathbf{u} \in [0,1)^2$ or $r^{[0,1]}(\mathbf{u}) < 1$. (b) $\tilde{r}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ such that $\mathbf{u} \notin [0,1)^2$ and $r^{[0,1]}(\mathbf{u}) = 1$. (c) $\tilde{r}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in [0,3]^2 \setminus [1,3]^2$.

Note that the above description of f is explicit.

Example 62. *Let U and $\tilde{U}$ be exactly as in Example 61. Let $r \in L_2$ be such that $r^{[0,1]}$ is as in Figure 2.31(a). r is not 1-reproducing. $r^{[0,1]}$ has linear components $p_1, \dots, p_4$, and is linearized by U, as shown in Figure 2.31(b). Let $r_1 = r_2 = r_3 = X_2 \odot X_2$. Let $\tilde{r}$ be the map specified by the following pairs: $(\mathbf{u}_{22}, r_1)$, $(\mathbf{u}_{23}, r_2)$, $(\mathbf{u}_{24}, r_3)$, $(\mathbf{u}_{30}, r_4)$, $(\mathbf{u}_i, r)$ for all $i \notin \{22, 23, 24, 30\}$. Then, $\tilde{r}$ is a system for r. It is depicted in Figure 2.31(c).*

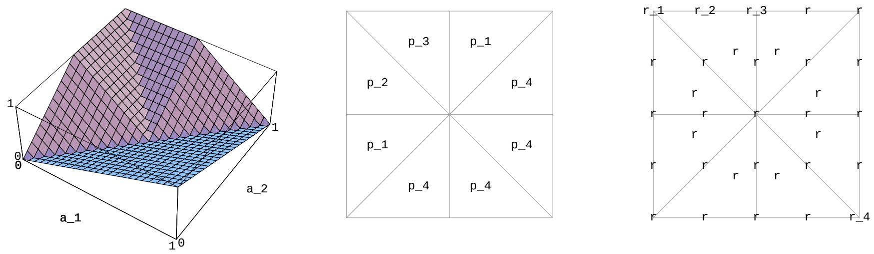
We claim that $\tilde{r}$ implements the binary function f over $[0,3]$ depicted in Figure 2.32. Let $\mathbf{b} \in [0,3]^2$. It is sufficient to prove that $\tilde{r}$ implements $f(\mathbf{b})$ for every $\mathbf{b} \in [0,3]^2$. Let $i \in [33]$ be such that $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}_i))$. Notice that $i \in \{22, 23, 24\}$ implies $0 = \lfloor b_1 \rfloor < \lfloor b_2 \rfloor$, and $i = 30$ implies $0 = \lfloor b_2 \rfloor < \lfloor b_1 \rfloor$. Then, $\tilde{r}$ implements $f(\mathbf{b})$, indeed, by Lemma 54 and Lemma 53 (see Figure 2.32), if $\mathbf{b} \in [1,3]^2$ then $f(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) = 0$, and otherwise:

$$f(\mathbf{b}) = \begin{cases} p_1(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{1,2,3,4,8,9,10,18,19,20\} \\ p_1(\mathbf{a}) = r^{[0,1]}(\mathbf{a}) = r^{[0,3]}(\mathbf{b}) & \text{if } i = 25 \text{ and } \mathrm{controller}(\mathbf{b}) = \mathbf{a} \\ p_2(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{5,6,21\} \\ p_3(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i = 7 \\ p_4(\mathbf{b}) = r^{[0,1]}(\mathbf{b}) = r^{[0,3]}(\mathbf{b}) & \text{if } i \in \{11,\dots,17,31,32,33\} \end{cases}$$



(a) (b) (c)

Figure 2.30: Example 61, Case $\mathbf{b} \in [1,3]^2$. In this case, $\tilde{r}$ delegates the implementation of f to the auxiliary system $\tilde{s}$. Let $\mathbf{u} \in \tilde{U}$, where $\tilde{U}$ is the quasipartition underlying $\tilde{s}$. (a) $\tilde{s}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ such that either $\mathbf{u} \in [0,1)^2 \cup \{\mathbf{1}\}$ or $s^{[0,1]}(\mathbf{u}) < 1$. (b) $\tilde{s}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, $\mathbf{u} \in \tilde{U}$, such that $\mathbf{u} \notin [0,1)^2 \cup \{\mathbf{1}\}$ and $s^{[0,1]}(\mathbf{u}) = 1$. (c) $\tilde{r}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in [1,3]^2$.



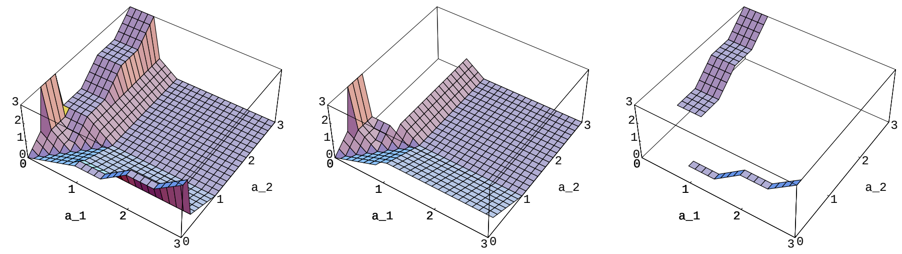
(a) $r^{[0,1]}$. (b) Decomposing $r^{[0,1]}$. (c) $\tilde{r}$.

Figure 2.31: r from Example 62. (a) $r^{[0,1]}$. (b) The linear decomposition of $r^{[0,1]}$ over U. (c) The system $\tilde{r}$ over $\tilde{U}$.

$$f(\mathbf{b}) = \begin{cases} p_4(\mathbf{a}) = r^{[0,1]}(\mathbf{a}) = r^{[0,3]}(\mathbf{b}) & \textit{if } i \in \{27, 28, 29\} \textit{ and } \mathrm{controller}(\mathbf{b}) = \mathbf{a} \\ (X_2 \odot X_2)^{[0,3]}(\mathbf{b}) = r_1^{[0,3]}(\mathbf{b}) & \textit{if } i = 22 \\ (X_2 \odot X_2)^{[0,3]}(\mathbf{b}) = r_2^{[0,3]}(\mathbf{b}) & \textit{if } i = 23 \\ (X_2 \odot X_2)^{[0,3]}(\mathbf{b}) = r_3^{[0,3]}(\mathbf{b}) & \textit{if } i = 24 \\ (X_1 \odot X_1)^{[0,3]}(\mathbf{b}) = r_4^{[0,3]}(\mathbf{b}) & \textit{if } i = 30 \end{cases}$$

Note that the above description of f is explicit.

In the next section, we finally get to the general case, $n \geq 1$.



(a) (b) (c)

Figure 2.32: The claim of Example 62 is that $\tilde{r}$ implements the function $f\colon [0,3]^2 \to [0,3]$ depicted in (a). Plot (b) shows that $\tilde{r}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ such that either $\mathbf{u} \in [0,1)^2 \cup \{\mathbf{1}\}$ or $r^{[0,1]}(\mathbf{u}) < 1$. Plot (c) shows that $\tilde{r}$ implements $f(\mathbf{b})$ for each $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ such that $\mathbf{u} \notin [0,1)^2$ and $r^{[0,1]}(\mathbf{u}) = 1$.

2.3.4 General Case

In this section, we show that the class F_n of n-ary BL-functions in Definition 36 has an explicit description in terms of Definition 15, and coincides with the class B_n of the truthfunctions of the n-variate fragment of Basic logic.

First, we enrich the signature of the algebra $[0, n+1]$ of Definition 3 by adding the operations $\vee$, $\wedge$, $\neg$, and $\top$, of arity 2, 2, 1, and 0 respectively, defined as in Definition 37. For $\circ \in \{\vee, \wedge, \odot, \to, \neg, \top, \bot\}$, we let $\circ^{[0,n+1]}$ denote the realization of the symbol $\circ$ in the algebra $[0, n+1]$ over the enriched signature.

Fact 63. *For every* $a_1, a_2 \in [0, n+1]$, $a_1 \wedge^{[0,n+1]} a_2 = \min(a_1, a_2)$, $a_1 \vee^{[0,n+1]} a_2 = \max(a_1, a_2)$, $\top^{[0,n+1]} = n+1$, *and,*

$$\neg^{[0,n+1]} a_1 = \begin{cases} n+1 & \textit{if } a_1 = 0 \\ \neg^{[0,1]} a_1 & \textit{if } 0 < a_1 < 1 \\ 0 & \textit{otherwise} \end{cases}$$

Note that, by Notation 10, for every $r, s \in L_n$ and every $\mathbf{a} \in [0, n+1]^n$, $(\neg r)^{[0,n+1]}(\mathbf{a}) = \neg^{[0,n+1]} r^{[0,n+1]}(\mathbf{a})$, $(r \wedge s)^{[0,n+1]}(\mathbf{a}) = r^{[0,n+1]}(\mathbf{a}) \wedge^{[0,n+1]} s^{[0,n+1]}(\mathbf{a})$, $(r \vee s)^{[0,n+1]}(\mathbf{a}) = r^{[0,n+1]}(\mathbf{a}) \vee^{[0,n+1]} s^{[0,n+1]}(\mathbf{a})$, and $\top^{[0,n+1]}(\mathbf{a}) = \top^{[0,n+1]}$.

Fact 64. *For every* $a_1, a_2 \in [0, n+1]$,

$$a_1 \odot^{[0,n+1]} a_2 = \begin{cases} \min(a_1, a_2) & \text{if } \lfloor a_1 \rfloor \neq \lfloor a_2 \rfloor \\ (a_1 - j \odot^{[0,1]} a_2 - j) + j & \text{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = j \end{cases}$$

$$a_1 \to^{[0,n+1]} a_2 = \begin{cases} n+1 & \text{if } a_1 \leq a_2 \\ a_2 & \text{if } \lfloor a_2 \rfloor < \lfloor a_1 \rfloor \\ (a_1 - j \to^{[0,1]} a_2 - j) - j & \text{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = j \end{cases}$$

Let $n \geq 1$. We mentioned that the truthfunctions of the n-variate fragment of Basic logic, B_n,

$$B_n = \{t^{[0,n+1]} \mid t \in L_n\} \subseteq [0, n+1]^{[0,n+1]^n},$$

coincide with the smallest set of n-ary functions over $[0, n+1]^n$ that contains the projections $x_1, \ldots, x_n$, the constant 0, and is closed under pointwise application of the operations $\odot^{[0,n+1]}$ and $\to^{[0,n+1]}$. Our goal is to provide an explicit description of B_n, in terms of Definition 15.

The solution schema sketched in Section 2.1 has two stages. The first stage consists in guessing an explicit class of n-ary functions over $[0, n+1]$. We guess that the required class of functions is F_n, the class of n-ary BL-functions in Definition 36. The second stage consists in checking that the guessed class is equal to B_n.

In the next section, we show that F_n is explicit, and that the inclusion $B_n \subseteq F_n$ holds.

Explicitness and Closure

In this section, we prove the inclusion $B_n \subseteq F_n$, and then we prove that F_n has an explicit description ($n \geq 1$).

Lemma 65 (Closure)**.** *Let t be a term in L_n. Then, there exists a function $f \in F_n$ such that $t^{[0,n+1]} = f$.*

Proof. The proof is by induction on t.

For the base case, let $i \in [n]$ and let $t = X_i$. By definition, $X_i^{[0,n+1]}(\mathbf{b}) = b_i$ for every $\mathbf{b} = (b_1, \ldots, b_n) \in [0, n+1]^n$, that is $X_i^{[0,n+1]}$ is the projection function x_i over $[0, n+1]^n$. Take $r, s = X_i$ and fix a unimodular triangulation U of $[0, 1]^n$ linearizing $X_i^{[0,1]} : [0, 1]^n \to [0, 1]$. Then, the map $\tilde{r}$ that sends every $\mathbf{u} \in \tilde{U}$ to X_i is a system for r, and the map $\tilde{s}$ that sends every $\mathbf{u} \in \tilde{U}$ to X_i is a system for s. But, $\tilde{r}$ implements x_i. Thus, $x_i \in F_n$. Now, let $t = \bot$. By definition, $\bot^{[0,n+1]}(\mathbf{b}) = 0$ for every $\mathbf{b} \in [0, n+1]^n$, that is $\bot^{[0,n+1]}$ is the constant function 0 over $[0, n+1]^n$. Take $r = \bot$ and fix a unimodular triangulation U of $[0,1]^n$ linearizing $\bot^{[0,1]} : [0,1]^n \to [0,1]$. Then, the map $\tilde{r}$ that sends every $\mathbf{u} \in \tilde{U}$ to $\bot$ is a system for r. But, $\tilde{r}$ implements 0. Thus, $0 \in F_n$. The base case is settled.

For the inductive step, let $t = t_1 \circ t_2$ for $\circ \in \{\odot, \to\}$. By the induction hypothesis, there exist functions $f_1, f_2 \in F_n$ such that $t_1^{[0,n+1]} = f_1$ and $t_2^{[0,n+1]} = f_2$. By definition,

$$t^{[0,n+1]} = t_1^{[0,n+1]} \circ^{[0,n+1]} t_2^{[0,n+1]} = f_1 \circ^{[0,n+1]} f_2.$$

Let $f = f_1 \circ^{[0,n+1]} f_2$. We define a system $\tilde{r}$ (over a certain $\tilde{U}$) that implements f, thus proving that $f \in F_n$.

Since $f_1, f_2 \in F_n$, there exist systems $\tilde{r_1}$ (say over $\tilde{U_1}$) and $\tilde{r_2}$ (say over $\tilde{U_2}$) implementing f_1 and f_2. On the basis of $\tilde{r_1}$ and $\tilde{r_2}$, we define $\tilde{r}$, as follows. There are eight cases ($\circ$ is equal to $\odot$ or to $\to$; both r_1 and r_2 are 1-reproducing, only r_1 is 1-reproducing, only r_2 is 1-reproducing, neither r_1 nor r_2 are 1-reproducing).

Case 1 (r_1, r_2 1-reproducing and $\circ = \odot$): In this case, by Definition 35, $r_1, r_2 \in L_n^+$, $\tilde{r_1}(\mathbf{1}) = s_1$, $\tilde{r_2}(\mathbf{1}) = s_2$, with $s_1, s_2 \in L_n^+$. We let $r = r_1 \odot r_2$, and $s = s_1 \odot s_2$. Note that both r and s are 1-reproducing in L_n^+. We define $\tilde{r}$ and $\tilde{s}$, as follows. First we put $\tilde{r}(\mathbf{1}) = s$. By Fact 27, let U be a unimodular triangulation refining both U_1 and U_2 and linearizing r, and let V be a unimodular triangulation refining both V_1 and V_2 and linearizing s. Note that for every $\mathbf{1} \neq \mathbf{u} \in \tilde{U}$, there exists exactly one pair $(\mathbf{u}_1, \mathbf{u}_2) \in \tilde{U_1} \times \tilde{U_2}$ such that $\mathbf{u} \in \mathrm{sibl}(\mathbf{u}_1) \cap \mathrm{sibl}(\mathbf{u}_2)$. Note that $\mathrm{par}(\mathbf{u}) = \mathrm{par}(\mathbf{u}_1) = \mathrm{par}(\mathbf{u}_2)$. In this case, we put,

$$\tilde{r}(\mathbf{u}) = \begin{cases} t_1 \odot t_2 & \text{if } \tilde{r_1}(\mathbf{u}_1) = t_1 \in L_{\mathrm{par}(\mathbf{u}_1)} \text{ and } \tilde{r_2}(\mathbf{u}_2) = t_2 \in L_{\mathrm{par}(\mathbf{u}_2)} \\ r & \text{otherwise} \end{cases}$$

where t_1 and t_2 are 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, so that $t_1 \odot t_2$ is 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$. Similarly, for every $\mathbf{v} \in \tilde{V}$, there exists exactly one pair $(\mathbf{v}_1, \mathbf{v}_2) \in \tilde{V_1} \times \tilde{V_1}$ such that $\mathbf{v} \in \mathrm{sibl}(\mathbf{v}_1) \cap \mathrm{sibl}(\mathbf{v}_2)$. Note that $\mathrm{par}(\mathbf{v}) = \mathrm{par}(\mathbf{v}_1) = \mathrm{par}(\mathbf{v}_2)$. In this case, we put $\tilde{s}(\mathbf{1}) = s$ and,

$$\tilde{s}(\mathbf{v}) = \begin{cases} w_1 \odot w_2 & \text{if } \tilde{s_1}(\mathbf{v}_1) = w_1 \in L_{\mathrm{par}(\mathbf{v}_1)} \text{ and } \tilde{s_2}(\mathbf{v}_2) = w_2 \in L_{\mathrm{par}(\mathbf{v}_2)} \\ s & \text{otherwise} \end{cases}$$

where w_1 and w_2 are 1-reproducing in $L_{\mathrm{par}(\mathbf{v})}$, so that $w_1 \odot w_2$ is 1-reproducing in $L_{\mathrm{par}(\mathbf{v})}$.

Case 2 (r_1 only 1-reproducing and $\circ = \odot$): In this case, by Definition 35, $r_1 \in L_n^+$, and $r_2 = \neg r_2'$ with $r_2' \in L_n^+$. Note that $r_1 \odot \neg r_2'$ is not 1-reproducing, and $(r_1 \odot \neg r_2')^{[0,1]} = (\neg(r_1 \to r_2'))^{[0,1]}$. We let $r = \neg(r_1 \to r_2')$, where $(r_1 \to r_2') \in L_n^+$. We define $\tilde{r}$, as follows. First we put $\tilde{r}(\mathbf{1}) = r$. Next, letting $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2$ as in the previous case, we put,

$$\tilde{r}(\mathbf{u}) = \begin{cases} t_1 \odot t_2 & \text{if } \tilde{r_1}(\mathbf{u}_1) = t_1 \in L_{\mathrm{par}(\mathbf{u}_1)} \text{ and } \tilde{r_2}(\mathbf{u}_2) = t_2 \in L_{\mathrm{par}(\mathbf{u}_2)} \\ r & \text{otherwise} \end{cases}$$

where t_1 and t_2 are 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, so that $t_1 \odot t_2$ is 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$.

Case 3 (r_2 only 1-reproducing and $\circ = \odot$): Here, $r_2 \in L_n^+$ and $r_1 = \neg r_1'$ with $r_1' \in L_n^+$. We let $r = \neg(r_2 \to r_1')$. The rest is similar to Case 2.

Case 4 (neither r_1 nor r_2 1-reproducing and $\circ = \odot$): Here, $r_1 = \neg r_1'$ with $r_1' \in L_n^+$, and $r_2 = \neg r_2'$ with $r_2' \in L_n^+$. Noticing that $(\neg r_1' \odot \neg r_2')^{[0,1]} = (\neg((r_1' \to (r_1' \odot r_2')) \to r_2'))^{[0,1]}$, we let $r = \neg((r_1' \to (r_1' \odot r_2')) \to r_2')$. The rest is similar to Case 2.

Case 5 (r_1, r_2 1-reproducing and $\circ = \to$): Here, $r_1, r_2 \in L_n^+$, $\tilde{r_1}(\mathbf{1}) = s_1$, $\tilde{r_2}(\mathbf{1}) = s_2$, with $s_1, s_2 \in L_n^+$. We let $r = r_1 \to r_2$, and $s = s_1 \to s_2$. Note that both r and s are 1-reproducing in L_n^+. We define $\tilde{r}$ and $\tilde{s}$, as follows. First we put $\tilde{r}(\mathbf{1}) = s$. Next, letting $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2$ be mutatis mutandis as in Case 1, we put,

$$\tilde{r}(\mathbf{u}) = \begin{cases} t_1 \to t_2 & \text{if } \tilde{r_1}(\mathbf{u}_1) = t_1 \in L_{\mathrm{par}(\mathbf{u}_1)} \text{ and } \tilde{r_2}(\mathbf{u}_2) = t_2 \in L_{\mathrm{par}(\mathbf{u}_2)} \\ \top & \text{if } \tilde{r_1}(\mathbf{u}_1) = r_1 \text{ and } \tilde{r_2}(\mathbf{u}_2) = t_2 \in L_{\mathrm{par}(\mathbf{u}_2)} \\ r & \text{otherwise} \end{cases}$$

where t_1 and t_2 are 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, so that $t_1 \odot t_2$ is 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, and $\top$ is trivially 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$. Finally, letting $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ be mutatis mutandis as in Case 1, we put $\tilde{s}(\mathbf{1}) = s$ and,

$$\tilde{s}(\mathbf{v}) = \begin{cases} w_1 \to w_2 & \text{if } \tilde{s_1}(\mathbf{v}_1) = w_1 \in L_{\mathrm{par}(\mathbf{v}_1)} \text{ and } \tilde{s_2}(\mathbf{v}_2) = w_2 \in L_{\mathrm{par}(\mathbf{v}_2)} \\ \top & \text{if } \tilde{s_1}(\mathbf{v}_1) = s_1 \text{ and } \tilde{s_2}(\mathbf{v}_2) = w_2 \in L_{\mathrm{par}(\mathbf{v}_2)} \\ s & \text{otherwise} \end{cases}$$

where w_1 and w_2 are 1-reproducing in $L_{\mathrm{par}(\mathbf{v})}$, so that $w_1 \to w_2$ is 1-reproducing in $L_{\mathrm{par}(\mathbf{v})}$, and $\top$ is trivially 1-reproducing in $L_{\mathrm{par}(\mathbf{v})}$.

Case 6 (r_1 only 1-reproducing and $\circ = \to$): Here, $r_1 \in L_n^+$ and $r_2 = \neg r_2'$ with $r_2' \in L_n^+$. Noticing that $r_1 \to \neg r_2'$ is not 1-reproducing, that $(r_1 \to \neg r_2')^{[0,1]} = (\neg(r_1 \odot r_2'))^{[0,1]}$, we let $r = \neg(r_1 \odot r_2')$, where $(r_1 \odot r_2') \in L_n^+$. We define $\tilde{r}$ as follows. First we put $\tilde{r}(\mathbf{1}) = r$. Next, letting $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2$ be mutatis mutandis as in Case 1, we put,

$$\tilde{r}(\mathbf{u}) = \begin{cases} t_1 \to t_2 & \text{if } \tilde{r_1}(\mathbf{u}_1) = t_1 \in L_{\mathrm{par}(\mathbf{u}_1)} \text{ and } \tilde{r_2}(\mathbf{u}_2) = t_2 \in L_{\mathrm{par}(\mathbf{u}_2)} \\ \top & \text{if } \tilde{r_1}(\mathbf{u}_1) = r_1 \text{ and } \tilde{r_2}(\mathbf{u}_2) = t_2 \in L_{\mathrm{par}(\mathbf{u}_2)} \\ r & \text{otherwise} \end{cases}$$

where t_1 and t_2 are 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, so that $t_1 \odot t_2$ is 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, and $\top$ is trivially 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$.

Case 7 (r_2 only 1-reproducing and $\circ = \to$): Here, $r_1 = \neg r_1'$ with $r_1' \in L_n^+$, $\tilde{r_1}(\mathbf{1}) = r_1$, $r_2 \in L_n^+$, and $\tilde{r_2}(\mathbf{1}) = s_2$ with $s_2 \in L_n^+$. Noticing that $\neg r_1' \to r_2$ is 1-reproducing, that $(\neg r_1' \to r_2)^{[0,1]} = ((r_1' \to (r_1' \odot r_2)) \to r_2)^{[0,1]}$, and that $(\neg r_1' \to s_2)^{[0,1]} = ((r_1' \to (r_1' \odot s_2)) \to s_2)^{[0,1]}$, we let $r = ((r_1' \to (r_1' \odot r_2)) \to r_2)$, and $s = ((r_1' \to (r_1' \odot s_2)) \to s_2)$, where both r and s are in L_n^+. The rest is similar to Case 5.

Case 8 (neither r_1 nor r_2 1-reproducing and $\circ = \to$): Here, $r_1 = \neg r_1'$ with $r_1' \in L_n^+$, $\tilde{r_1}(\mathbf{1}) = r_1$, $r_2 = \neg r_2'$ with $r_2' \in L_n^+$, $\tilde{r_2}(\mathbf{1}) = r_2$. Noticing that $\neg r_1' \to \neg r_2'$ is 1-reproducing, and that $(\neg r_1' \to \neg r_2')^{[0,1]} = (r_2' \to (r_1' \odot (r_1' \to r_2')))^{[0,1]}$, we let $r = s = (r_2' \to (r_1' \odot (r_1' \to r_2'))) \in L_n^+$. The rest is similar to Case 5.

We conclude the proof noticing that, by construction, $\tilde{r}$ is a system that implements the function f. Thus, $f \in F_n$, and the inductive step is settled. □

Now we consider the explicitness of F_n, and we claim that Definition 36 provides an explicit description of F_n: if the function $f \in F_n$ is implemented by the system $\tilde{r}$, then $\tilde{r}$ describes explicitly f in the sense of Definition 15. In order to prove the claim, we preliminarily need two technical lemmas, that deal in full generality with the phenomena of *extension* and *lifting* we have already encountered.

The first lemma deals with the extension mechanism. Modulo technicalities, the underlying intuition is that if $t \in L_n$ is a term such that at most $m < n$ variables occur in t, say among $X_1, \dots, X_m$, then, reasoning on Definition 3, the function,

$$t^{[0,n+1]} \colon [0, n+1]^n \to [0, n+1],$$

can be described in terms of the function,

$$t^{[0,m+1]} \colon [0, m+1]^m \to [0, m+1].$$

But then, under the assumption that $t^{[0,m+1]}$ has an explicit description, $t^{[0,n+1]}$ itself has an explicit description. The assumption is justified by an inductive argument, with the case $n = 1$ in Section 2.3.2 acting as induction basis. The technical details follow.

Definition 66 (Projection)**.** *Let $n \geq 2$. Let $I = \{i_1, \dots, i_m\} \subset [n]$ be such that $i_1 < \cdots < i_m$. Let $\mathbf{b} = (b_1, \dots, b_n) \in [0, n+1]^n$, and let π be a permutation of I such that $\lfloor b_{\pi(i_1)} \rfloor \leq \cdots \leq \lfloor b_{\pi(i_m)} \rfloor$. Let $k \geq 0$ and $(\lhd_1, \dots, \lhd_{m-k+1}) \in \{<, =\}^{m-k+1}$ be such that:*

$$0 = \lfloor b_{\pi(i_1)} \rfloor = \cdots = \lfloor b_{\pi(i_k)} \rfloor < 1 \lhd_1 \lfloor b_{\pi(i_{k+1})} \rfloor \lhd_2 \cdots \lhd_{m-k} \lfloor b_{\pi(i_m)}) \rfloor \lhd_{m-k+1} n+1.$$

Let ρ be the permutation of $[m]$ such that $\rho(j) = j'$ if and only if $\pi(i_j) = i_{j'}$. Let $\mathbf{c} = (c_1, \dots, c_m) \in [0, m+1]^m$ be the unique element in $[0, m+1]^m$ that satisfies $c_1 - \lfloor c_1 \rfloor = b_{i_1} - \lfloor b_{i_1} \rfloor, \dots, c_m - \lfloor c_m \rfloor = b_{i_m} - \lfloor b_{i_m} \rfloor$, satisfies,

$$0 = \lfloor c_{\rho(1)} \rfloor = \cdots = \lfloor c_{\rho(k)} \rfloor < 1 \lhd_1 \lfloor c_{\rho(k+1)} \rfloor \lhd_2 \cdots \lhd_{m-k} \lfloor c_{\rho(m)} \rfloor \lhd_{m-k+1} m+1,$$

*and minimizes $\lfloor c_1 \rfloor + \cdots + \lfloor c_m \rfloor$. We call $\mathbf{c}$ the I-*projection *of $\mathbf{b}$ over $[0, m+1]^m$.*

Lemma 67 (Extension). *Let $n \geq 2$, and suppose that F_i has an explicit description for all $i < n$. Let $\emptyset \subset I \subset [n]$ and let t be a 1-reproducing term in L_I. Then, $t^{[0,n+1]}$ has an explicit description in terms of Definition 15.*

Proof. Suppose that $I = \{i_1, \ldots, i_m\}$ with $i_1 < \cdots < i_m$, and let $\bar{t} = t_{\{i_1 \backslash 1, \ldots, i_m \backslash m\}}$. A routine induction on t, applying Definition 3 and Definition 66, shows that, for every $\mathbf{b} = (b_1, \ldots, b_n) \in [0, n+1]^n$, letting $\mathbf{c} = (c_1, \ldots, c_m) \in [0, m+1]^m$ be the I-projection of $\mathbf{b}$ over $[0, m+1]^m$:

$$t^{[0,n+1]}(\mathbf{b}) = \begin{cases} 0 & \text{if } \bar{t}^{[0,m+1]}(\mathbf{c}) = 0 \\ n+1 & \text{if } \bar{t}^{[0,m+1]}(\mathbf{c}) = m+1 \\ \bar{t}^{[0,m+1]}(\mathbf{c}) - \lfloor \bar{t}^{[0,m+1]}(\mathbf{c}) \rfloor + j & \text{otherwise,} \end{cases} \tag{2.17}$$

where j in the third clause is settled as follows: if $0 < \bar{t}^{[0,m+1]}(\mathbf{c}) < m+1$, then there exists $k \in [m]$ such that $\lfloor \bar{t}^{[0,m+1]}(\mathbf{c}) \rfloor = \lfloor c_k \rfloor$; then, let $k \in [m]$ be such that $\lfloor \bar{t}^{[0,m+1]}(\mathbf{c}) \rfloor = \lfloor c_k \rfloor$, and settle $j = \lfloor b_{i_k} \rfloor$.

By Lemma 65, $\bar{t}^{[0,m+1]} \in B_m \subseteq F_m$. Since by hypothesis $m < n$, by hypothesis, F_m has an explicit description. Hence, we conclude that (2.17) describes explicitly $t^{[0,n+1]}$. □

The second lemma deals with the lifting mechanism.

Lemma 68 (Lifting). *Let $n \geq 2$, and suppose that F_i has an explicit description for all $i < n$. Let $t \in L_n$ be such that $t \in L_n^+$ if t is 1-reproducing, and $t = \neg t'$ with $t' \in L_n^+$ otherwise. Then, $t^{[0,n+1]}$ has an explicit description.*

Proof. Let $\tilde{U}$ be a quasipartition for t, let $\mathbf{b} = (b_1, \ldots, b_n) \in [0, n+1]^n$, and let $\mathbf{u}$ in $\tilde{U}$ be such that $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$. We show that $t^{[0,n+1]}(\mathbf{b})$ is described explicitly by equation (2.18) if t is 1-reproducing, and by equation (2.19) if t is not 1-reproducing.

First suppose that t is 1-reproducing, so that $t \in L_n^+$. By induction on t, applying Definition 3, we get,

$$t^{[0,n+1]}(\mathbf{b}) = \begin{cases} t^{[0,1]}(\mathbf{b} - \mathbf{j}) + j & j = \lfloor b_1 \rfloor = \cdots = \lfloor b_n \rfloor,\ t^{[0,1]}(\mathbf{u}) < 1 \\ n+1 & j = \lfloor b_1 \rfloor = \cdots = \lfloor b_n \rfloor,\ t^{[0,1]}(\mathbf{u}) = 1 \\ t^{[0,1]}(\mathrm{controller}(\mathbf{b})) + j & j = \min\{\lfloor b_1 \rfloor, \ldots, \lfloor b_n \rfloor\},\ t^{[0,1]}(\mathbf{u}) < 1 \\ \bar{t}^{[0,n+1]}(\mathbf{b}) & \text{otherwise,} \end{cases} \tag{2.18}$$

where $\bar{t}$ is a 1-reproducing term in $L_{\mathrm{par}(\mathbf{u})}$. Noticing that $\emptyset \subset \mathrm{par}(\mathbf{u}) \subset [n]$ in the last clause, and that F_i has an explicit description for all $i < n$ by hypothesis, we apply Lemma 67 to $\bar{t}$, and we have that $\bar{t}^{[0,n+1]}$ has an explicit description. Moreover, by Fact 19, $t^{[0,1]}$ has an explicit description. Hence, $t^{[0,n+1]}(\mathbf{b})$ has an explicit description, precisely, the description given by equation (2.18).

Next suppose that t is not 1-reproducing, so that $t = \neg t'$ for $t' \in L_n^+$. By the previous part, t' is described explicitly by (2.18). Hence, by the definition of $\neg^{[0,n+1]}$ in Fact 63, we get,

$$t^{[0,n+1]}(\mathbf{b}) = \begin{cases} t^{[0,1]}(\mathbf{b}) & \mathbf{b} \in [0,1)^n,\, t^{[0,1]}(\mathbf{u}) < 1 \\ n+1 & \mathbf{b} \in [0,1)^n,\, t^{[0,1]}(\mathbf{u}) = 1 \\ 0 & \mathbf{b} \in [1,n+1]^n \\ t^{[0,1]}(\mathrm{controller}(\mathbf{b})) & t^{[0,1]}(\mathbf{u}) < 1 \\ n+1 & \text{otherwise,} \end{cases} \tag{2.19}$$

By Fact 19, $t^{[0,1]}$ has an explicit description. Hence, $t^{[0,n+1]}(\mathbf{b})$ has an explicit description, precisely, the description given by equation (2.19). $\square$

It is now possible to show that F_n has an explicit description ($n \geq 1$).

Proposition 69 (Explicitness)**.** *For every $n \geq 1$, F_n has an explicit description in terms of Definition 15.*

Proof. The proof is by induction on n. For the base case $n = 1$, we have that F_1 has an explicit description by Proposition 41. For the inductive step, let $n \geq 2$ and suppose that F_i has an explicit description for every $i < n$. Let $f \in F_n$ be implemented by the system $\tilde{r}$ over the term $r \in L_n$ and the quasipartition $\tilde{U}$. We distinguish two cases.

If r is 1-reproducing, then $\tilde{r}(\mathbf{1}) = s$, with s is 1-reproducing in L_n, and there exists a system $\tilde{s}$ for s, say over the quasipartition $\tilde{V}$. Both r and s are in L_n^+. Let $\mathbf{b} \in [0, n+1]^n$. We have to show that $f(\mathbf{b})$ has an explicit description. First suppose that $\mathbf{b} \notin [1, n+1]^n$. Let $\mathbf{a} \in [0,1]^n$ be the controller of $\mathbf{b}$, let $\mathbf{u}$ be the parent of $\mathbf{a}$ in $\tilde{U}$, and let $\tilde{r}(\mathbf{u}) = t$. By Definition 35, $f(\mathbf{b}) = t^{[0,n+1]}(\mathbf{b})$. If $\mathbf{u} \in [0,1)^n$ or $r^{[0,1]}(\mathbf{u}) < 1$, we have that $t = r$, and, by Lemma 68 we have that $r^{[0,n+1]}(\mathbf{b})$ has an explicit description; otherwise, if $\mathbf{u} \notin [0,1)^n$ and $r^{[0,1]}(\mathbf{u}) = 1$, we have that t is

1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, with $\emptyset \subset \mathrm{par}(\mathbf{u}) \subset [n]$. Hence, exploiting the induction hypothesis, Lemma 67 applies and we have that $t^{[0,n+1]}(\mathbf{b})$ has an explicit description. Next suppose that $\mathbf{b} \in [1, n+1]^n$. Let $\mathbf{a} \in [0,1]^n$ be the controller of $\mathbf{b}$, let $\mathbf{v}$ be the parent of $\mathbf{a}$ in $\tilde{V}$, and let $\tilde{s}(\mathbf{v}) = t$. By Definition 35, $f(\mathbf{b}) = t^{[0,n+1]}(\mathbf{b})$. Reasoning as above, we have that $t^{[0,n+1]}(\mathbf{b})$ has an explicit description. Therefore, if r is 1-reproducing, then f has an explicit description.

Otherwise, suppose that r is not 1-reproducing ($r = \neg r'$ with $r' \in L_n^+$). Let $\mathbf{b} \in [0, n+1]^n$. We have to show that $f(\mathbf{b})$ has an explicit description. First suppose that $\mathbf{b} \notin [1, n+1]^n$. Let $\mathbf{a} \in [0,1]^n$ be the controller of $\mathbf{b}$, let $\mathbf{u}$ be the parent of $\mathbf{a}$ in $\tilde{U}$, and let $\tilde{r}(\mathbf{u}) = t$. By Definition 35, $f(\mathbf{b}) = t^{[0,n+1]}(\mathbf{b})$. If $\mathbf{u} \in [0,1)^n$ or $r^{[0,1]}(\mathbf{u}) < 1$, we have that $t = r$, and, by Lemma 68 we have that $r^{[0,n+1]}(\mathbf{b})$ has an explicit description; otherwise, if $\mathbf{u} \notin [0,1)^n$ and $r^{[0,1]}(\mathbf{u}) = 1$, we have that t is 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$, with $\emptyset \subset \mathrm{par}(\mathbf{u}) \subset [n]$. Hence, by the induction hypothesis, Lemma 67 applies and we have that $t^{[0,n+1]}(\mathbf{b})$ has an explicit description. Next suppose that $\mathbf{b} \in [1, n+1]^n$. Since r is not 1-reproducing, $\tilde{r}(\mathbf{1}) = r$, and by Definition 35, $f(\mathbf{b}) = r^{[0,n+1]}(\mathbf{b})$. But, by Lemma 68, $r^{[0,n+1]}(\mathbf{b}) = 0$. Therefore, if r is not 1-reproducing, f has an explicit description. □

In the next section, we perform the last step of our solution schema, proving the inclusion $F_n \subseteq B_n$.

Normal Forms

In this section, we provide a constructive proof of the inclusion $F_n \subseteq B_n$. To this aim, we preliminarily implement in full generality the *isolation* mechanism.

Definition 70 (Isolation). *Let $n \geq 2$, let $t, s \in L_n$, and let $D \subseteq [0, n+1]^n$. We say that t* isolates *s over the D if $t^{[0,n+1]}(\mathbf{b}) = s^{[0,n+1]}(\mathbf{b})$ if $\mathbf{b} \in D$, and otherwise $t^{[0,n+1]}(\mathbf{b}) = \top^{[0,n+1]}$.*

The intuition underlying the isolation mechanism is the following. Let $f \in F_n$ be an n-ary BL-function, and let $\tilde{r}$ be a system, say over the quasipartition $\tilde{U}$, that implements f. For the sake of clarity, suppose that r is 1-reproducing in L_n^+ (the case where $r = \neg r'$ with r' in L_n^+ is basically subsumed). Then, by Definition 34, $\tilde{r}(\mathbf{1}) = s \in L_n^+$, and, by Definition 35, there exists an auxiliary system $\tilde{s}$, say over the quasipartition $\tilde{V}$. The isolation mechanism works as follows: for every $\mathbf{u} \in \tilde{U}$, it computes an isolating term $t_{\mathbf{u}}$, that is, a term that isolates $\tilde{r}(\mathbf{u})$ over $\mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \cap ([0, n+1]^n \setminus [1, n+1]^n)$; and similarly, for every $\mathbf{v} \in \tilde{V}$, it computes an isolating term $w_{\mathbf{v}}$, that is, a term that isolates $\tilde{s}(\mathbf{v})$ over $\mathrm{realm}(\mathrm{sibl}(\mathbf{v})) \cap [1, n+1]^n$. But then the term,

$$\bigwedge_{\mathbf{u}\in\tilde{U}} t_{\mathbf{u}} \wedge \bigwedge_{\mathbf{v}\in\tilde{V}} w_{\mathbf{v}},$$

computes the input function f.

By Definition 34 and Definition 35, $\tilde{r}(\mathbf{u})$ is either r, that is, a 1-reproducing term in L_n^+, or is a 1-reproducing term in L_I for some $\emptyset \subset I \subset [n]$, and similarly $\tilde{s}(\mathbf{v})$ is either s, that is, a 1-reproducing term in L_n^+, or is a 1-reproducing term in L_I for some $\emptyset \subset I \subset [n]$. The mechanism isolates the variables first, and then easily extends to terms. For instance, consider the case where $\mathbf{u} \in [0,1)^n$, so that $\tilde{r}(\mathbf{u}) = r$. It turns out that, by Lemma 71(i), we are able to isolate variable X_1, variable X_2, ..., variable X_n over $D = \mathrm{realm}(\mathrm{neigh}(\mathbf{u})) \cap ([0, n+1]^n \setminus [1, n+1]^n)$. Let $t_1, \ldots, t_n \in L_n$ be terms that isolate variables $X_1, \ldots, X_n$ respectively over D. Then the term,

$$r_{\{X_1 \backslash t_1, \ldots, X_n \backslash t_n\}},$$

isolates the term r over D, indeed, for every $\mathbf{b} \in D$,

$$r^{[0,n+1]}_{\{X_1 \backslash t_1, \ldots, X_n \backslash t_n\}}(\mathbf{b}) = r^{[0,n+1]}_{\{X_1 \backslash X_1, \ldots, X_n \backslash X_n\}}(\mathbf{b}) = r^{[0,n+1]}(\mathbf{b}),$$

and for every $\mathbf{b} \notin D$,

$$r^{[0,n+1]}_{\{X_1 \backslash t_1, \ldots, X_n \backslash t_n\}}(\mathbf{b}) = r^{[0,n+1]}_{\{X_1 \backslash \top, \ldots, X_n \backslash \top\}}(\mathbf{b}) = \top^{[0,n+1]},$$

noticing that, if r is 1-reproducing, then $r^{[0,n+1]}(\mathbf{n+1}) = \top^{[0,n+1]}$.

The technical details, along with the other relevant cases, follow. The first lemma deals with variables isolation.

Lemma 71 (Variable Isolation)**.** *Let $n \geq 2$ and let $i \in [n]$.*

(i) Let $\mathbf{a} \in [0,1]^n$ be such that $i \in \mathrm{par}(\mathbf{a})$. There exists terms $r_{(i,\mathbf{a})}, s_{(i,\mathbf{a})} \in L_n$ such that, $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ isolate X_i over $D_{(i,\mathbf{a})}$ and $E_{(i,\mathbf{a})}$ respec-

tively, where,

$$D_{(i,\mathbf{a})} = \{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))\},$$
$$E_{(i,\mathbf{a})} = \{\mathbf{b} \in [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))\}.$$

(ii) Let $\tilde{U}$ be a quasipartition of $[0,1]^n$ and let $\mathbf{u} \in \tilde{U}$ be such that $i \in \mathrm{par}(\mathbf{u})$. There exists terms $r_{(i,\mathbf{u})}, s_{(i,\mathbf{u})} \in L_n$ such that, $r_{(i,\mathbf{u})}$ and $s_{(i,\mathbf{u})}$ isolate X_i over $D_{(i,\mathbf{u})}$ and $E_{(i,\mathbf{u})}$ respectively, where,

$$D_{(i,\mathbf{u})} = \{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\},$$
$$E_{(i,\mathbf{u})} = \{\mathbf{b} \in [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\}.$$

(iii) There exists terms $v_i, w_i \in L_n$ such that, v_i and w_i isolate X_i over J_i and K_i respectively, where,

$$J_i = \{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a})), \mathbf{a} \in [0,1]^n, i \notin \mathrm{par}(\mathbf{a})\},$$
$$K_i = \{\mathbf{b} \in [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a})), \mathbf{a} \in [0,1]^n, i \notin \mathrm{par}(\mathbf{a})\}.$$

Proof. (i) If $\mathbf{a}$ is equal to $\mathbf{1}$, then $r_{(i,\mathbf{1})} = s_{(i,\mathbf{1})} = \top$ settles the claim, simply noticing that $\mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{1}))$ if and only if $\mathbf{b} = \mathbf{n} + \mathbf{1}$. Otherwise, we proceed as follows. First, we define the following terms in L_n, for $i \neq j \in [n]$:

$$t_{(1,i)} = \neg\neg X_i \tag{2.20}$$
$$t_{(2,i)} = t_{(1,i)} \to X_i \tag{2.21}$$
$$t_{(3,i,j)} = t_{(1,j)} \to t_{(2,i)} \tag{2.22}$$
$$t_{(4,i,j)} = ((t_{(2,i)} \to t_{(2,j)}) \to t_{(2,j)}) \vee ((t_{(2,j)} \to t_{(2,i)}) \to t_{(2,i)}) \tag{2.23}$$
$$t_{(5,i,j)} = ((X_i \to X_j) \to X_j) \to (t_{(2,i)} \vee t_{(2,j)}) \tag{2.24}$$
$$t_{(6,i,j)} = t_{(1,j)} \wedge (t_{(5,j,i)} \to t_{(3,j,i)}) \tag{2.25}$$

Claim 72. *The following facts hold.*

(i) $t_{(1,i)}$ isolates X_i over $\{\mathbf{b} \mid b_i < 1\}$.

(ii) $t_{(2,i)}$ isolates X_i over $\{\mathbf{b} \mid 1 \leq b_i\}$.

(iii) $t_{(3,i,j)}$ isolates X_i over $\{\mathbf{b} \mid 1 \leq b_i, b_j\}$.

(iv) $t_{(4,i,j)}$ isolates $X_i \vee X_j$ over $\{\mathbf{b} \mid 1 \leq \lfloor b_i \rfloor = \lfloor b_j \rfloor\}$.

(v) $t_{(5,i,j)}$ *isolates* X_i *over* $\{\mathbf{b} \mid 1 \leq \lfloor b_j \rfloor < \lfloor b_i \rfloor\}$.

(vi) $t_{(6,i,j)}$ *isolates* X_j *over* $\{\mathbf{b} \mid 0 \leq \lfloor b_j \rfloor \leq \lfloor b_i \rfloor\}$.

Proof. See Appendix, page 115. □

Now, let $\mathbf{a} \in [0,1]^n$ be such that $i \in \mathrm{par}(\mathbf{a})$ and $j \in \mathrm{par}(\mathbf{a})' = [n] \setminus \mathrm{par}(\mathbf{a})$. We define the terms $r_{(i,\mathbf{a})}, s_{(i,\mathbf{a})} \in L_n$ as follows:

$$r_{(i,\mathbf{a})} = \bigvee_{k \in \mathrm{par}(\mathbf{a})'} t_{(1,k)} \vee \bigvee_{k \in \mathrm{par}(\mathbf{a}) \setminus \{i\}} t_{(3,i,k)} \tag{2.26}$$

$$s_{(i,\mathbf{a})} = t_{(5,i,j)} \vee \bigvee_{k < k' \in \mathrm{par}(\mathbf{a})'} t_{(4,k,k')} \vee \bigvee_{k \in \mathrm{par}(\mathbf{a}) \setminus \{i\}} (t_{(6,j,k)} \to t_{(5,i,j)}) \tag{2.27}$$

stipulating that, if $\mathrm{par}(\mathbf{a}) = \{i\}$, then $t_{(2,i)}$ substitutes $\bigvee_{k \in \mathrm{par}(\mathbf{a}) \setminus \{i\}} t_{(3,i,k)}$ in equation (2.26).

Claim 73. $r_{(i,\mathbf{a})}$ *isolates* X_i *over* $D_{(i,\mathbf{a})}$*, and* $s_{(i,\mathbf{a})}$ *isolates* X_i *over* $E_{(i,\mathbf{a})}$.

Proof. See Appendix, page 117. □

The previous claim concludes the proof of the first part.

(ii) If $\mathbf{u}$ is equal to $\mathbf{1}$, then $r_{(i,\mathbf{1})} = s_{(i,\mathbf{1})} = \top$ settles the claim, simply noticing that $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{1}))$ if and only if $\mathbf{b} = \mathbf{n} + \mathbf{1}$. Otherwise, $\mathrm{par}(\mathbf{u}) = \{i_1, \ldots, i_m\} \subset [n]$, with $i_1 < \cdots < i_m$, and $\mathrm{par}(\mathbf{u})' = [n] \setminus \mathrm{par}(\mathbf{u}) = \{j_1, \ldots, j_{n-m}\}$, with $j_1 < \cdots < j_{n-m}$. Suppose w.l.o.g. that $i = i_1$.

Recall that $\mathrm{sibl}(\mathbf{u})$ is defined as the relative interior of a face of a simplex in the unimodular triangulation U of $[0,1]^n$, so that $\mathrm{sibl}(\mathbf{u})$ has dimension $0 \leq d \leq n$. By induction on d, we show that for every $\mathrm{sibl}(\mathbf{u})$, there exist terms $r_{(i,\mathbf{u})}$ and $s_{(i,\mathbf{u})}$ isolating X_i over $D_{(i,\mathbf{u})}$ and $E_{(i,\mathbf{u})}$ respectively.

For the base case, suppose that $\mathrm{sibl}(\mathbf{u})$ has dimension 0. Then, $\mathrm{sibl}(\mathbf{u}) = \{\mathbf{v}\}$, where $\mathbf{v} = (v_1, \ldots, v_n)$ is a rational vertex of the unimodular triangulation U. The projection of U onto coordinates $j_1, \ldots, j_{n-m}$ is a unimodular triangulation, U', of $[0,1]^{m-n}$, such that $\mathbf{v}' = (v_{j_1}, \ldots, v_{j_{m-n}})$ is a vertex of U'. Fix $t \in L^+_{n-m}$ such that $t^{[0,1]}(\mathbf{c}) =$

1 if and only if $\mathbf{c} = \mathbf{v}'$ or $\mathbf{c} = \mathbf{1}$. Such t exists by Theorem 28 and Corollary 30. Let,

$$r_{(i,\mathbf{u})} = t_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}} \to r_{(i,\mathbf{a})}, \tag{2.28}$$

$$s_{(i,\mathbf{u})} = t_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}} \to s_{(i,\mathbf{a})}, \tag{2.29}$$

where $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ are the terms given by part (i) with $\mathbf{a} = \mathbf{u}$.

Claim 74. *$r_{(i,\mathbf{u})}$ isolates X_i over $D_{(i,\mathbf{u})}$, and $s_{(i,\mathbf{u})}$ isolates X_i over $E_{(i,\mathbf{u})}$.*

Proof. See Appendix, page 119. □

For the inductive step, suppose that $\mathrm{sibl}(\mathbf{u})$ has dimension $d \geq 1$. In this case, there is a d-dimensional face $F = \mathrm{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_{d+1}\}$ of some simplex in U, with $\mathbf{v}_1 = (v_{1,1}, \dots, v_{1,n}), \dots, \mathbf{v}_{d+1} = (v_{d+1,1}, \dots, v_{d+1,n})$ rational vertices, such that $\mathrm{sibl}(\mathbf{u}) = \mathrm{relint}\ F$. Let $F_1, \dots, F_k$ be the faces of F of dimension $\leq d-1$, where $k = \sum_{i=1}^{d} \binom{d+1}{i}$, and let $\mathbf{u}_1, \dots, \mathbf{u}_k$ be respectively the parents in $\tilde{U}$ of $\mathrm{sibl}(\mathbf{u}_1) = \mathrm{relint}\ F_1, \dots, \mathrm{sibl}(\mathbf{u}_k) = \mathrm{relint}\ F_k$. By the induction hypothesis, for every $\mathbf{u}' \in \tilde{U}$ such that $\mathrm{sibl}(\mathbf{u}')$ has dimension less than or equal to $d-1$, there exist terms $r_{(i,\mathbf{u}')}$ and $s_{(i,\mathbf{u}')}$ that isolate X_i over $D_{(i,\mathbf{u}')}$ and $E_{(i,\mathbf{u}')}$ respectively. Hence, there exist terms $r_{(i,\mathbf{u}_1)}, \dots, r_{(i,\mathbf{u}_k)}$, and $s_{(i,\mathbf{u}_1)}, \dots, s_{(i,\mathbf{u}_k)}$ that isolate X_i over, respectively, $D_{(i,\mathbf{u}_1)}, \dots, D_{(i,\mathbf{u}_k)}$, and $E_{(i,\mathbf{u}_1)}, \dots, E_{(i,\mathbf{u}_k)}$.

The projection of U onto coordinates $j_1, \dots, j_{n-m}$ is a unimodular triangulation, U', of $[0,1]^{m-n}$, such that $\mathbf{v}'_1 = (v_{1,j_1}, \dots, v_{1,j_{m-n}}), \dots, \mathbf{v}'_{d+1} = (v_{d+1,j_1}, \dots, v_{d+1,j_{m-n}})$ are vertices of U', and $F' = \mathrm{conv}\{\mathbf{v}'_1, \dots, \mathbf{v}'_{d+1}\}$ is a face of a simplex in U'. Hence, by Theorem 28 and Corollary 30, there exists a term $t \in L^+_{n-m}$ such that $t^{[0,1]}(\mathbf{c}) = 1$ if and only if $\mathbf{c} \in F'$ or $\mathbf{c} = \mathbf{1}$. Let:

$$r_{(i,\mathbf{u})} = \left(\bigwedge_{j=1}^{k} r_{(i,\mathbf{u}_j)} \right) \to \left(t_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}} \to r_{(i,\mathbf{a})} \right) \tag{2.30}$$

$$s_{(i,\mathbf{u})} = \left(\bigwedge_{j=1}^{k} s_{(i,\mathbf{u}_j)} \right) \to \left(t_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}} \to s_{(i,\mathbf{a})} \right), \tag{2.31}$$

where $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ are the terms given by part (i) with $\mathbf{a} = \mathbf{u}$.

Claim 75. *$r_{(i,\mathbf{u})}$ isolates X_i over $D_{(i,\mathbf{u})}$, and $s_{(i,\mathbf{u})}$ isolates X_i over $E_{(i,\mathbf{u})}$.*

Proof. See Appendix, page 120. □

The previous claim concludes the proof of the second part.

(iii) Put,

$$v_i = \neg\neg X_i \tag{2.32}$$

$$w_i = \left(\bigwedge_{\mathbf{a} \in A} (r_{(i,\mathbf{a})} \wedge s_{(i,\mathbf{a})}) \right) \to (\neg\neg X_i \to X_i), \tag{2.33}$$

where $A = \{\mathbf{a} \in [0,1]^n \mid i \in \mathrm{par}(\mathbf{a})\}$, and $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ are as in part (i).

Claim 76. *v_i isolates X_i over J_i, and w_i isolates X_i over K_i.*

Proof. See Appendix, page 121. □

The previous claim concludes the proof of the third part. □

The second lemma deals with terms isolation.

Lemma 77 (Term Isolation)**.** *Let $n \geq 1$.*

(i) Let $t \in L_n^+$, and let $\tilde{U}$ be a quasipartition for t. Then, there exist terms $\hat{t}$ and $\check{t}$ in L_n such that, $\hat{t}$ isolates t over:

$\{\mathbf{b} \notin [1, n{+}1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^n \textit{ or } t^{[0,1]}(\mathbf{u}) < 1\}$,

and $\check{t}$ isolates t over:

$\{\mathbf{b} \in [1, n{+}1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^n \textit{ or } t^{[0,1]}(\mathbf{u}) < 1\}$.

(ii) Let $t = \neg t'$ with $t' \in L_n^+$, and let $\tilde{U}$ be a quasipartition for t. Then, there exists a term $\hat{t}$ in L_n such that, $\hat{t}$ isolates t over

$[1, n{+}1]^n \cup \{\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \mid \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^n \textit{ or } t^{[0,1]}(\mathbf{u}) < 1\}$.

(iii) Let $\tilde{U}$ be a quasipartition of $[0,1]^n$, let $\mathbf{u} \in \tilde{U}$ be such that $0 < |\mathrm{par}(\mathbf{u})| < n$, and let t be a 1-reproducing term in $L_{\mathrm{par}(\mathbf{u})}$. Then, there exist terms $\dot{t}_{\mathbf{u}}$ and $\ddot{t}_{\mathbf{u}}$ in L_n such that, $\dot{t}_{\mathbf{u}}$ isolates t over:

$$\{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\},$$

and $\ddot{t}_{\mathbf{u}}$ isolates t over:

$$\{\mathbf{b} \in [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\}.$$

Proof. (i) Put,

$$\hat{t} = t_{\{1\backslash v_1,\ldots,n\backslash v_n\}}, \tag{2.34}$$

$$\check{t} = t_{\{1\backslash w_1,\ldots,n\backslash w_n\}}. \tag{2.35}$$

Claim 78. *$\hat{t}$ and $\check{t}$ satisfy the statement.*

Proof. See Appendix, page 122. □

The previous claim concludes the proof of the first part.

(ii) Put,

$$\hat{t} = \neg t'_{\{1\backslash v_1,\ldots,n\backslash v_n\}}. \tag{2.36}$$

Claim 79. *$\hat{t}$ satisfies the statement.*

Proof. See Appendix, page 123. □

The previous claim concludes the proof of the second part.

(iii) Let $\mathrm{par}(\mathbf{u}) = \{i_1, \ldots, i_m\}$, and let $t \in L_{\mathrm{par}(\mathbf{u})}$ be 1-reproducing. Put,

$$\dot{t}_{\mathbf{u}} = t_{\{i_1\backslash r_{(i_1,\mathbf{u})},\ldots,i_m\backslash r_{(i_m,\mathbf{u})}\}}, \tag{2.37}$$

$$\ddot{t}_{\mathbf{u}} = t_{\{i_1\backslash s_{(i_1,\mathbf{u})},\ldots,i_m\backslash s_{(i_m,\mathbf{u})}\}}, \tag{2.38}$$

where $r_{(i_1,\mathbf{u})}, \ldots, r_{(i_m,\mathbf{u})}$, and $s_{(i_1,\mathbf{u})}, \ldots, s_{(i_m,\mathbf{u})}$ are the terms in L_n given by part (ii).

Claim 80. *$\dot{t}_{\mathbf{u}}$ and $\ddot{t}_{\mathbf{u}}$ satisfy the statement.*

Proof. See Appendix, page 124. □

The previous claim concludes the proof of the third part. □

The terms isolation mechanism allows to conclude that $F_n \subseteq B_n$.

Lemma 81 (Normal Form)**.** *For every function $f \in F_n$, there exists a term $t \in L_n$ such that $f = t^{[0,n+1]}$.*

Proof. Let $f \in F_n$ be implemented by the system $\tilde{r}$ over the quasipartition $\tilde{U}$, for $r \in L_n$. We distinguish two cases.

First suppose that r is not 1-reproducing. Put,

$$t = \hat{r} \wedge \bigwedge_{\tilde{r}(\mathbf{u})=p} \dot{p}, \tag{2.39}$$

where: $\hat{r}$ is given by the application of Lemma 77(i) to r; $\mathbf{u}$ ranges over all the $\mathbf{u}' \in \tilde{U}$ such that $\mathbf{u}' \notin [0,1)^n$ and $r^{[0,1]}(\mathbf{u}') = 1$; $\dot{p}$ is given by the application of Lemma 77(iii) to p. But then, by Lemma 77(i) and (iii), and Definition 35, we have that $t^{[0,n+1]}$ coincides with the function implemented by the system $\tilde{r}$, that is, $t^{[0,n+1]} = f$.

Next suppose that r is 1-reproducing. By Definition 34, $\tilde{r}(\mathbf{1}) = s$ with $s \in L_n$ 1-reproducing, and, by Definition 35, there exists an auxiliary system $\tilde{s}$, say over quasipartition $\tilde{V}$. Put,

$$t = \left(\hat{r} \wedge \bigwedge_{\tilde{r}(\mathbf{u})=p} \dot{p}\right) \wedge \left(\check{s} \wedge \bigwedge_{\tilde{s}(\mathbf{v})=q} \ddot{q}\right), \tag{2.40}$$

where: $\hat{r}$ is given by the application of Lemma 77(ii) to r; $\mathbf{u}$ ranges over all the $\mathbf{u}' \in \tilde{U}$ such that $\mathbf{u}' \notin [0,1)^n$ and $r^{[0,1]}(\mathbf{u}') = 1$; $\dot{p}$ is given by the application of Lemma 77(iii) to p; $\check{s}$ is given by the application of Lemma 77(ii) to s; $\mathbf{v}$ ranges over all the $\mathbf{v}' \in \tilde{V}$ such that $\mathbf{v}' \notin [0,1)^n$ and $s^{[0,1]}(\mathbf{v}') = 1$; $\ddot{q}$ is given by the application of Lemma 77(iii) to q. But then, by Lemma 77(ii) and (iii), and Definition 35, we have that $t^{[0,n+1]}$ coincides with the function implemented by the system $\tilde{r}$ (with the support of the auxiliary system $\tilde{s}$), that is, $t^{[0,n+1]} = f$. $\square$

Notice that, as an additional benefit, the construction of the term $t \in L_n$ computing a BL-function $f \in F_n$, given by its implementing system, is effective.

In the next section, we recast our result in universal algebraic terms.

2.4 Free BL-Algebra

On the basis of the explicit description of the truthfunctions of the n-variate fragment of Basic logic, in this section we obtain a functional representation of the free n-generated BL-algebra in terms of n-ary BL-functions. This result accounts as the BL-algebraic counterpart (of the constructive version) of the functional representation of the free n-generated MV-algebra in terms of n-ary McNaughton functions.

Let $n \geq 1$. In the introductory discussion of this chapter, we mentioned that the BL-chain $[0, n+1]'$ of Definition 12 generates (as a quasivariety) the variety generated by the class of all n-generated BL-algebras (Theorem 13). Thus, by universal algebra, the free n-generated BL-algebra is isomorphic to the smallest subalgebra of n-ary functions over $[0, n+1]$ that contains exactly the n-ary constant functions 0 and $n+1$, the n-ary projections $x_1, \ldots, x_n$, and is closed under pointwise applications of the operation $\circ^{[0,n+1]'}$ for $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$. This class of functions is exactly the class B_n, that contains the n-ary functions over $[0, n+1]$ of the form $t^{[0,n+1]}$ for some $t \in L_n$. In the previous section, we defined an explicit class of functions F_n, the class of BL-functions (Definition 36), and we proved that F_n coincides with B_n:

Theorem 82. *$B_n = F_n$ for every $n \geq 1$.*

Proof. By Lemma 65 and Lemma 81. □

As an immediate consequence, we obtain the following explicit functional representation of the free n-generated BL-algebra in terms of n-ary BL-functions.

Theorem 83 (Functional Representation)**.** *Let $n \geq 1$. The free n-generated BL-algebra is isomorphic to the algebra,*

$$(F_n, \vee, \wedge, \odot, \rightarrow, \top, \bot),$$

of type $(2, 2, 2, 2, 0, 0)$, where $\bot$ and $\top$ are realized by the constant functions 0 and $n+1$ respectively, and each $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$ is realized by the binary operation $\circ^{[0,n+1]'}$ defined pointwise.

In light of the constructiveness of the normal form lemma given in Section 2.3.4, the previous theorem accounts as the BL-algebraic counterpart of Mundici's constructive version of McNaughton's theorem for MV-algebras [Mun94].

3 Conclusion

In this conclusion we summarize the representation of the free n-generated BL-algebra in terms of n-ary BL-functions, collecting some corollaries (Section 3.1). Moreover, we discuss further developments of the present work, namely, the combinatorial representation of locally finite subvarieties of BL-algebras, the identification of tight finite countermodels to BL-equations, and the construction of deductive interpolants in Basic logic (Section 3.2).

A natural development of the present work, not discussed extensively in this conclusion, is the generalization of De Finetti coherence criterion to Basic logic and BL-algebras, by exploiting the recent works of Kühr and Mundici [KM07] on MV-algebras, and of Aguzzoli et al. [AGM] on Gödel algebras.

3.1 Summary

We summarize our functional representation of the free n-generated BL-algebra in terms of n-ary BL-functions, and we relate our result with other known and unknown functional representation results in subvarieties of BL-algebras.

3.1.1 BL-Functions

In this thesis, we provided a functional representation of the free n-generated BL-algebra in terms of n-ary BL-functions ($n \geq 1$).

An n-ary BL-function f is a *discontinuous* function from $[0, n+1]^n$ to $[0, n+1]$ such that there exists a polyhedral partition C of $[0, n+1]^n$, [1] each polyhedron in C having rational vertices, that *linearizes* f in

[1] A *polyhedral complex* in $[0, n+1]^n$ is a finite set C of polyhedra in $[0, n+1]^n$ such

the following sense: C is such that f coincides with a linear n-variate real-valued polynomial with integer coefficients over the *relative interior* of every polyhedron $P \in C$. Therefore, possibly f has discontinuity points at the boundaries of some polyhedron in C.

In Chapter 2, we refined the previous sketch and we attained a complete description of BL-functions, characterizing the general form of the partitions of $[0, n+1]^n$ that linearize n-ary BL-functions, and listing the dependencies that link the behavior of n-ary BL-functions over different blocks of such partitions. Below, we summarize our definition of BL-functions, adopting the following terminology.

Definition 84 (Cell)**.** *Let $n, k \geq 1$, let $B_1 < \cdots < B_k$ be an ordered partition of $[n]$ into k nonempty blocks, and let $0 \leq j_1 < \cdots < j_k \leq n+1$ be an increasing sequence of k nonnegative integers between 0 and $n+1$. We call,*

$$C^{B_1 < \cdots < B_k}_{j_1 < \cdots < j_k} = \{(b_i)_{i \in [n]} \mid i \in B_l \textit{ implies } \lfloor b_i \rfloor = j_l,\, l \in [k]\} \subseteq [0, n+1]^n,$$

the cell *corresponding to $B_1 < \cdots < B_k$ and $j_1 < \cdots < j_k$.*

Clearly, the set containing all cells of the form $C^{B_1 < \cdots < B_k}_{j_1 < \cdots < j_k}$, with $k \geq 1$, $B_1 < \cdots < B_k$ ranging over all the ordered partitions of $[n]$ into k nonempty blocks, and $0 \leq j_1 < \cdots < j_k \leq n+1$ ranging over all the increasing sequences of k nonnegative integers between 0 and $n+1$, forms a partition of $[0, n+1]^n$ into disjoint blocks. Then, we define BL-functions by specifying their behavior cellwise, as follows.

An n-ary function $f\colon [0, n+1]^n \to [0, n+1]$ is an *n-ary BL-function*, that is, f is in the class F_n of Definition 36, if and only if the behavior of f satisfies the following two constraints. [2]

Constraint 1. The first constraint links the behavior of $f \in F_n$ over cells in $[0, n+1]^n$ that correspond to the same ordered partition $B_1 <$

that each face of each polyhedron in C belongs to C, and any two polyhedra of C intersect in a common face. A polyhedral complex C in $[0, n+1]^n$ forms a *polyhedral partition* of $[0, n+1]^n$, if $[0, n+1]^n$ is the union of all polyhedra in C. By Definition 26, a unimodular triangulation of $[0, n+1]^n$ is a polyhedral partition of $[0, n+1]^n$ such that each polyhedron in C is a unimodular simplex.

[2] The definition of BL-functions given in this section is alternative, but equivalent, to Definition 36.

$\cdots < B_k$ of $[n]$, as follows. Let $B_1 < \cdots < B_k$ be an ordered partition of $[n]$ into $k \geq 1$ nonempty blocks.

Case 1: As regards to cells lying outside $[1, n+1]^n$, the behavior of f over the cell,

$$C^{B_1 < \cdots < B_k}_{j_1 < \cdots < j_k},$$

with $j_1 = 0$, $j_2 = 1, \ldots, j_{k-1} = k-2$, $j_k \in \{k-1, n+1\}$, *determines* the behavior of f over cells of the form,

$$C^{B_1 < \cdots < B_k}_{i_1 < \cdots < i_k},$$

with $i_1 = 0$, and $i_k = n+1$ if and only if $j_k = n+1$, as follows. Let $\mathbf{b} \in C^{B_1 < \cdots < B_k}_{j_1 < \cdots < j_k}$ and $\mathbf{c} \in C^{B_1 < \cdots < B_k}_{i_1 < \cdots < i_k}$, with $j_1 = i_1 = 0$, and $j_k, i_k < n+1$ or $j_k = i_k = n+1$. Then, either $\lfloor f(\mathbf{b}) \rfloor \in \{j_{k-1}, j_k\}$ or $f(\mathbf{b}) \in \{0, n+1\}$ and,

$$f(\mathbf{c}) = \begin{cases} f(\mathbf{b}) - j_{k-1} + i_{k-1} & \text{if } \lfloor f(\mathbf{b}) \rfloor = j_{k-1} \\ f(\mathbf{b}) - j_k + i_k & \text{if } \lfloor f(\mathbf{b}) \rfloor = j_k \\ f(\mathbf{b}) & \text{otherwise} \end{cases}$$

Case 2: As regards to cells lying inside $[1, n+1]^n$, the behavior of f over the cell,

$$C^{B_1 < \cdots < B_k}_{j_1 < \cdots < j_k},$$

with $j_1 = 1$, $j_2 = 2, \ldots, j_{k-1} = k-1$, $j_k \in \{k, n+1\}$, *determines* the behavior of f over cells of the form,

$$C^{B_1 < \cdots < B_k}_{i_1 < \cdots < i_k},$$

with $i_1 \geq 1$, and $i_k = n+1$ if and only if $j_k = n+1$, as follows. Let $\mathbf{b} \in C^{B_1 < \cdots < B_k}_{j_1 < \cdots < j_k}$ and $\mathbf{c} \in C^{B_1 < \cdots < B_k}_{i_1 < \cdots < i_k}$, with $j_1 = 1 \leq i_1$, and $j_k, i_k < n+1$ or $j_k = i_k = n+1$. Then, either $\lfloor f(\mathbf{b}) \rfloor \in \{j_{k-1}, j_k\}$ or $f(\mathbf{b}) \in \{0, n+1\}$ and,

$$f(\mathbf{c}) = \begin{cases} f(\mathbf{b}) - j_{k-1} + i_{k-1} & \text{if } \lfloor f(\mathbf{b}) \rfloor = j_{k-1} \\ f(\mathbf{b}) - j_k + i_k & \text{if } \lfloor f(\mathbf{b}) \rfloor = j_k \\ f(\mathbf{b}) & \text{otherwise} \end{cases}$$

Therefore, by Constraint 1, it is possible to describe the behavior of f over $[0, n+1]^n$ by focusing only on cells outside $[1, n+1]^n$ of the form,

$$C_{0<1<\cdots<k-1}^{B_1<B_2<\cdots<B_k} \text{ or } C_{0<1<\cdots<k-2<n+1}^{B_1<B_2<\cdots<B_{k-1}<B_k}, \tag{3.1}$$

and on cells inside $[1, n+1]^n$ of the form,

$$C_{1<2<\cdots<k}^{B_1<B_2<\cdots<B_k} \text{ or } C_{1<2<\cdots<k-1<n+1}^{B_1<B_2<\cdots<B_{k-1}<B_k}, \tag{3.2}$$

where $B_1 < B_2 < \cdots < B_{k-1} < B_k$ ranges over all the ordered partitions of $[n]$ into k nonempty blocks, for all $k \geq 1$. Cells of the form (3.1) or (3.2) are said *unavoidable*. [3]

Constraint 2. The second constraint links the behavior of $f \in F_n$ over pairs of unavoidable cells, outside or inside $[1, n+1]^n$, such that the second cell *refines* the first, that is, either pairs of cells of the form ($k \geq 0$),

$$(C_{0<\cdots<k-1}^{B_1<\cdots<B_k \cup B_{k+1}}, C_{0<\cdots<k-1<j_k}^{B_1<\cdots<B_k<B_{k+1}}),$$

with $j_k \in \{k, n+1\}$, or pairs of cells of the form ($k \geq 0$),

$$(C_{1<\cdots<k}^{B_1<\cdots<B_k \cup B_{k+1}}, C_{1<\cdots<k<j_k}^{B_1<\cdots<B_k<B_{k+1}}),$$

with $j_k \in \{k+1, n+1\}$. In this case, the behavior of f over the second cell is *partially* determined by the behavior of f over the first cell, as follows (we proceed by induction on $k \geq 0$).

Base Case ($k = 0$). The behavior of $f \in F_n$ over cells $C_0^{[n]}$, $C_1^{[n]}$, and $C_{n+1}^{[n]} = \{\mathbf{n+1}\}$ is defined as follows. There exist a pair of n-ary McNaughton function $g, h\colon [0,1]^{[n]} \to [0,1]$, such that either $g(\mathbf{1}) = h(\mathbf{1}) = 1$, or $g(\mathbf{1}) = 0$ and h is the constant 0 function, and a pair of unimodular triangulations U and V of $[0,1]^{[n]}$ that linearize g and h respectively, such that, for every $\mathbf{b} = (b_i)_{i \in [n]} \in$

[3] Figure 3.2 depicts unavoidable cells in the case $n = 3$.

$C_0^{[n]}$ and every $\mathbf{c} = (c_i)_{i\in[n]} \in C_1^{[n]}$,

$$f(\mathbf{b}) = \begin{cases} g((b_i - \lfloor b_i \rfloor)_{i\in[n]}) + 0 & \text{if } g((b_i - \lfloor b_i \rfloor)_{i\in[n]}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

$$f(\mathbf{c}) = \begin{cases} 0 & \text{if } h = 0 \\ h((c_i - \lfloor c_i \rfloor)_{i\in[n]}) + 1 & \text{if } h((c_i - \lfloor c_i \rfloor)_{i\in[n]}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

and,

$$f(\mathbf{n+1}) = \begin{cases} n+1 & \text{if } g(\mathbf{1}) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Inductive Case ($k \geq 1$). Let $1 \leq k \leq n$. We define the behavior of $f \in F_n$ over the unavoidable cells,

$$C^{B_1<\cdots<B_k<B_{k+1}}_{0<\cdots<k-1<k} \text{ and } C^{B_1<\cdots<B_k<B_{k+1}}_{0<\cdots<k-1<n+1},$$

refining the unavoidable cell $C^{B_1<\cdots<B_k\cup B_{k+1}}_{0<\cdots<k-1}$, and over the unavoidable cells,

$$C^{B_1<\cdots<B_k<B_{k+1}}_{1<\cdots<k<k+1} \text{ and } C^{B_1<\cdots<B_k<B_{k+1}}_{1<\cdots<k<n+1},$$

refining the unavoidable cell $C^{B_1<\cdots<B_k\cup B_{k+1}}_{1<\cdots<k}$.

As for the first couple of cells, outside $[1, n+1]^n$, f satisfies the following constraint. By the induction hypothesis, there exist a $|B_k \cup B_{k+1}|$-ary McNaughton function g, and a unimodular triangulation U of $[0,1]^{B_k\cup B_{k+1}}$ linearizing g, such that, for every $\mathbf{b} \in C^{B_1<\cdots<B_k\cup B_{k+1}}_{0<\cdots<k-1}$,

$$f(\mathbf{b}) = \begin{cases} g((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) + (k-1) & \text{if } g((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

For every unimodular simplex $S \in U$ such that $g|_S = 1$ and,

$$\text{relint } S \cap \{(a_i)_{i\in B_k\cup B_{k+1}} \in [0,1]^{B_k\cup B_{k+1}} \mid a_i = 1 \text{ if } i \in B_{k+1}\} \neq \emptyset,$$

there exists a $|B_{k+1}|$-ary McNaughton function $g_S\colon [0,1]^{B_{k+1}} \to [0,1]$, and a unimodular triangulation U_S of $[0,1]^{B_{k+1}}$ linearizing g_S, such that, for every $\mathbf{b} = (b_i)_{i\in[n]} \in C^{B_1<\cdots<B_k<B_{k+1}}_{0<\cdots<k-1<k}$,

$$f(\mathbf{b}) = \begin{cases} g((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) + (k-1) & \text{if } g((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) < 1 \\ g'((b_i - \lfloor b_i \rfloor)_{i\in B_{k+1}}) + k & \text{if } g'((b_i - \lfloor b_i \rfloor)_{i\in B_{k+1}}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

and for every $\mathbf{b} = (b_i)_{i\in[n]} \in C^{B_1<\cdots<B_k<B_{k+1}}_{0<\cdots<k-1<n+1}$,

$$f(\mathbf{b}) = \begin{cases} g((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) + (k-1) & \text{if } g((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

As for the second couple of cells, that is, inside $[1, n+1]^n$, f satisfies the following constraint. By the induction hypothesis, there exist a $|B_k \cup B_{k+1}|$-ary McNaughton function h, such that $h(\mathbf{1}) = 1$ of h is the constant 0 function, and a unimodular triangulation V of $[0,1]^{B_k\cup B_{k+1}}$ linearizing h, such that, for every $\mathbf{b} \in C^{B_1<\cdots<B_k\cup B_{k+1}}_{1<\cdots<k}$,

$$f(\mathbf{b}) = \begin{cases} 0 & \text{if } h = 0 \\ h((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) + k & \text{if } h((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

For every unimodular simplex $T \in V$ such that $h|_T = 1$ and,

$$\operatorname{relint} T \cap \{(a_i)_{i\in B_k\cup B_{k+1}} \in [0,1]^{B_k\cup B_{k+1}} \mid a_i = 1 \text{ if } i \in B_{k+1}\} \neq \emptyset,$$

there exists a $|B_{k+1}|$-ary McNaughton function $h_T\colon [0,1]^{B_{k+1}} \to [0,1]$, and a unimodular triangulation V_T of $[0,1]^{B_{k+1}}$ linearizing h_T, such that, for every $\mathbf{b} = (b_i)_{i\in[n]} \in C^{B_1<\cdots<B_k<B_{k+1}}_{1<\cdots<k<k+1}$,

$$f(\mathbf{b}) = \begin{cases} 0 & \text{if } h = 0 \\ h((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) + k & \text{if } h((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) < 1 \\ h'((b_i - \lfloor b_i \rfloor)_{i\in B_{k+1}}) + (k+1) & \text{if } h'((b_i - \lfloor b_i \rfloor)_{i\in B_{k+1}}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

and for every $\mathbf{b} = (b_i)_{i\in[n]} \in C^{B_1<\cdots<B_k<B_{k+1}}_{1<\cdots<k<n+1}$,

$$f(\mathbf{b}) = \begin{cases} 0 & \text{if } h = 0 \\ h((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) + k & \text{if } h((b_i - \lfloor b_i \rfloor)_{i\in B_k\cup B_{k+1}}) < 1 \\ n+1 & \text{otherwise} \end{cases}$$

The above cellwise definition of $f \in F_n$ is clearly explicit in terms of Definition 15, because McNaughton functions are explicit by Fact 19. Moreover, the free n-generated BL-algebra is isomorphic to the algebra of n-ary BL-functions, equipped with pointwise defined operations $\odot^{[0,n+1]}$ and $\to^{[0,n+1]}$ in Definition 3, indeed we proved that,

$$F_n = B_n,$$

that is: for every term $t \in L_n$, the term function $t^{[0,n+1]}$, corresponding to t in $[0, n+1]$, is a BL-function (Lemma 65); and conversely, every n-ary BL-function f is equal to the term function $t^{[0,n+1]}$ for some term $t \in L_n$ (Lemma 81). As an additional benefit, given a suitable encoding of $f \in F_n$, say, a system implementing f, Lemma 81 gives an effective constrution of the term $t \in L_n$ such that $f = t^{[0,n+1]}$.

3.1.2 SBL-Functions

In this section, we recover from our functional representation result analogous well known representation results of free algebras in subvarieties of BL-algebras, namely MV-algebras and Gödel algebras. The same method furnishes new functional representations of free algebras in subvarieties of BL-algebras, such as, for instance, the functional representation of the free n-generated SBL-algebra in terms of n-ary SBL-functions.

As a first exercise, we recover from Theorem 83 the well known explicit representation of the free n-generated MV-algebra in terms of McNaughton functions.

Definition 85 (MV-Algebra)**.** *An* MV-algebra *is an* involutive *BL-algebra, that is, a BL-algebra* $\mathbf{A} = (A, \vee, \wedge, \odot, \to, \top, \bot)$ *such that* $(a \to \bot) \to \bot = a$ *holds for every* $a \in A$.

Corollary 86. *The free n-generated MV-algebra is isomorphic to the algebra,*

$$(F_n|_{[0,1)\cup\{n+1\}}, \vee, \wedge, \odot, \to, \top, \bot),$$

of type $(2, 2, 2, 2, 0, 0)$*, where,*

$$F_n|_{[0,1)\cup\{n+1\}} = \{f|_{([0,1)\cup\{n+1\})^n} \mid f \in F_n\},$$

$\bot$ *and* $\top$ *are realized respectively by the constant functions* 0 *and* $n+1$*, and each* $\circ \in \{\vee, \wedge, \odot, \to\}$ *is realized by the restriction of the binary operation* $\circ^{[0,n+1]'}$ *of Definition 12 to* $([0,1) \cup \{n+1\})^2$ *defined pointwise.*

Proof (Sketch). By Theorem 25, the free n-generated MV-algebra is the algebra of n-ary McNaughton functions M_n with $\bot$ and $\top$ realized by the constant functions 0 and 1 respectively, and the binary operation $\circ \in \{\vee, \wedge, \odot, \to\}$ realized by $\circ^{[0,1]}$ defined pointwise. For every $\mathbf{b} = (b_1, \ldots, b_n) \in ([0,1) \cup \{n+1\})^n$, let $\mathbf{b}' = (b'_1, \ldots, b'_n) \in [0,1]^n$ be such that $b'_i = b_i$ if $b_i < n+1$ and $b'_i = 1$ otherwise, $i \in [n]$. For every $f \in F_n|_{[0,1)\cup\{n+1\}}$, we let $h(f)$ be the n-ary McNaughton function $g \in M_n$ such that,

$$g(\mathbf{b}') = \begin{cases} f(\mathbf{b}) & \text{if } f(\mathbf{b}) < n+1 \\ 1 & \text{otherwise} \end{cases}$$

It is possible to check that h is an isomorphism of MV-algebras. □

As a second exercise, we recover from Theorem 83 an explicit representation of the n-generated algebra in the variety of Gödel algebras.

Definition 87 (Gödel Algebra)**.** *A* Gödel algebra *is an* idempotent *BL-algebra, that is, a BL-algebra* $\mathbf{A} = (A, \vee, \wedge, \odot, \to, \top, \bot)$ *such that* $a \odot a = a$ *holds for every* $a \in A$*.*

It is well known that the variety of Gödel algebras is *locally finite*, that is, the free n-generated Gödel algebra is finite for every $n \geq 1$. Thus, instead of representing the free n-generated Gödel algebra in terms of suitable n-ary functions over $[0,1]^n$ [Ger00], it is natural to embrace a combinatorial representation of the free n-generated Gödel algebra, in terms of maximal antichains in suitable posets [AG08].

Corollary 88. *The free n-generated Gödel algebra is isomorphic to the algebra,*

$$(F_n|_{\{0,1,\dots,n+1\}}, \vee, \wedge, \odot, \to, \top, \bot),$$

of type $(2,2,2,2,0,0)$*, where,*

$$F_n|_{\{0,1,\dots,n+1\}} = \{f|_{\{0,1,\dots,n+1\}^n} \mid f \in F_n\},$$

$\bot$ *and* $\top$ *are realized respectively by the constant functions* 0 *and* $n+1$*, and each* $\circ \in \{\vee, \wedge, \odot, \to\}$ *is realized by the restriction of the binary operation* $\circ^{[0,n+1]'}$ *of Definition 12 to* $\{0,1,\dots,n+1\}^2$ *defined pointwise.*

Proof (Sketch). As a sample case, we pick $n = 2$ (the general case $n \geq 1$ is similar). As shown in [AG08], the free 2-generated Gödel algebra is isomorphic to the algebra of maximal antichains in the poset P over $\{0 = 1 = 2, 0 = 1 = 2 < 3, 0 = 1, 0 = 1 < 2, 0 = 1 < 2 < 3, 0 = 2, 0 = 2 < 1, 0 = 2 < 1 < 3, 0, 0 < 1 = 2, 0 < 1 = 2 < 3, 0 < 1, 0 < 1 < 2, 0 < 1 < 2 < 3, 0 < 2, 0 < 2 < 1, 0 < 2 < 1 < 3\}$ given by the cover graph in Figure 3.1,

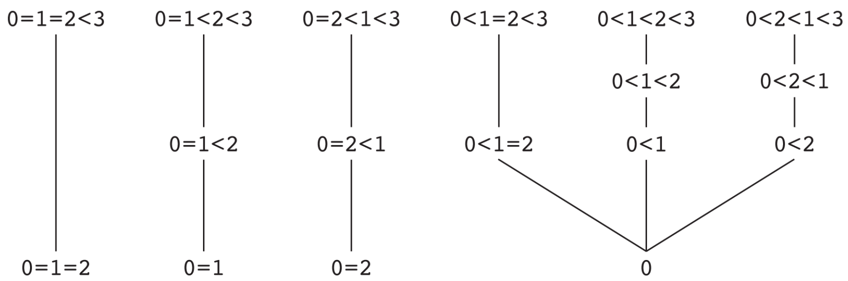


Figure 3.1: Poset representation of the free 2-generated Gödel algebra.

where $\bot$ is realized by the antichain $\{0 = 1 = 2, 0 = 1, 0 = 2, 0\}$, $\top$ is realized by the antichain $\{0 = 1 = 2 < 3, 0 = 1 < 2 < 3, 0 = 2 < 1 < 3, 0 < 1 = 2 < 3, 0 < 1 < 2 < 3, 0 < 2 < 1 < 3\}$, and the operations are defined chainwise, as follows. Let A and A' be maximal antichains in P, let $C \subseteq P$ be a maximal chain in P, having $c \in C$ as maximal element, let $a = A \cap C$, and let $a' = A' \cap C$. Then, $(A \odot A') \cap C = a$ if $a \leq a'$, and $(A \odot A') \cap C = a'$ otherwise; and, $(A \to A') \cap C = c$ if $a \leq a'$, and $(A \odot A') \cap C = a'$ otherwise.

We define a map h from $F_2|_{\{0,1,2,3\}}$ to the set of maximal antichains in P. Let $f \in F_2|_{\{0,1,2,3\}}$. Then, $h(f)$ is the maximal antichain in P

uniquely determined by the following stipulations. Let $\mathbf{b} = (b_1, b_2) \in \{0,1,2,3\}^2$, so that there exists a unique choice of a pair $(\lhd_1, \lhd_2) \in \{<, =\}^2$ and of a lexicographically minimal pair $(i,j) \in \{(1,2),(2,1)\}$ satisfying,

$$0 \lhd_1 b_i \lhd_2 b_j.$$

We let $C(\mathbf{b})$ denote the maximal chain in P having as maximal element $0 \lhd_1 i \lhd_2 j < 3$. Clearly, $\{C(\mathbf{b}) \mid \mathbf{b} \in \{0,1,2,3\}^2\}$ contains exactly the maximal chains in P. Note that $f(\mathbf{b}) \in \{0, b_1, b_2, 3\}$. We let $h(f)$ be the unique maximal antichain in P such that, for every $\mathbf{b} \in \{0,1,2,3\}^2$,

$$h(f) \cap C(\mathbf{b}) = \begin{cases} 0 \lhd_1 i \lhd_2 j \lhd_3 3 & \text{if } f(\mathbf{b}) = 3 \\ 0 \lhd_1 i \lhd_2 j & \text{if } f(\mathbf{b}) = b_j \\ 0 \lhd_1 i & \text{if } f(\mathbf{b}) = 0 = b_i < b_j \\ 0 & \text{otherwise} \end{cases}$$

It is possible to check that h is an isomorphism of Gödel algebras. $\square$

As a last corollary of Theorem 83, we obtain an explicit functional representation of the free n-generated SBL-algebra.

Definition 89 (SBL-Algebra)**.** *An* SBL-algebra *is a BL-algebra* $\mathbf{A} = (A, \vee, \wedge, \odot, \to, \top, \bot)$ *such that* $a \wedge (a \to \bot) = \bot$ *holds for every* $a \in A$*.*

The variety of SBL-algebras is not locally finite, hence we aim at a functional representation of the free n-generated SBL-algebra, in terms of n-ary finite piecewise linear discontinuous functions over a certain subset of $[0, n+1]^n$.

Definition 90. *Let* $n \geq 1$*. The algebra,*

$$\{0\} \cup [1, n+1] = (\{0\} \cup [1, n+1], \vee, \wedge, \odot, \to, \top, \bot),$$

is the algebra of type $(2,2,2,2,0,0)$*, where* $\bot$ *is realized by* 0*,* $\top$ *is realized by* $n+1$*, and every* $\circ \in \{\vee, \wedge, \odot, \to\}$ *is realized by the restriction of the binary operation* $\circ^{[0,n+1]'}$ *in Definition 12 to* $(\{0\} \cup [1, n+1])^2$*. For every* $\circ \in \{\vee, \wedge, \odot, \to, \top, \bot\}$*, we let* $\circ^{\{0\} \cup [1,n+1]}$ *denote the realization of* $\circ$ *in* $\{0\} \cup [1, n+1]$*.*

It is easy to check that the algebra $\{0\} \cup [1, n+1]$ is an SBL-algebra. Moreover, the algebra $\{0\} \cup [1, n+1]$ singly generates as a quasivariety the variety generated by the class of all n-generated SBL-algebras [AM03].

Theorem 91 (Aglianó and Montagna)**.** *Let $n \geq 1$. The algebra $\{0\} \cup [1, n+1]$ generates as a quasivariety the variety generated by the class of all n-generated SBL-algebras.*

Hence, by universal algebra, the free n-generated SBL-algebra is isomorphic to the smallest subalgebra of n-ary functions over $\{0\}\cup[1, n+1]$ that contains the constant functions 0 and $n+1$, the projection functions $x_1, \ldots, x_n$, and is closed under pointwise application of the basic operation $\circ^{\{0\}\cup[1,n+1]}$ of the generic algebra $\{0\} \cup [1, n+1]$, for every $\circ \in \{\vee, \wedge, \odot, \to\}$. As a direct consequence of Theorem 83, we can improve the previous implicit characterization via the following explicit functional respresentation of the free n-generated SBL-algebra.

Corollary 92. *The free n-generated SBL-algebra is isomorphic to the algebra,*

$$(F_n|_{\{0\}\cup[1,n+1]}, \vee, \wedge, \odot, \to, \top, \bot),$$

of type $(2, 2, 2, 2, 0, 0)$, where $F_n|_{\{0\}\cup[1,n+1]} = \{f|_{(\{0\}\cup[1,n+1])^2} \mid f \in F_n\}$, $\bot$ and $\top$ are realized respectively by the constant functions 0 and $n+1$, and each $\circ \in \{\vee, \wedge, \odot, \to\}$ is realized by the restriction of the binary operation $\circ^{[0,n+1]'}$ of Definition 12 to $\{0\} \cup [1, n+1]$ defined pointwise.

Proof (Sketch). Immediate by Theorem 83. □

In the next section, we discuss some natural developments of the work presented in this thesis.

3.2 Future Work

In this section, we discuss some ideas for future work on BL-algebras, namely, the combinatorial representation of locally finite subvarieties of BL-algebras (Section 3.2.1), the identification of tight finite countermodels to BL-equations (Section 3.2.2), and the construction of deductive interpolants in Basic logic (Section 3.2.3).

3.2.1 Locally Finite Subvarieties

In the previous section, we considered a locally finite subvariety of BL-algebras, namely, the well known variety of Gödel algebras. In this section, we introduce a natural locally finite subvariety of BL-algebras, which we call *BL_k-algebras*. BL_k-algebras form the BL-algebraic counterpart of the variety of Grigolia MV_k-algebras [Gri73, CDM99], which are the MV-algebras singly generated by the MV-chain,

$$\mathbf{C}_k = (\{0, 1/k, \ldots, (k-1)/k, 1\}, \odot, \to, \bot), \tag{3.3}$$

defined as the subalgebra of the MV-algebra $[0, 1]$ of Definition 16 generated by the rational numbers of denominator k between 0 and 1. These algebras constitute the natural search space for finite countermodels to BL-quasiequations.

Notation 93. *Let $\mathbf{A}$ be a BL-algebra. We shall adopt the following abbreviations: $\neg a$ stands for the term operation $a \to \bot$; $a \oplus a'$ stands for the term operation $(a \to (a \odot a')) \to a'$; for every $n \geq 0$, a^n stands for the term operation $a \odot \cdots \odot a$, with a occurring n times; for every $n \geq 0$, na stands for the term operation $a \oplus \cdots \oplus a$, with a occurring n times. We stipulate that $a^0 = \top$, $0a = \bot$, and $\odot$ applies before $\oplus$.*

Definition 94 (BL_k-Algebra)**.** *Let $k \geq 2$. A* BL_k-algebra *is a BL-algebra $\mathbf{A} = (A, \vee, \wedge, \odot, \to, \top, \bot)$ satisfying, for every $a \in A$,*

$$ka = (k+1)a,$$

and for every integer $h \geq 2$ that does not divide k,

$$(ha^{h-1})^{k+1} = (k+1)a^h.$$

Definition 95. *Let $n \geq 1$ and let $k \geq 2$. Let,*

$$A_{n,k} = \{a/k \mid 0 \leq a \leq k(n+1)\} \subseteq \mathbb{Q}.$$

The algebra,

$$\mathbf{A}_{n,k} = (A_{n,k}, \vee, \wedge, \odot, \to, \top, \bot),$$

is the algebra of type $(2, 2, 2, 2, 0, 0)$, where $\bot$ is realized by 0, $\top$ is realized by $n+1$, and every $\circ \in \{\vee, \wedge, \odot, \to\}$ is realized by the restriction of the binary operation $\circ^{[0,n+1]'}$ in Definition 12 to $A^2_{n,k}$. For every $\circ \in \{\vee, \wedge, \odot, \to, \top, \bot\}$, we let $\circ^{\mathbf{A}_{n,k}}$ denote the realization of $\circ$ in $\mathbf{A}_{n,k}$.

Theorem 96. *Let $n \geq 1$ and let $k \geq 2$. The algebra $\mathbf{A}_{n,k}$ generates as a quasivariety the variety generated by the class of all n-generated BL$_k$-algebras.*

Proof (Sketch). Immediate by Theorem 13. □

Again, by universal algebra, the free n-generated BL$_k$-algebra, in symbols $\mathbf{BL}_{k,n}$, is isomorphic to the smallest subalgebra of n-ary functions over $A_{n,k}$ that contains the constant functions 0 and $n+1$, the projection functions $x_1, \ldots, x_n$, and is closed under pointwise application of the basic operation $\circ^{\mathbf{A}_{n,k}}$ of the generic algebra $\mathbf{A}_{n,k}$, for every $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$. As a direct consequence of Theorem 83, we can improve the previous implicit characterization via the following explicit functional respresentation of $\mathbf{BL}_{k,n}$.

Corollary 97. *Let $n \geq 1$ and let $k \geq 2$. The free n-generated BL$_k$-algebra $\mathbf{BL}_{k,n}$ is isomorphic to the algebra,*

$$(F_n|_{A_{n,k}}, \vee, \wedge, \odot, \rightarrow, \top, \bot),$$

of type $(2,2,2,2,0,0)$, where $F_n|_{A_{n,k}} = \{f|_{A^n_{n,k}} \mid f \in F_n\}$, $\bot$ and $\top$ are realized respectively by the constant functions 0 and $n+1$, and each $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$ is realized by the restriction of the binary operation $\circ^{[0,n+1]'}$ of Definition 12 to $A^n_{n,k}$ defined pointwise.

Proof (Sketch). Immediate by Theorem 83. □

Thus, since $\mathbf{BL}_{k,n}$ is finite, the variety of BL$_k$-algebras is locally finite. It is therefore possible, and natural, to provide a combinatorial representation of $\mathbf{BL}_{k,n}$ in terms of (maximal antichains in suitable finite) posets, in the spirit of [dNL03, JM, AG08].

For a fixed $k \geq 2$, we shall consider finite posets having as domain the multiset,

$$K = \{\mathbf{d} \mid d \geq 1, d \text{ divisor of } k\}.$$

We define the following operations over such posets. Let P and Q be posets with cover graph $\mathbf{P}$ and $\mathbf{Q}$, respectively, with $\mathbf{Q} \neq \mathbf{1}$. The operation $\mathbf{P} + \mathbf{Q}$ returns the cover graph given by juxtaposition of $\mathbf{P}$ and $\mathbf{Q}$. For $n \geq 1$, we let $n \cdot \mathbf{P} = \sum_{i=1}^{n} \mathbf{P}$. If both $\mathbf{P}$ and $\mathbf{Q}$ have depth 0, and $\mathbf{Q}$ is a subgraph of $\mathbf{P}$, then we let $\mathbf{P} - \mathbf{Q}$ denote the cover graph $\mathbf{R}$ such that $\mathbf{P} + \mathbf{R} = \mathbf{Q}$. Let $l_1, \ldots, l_M$ be the leaves of $\mathbf{P}$, and let $r_1, \ldots, r_m$ be the roots of $\mathbf{Q}$. The operation,

$$\begin{pmatrix} \mathbf{Q} \\ \times \\ \mathbf{P} \end{pmatrix},$$

returns the cover graph given by taking a copy of $\mathbf{P}$ along with M copies $\mathbf{Q}_1, \ldots, \mathbf{Q}_M$ of $\mathbf{Q}$, the jth copy having roots $r_{j,1}, \ldots, r_{j,m}$ for all $j \in [M]$, and adding edges $(l_1, r_{1,1})$, $(l_1, r_{1,2})$, $\ldots$, $(l_M, r_{M,m-1})$, $(l_M, r_{M,m})$. We complete the definition by stipulating that,

$$\begin{pmatrix} \mathbf{1} \\ \times \\ \mathbf{P} \end{pmatrix} = \mathbf{P}.$$

Below we describe a recursive algorithm that, for every $k \geq 2$ and every $n \geq 1$, constructs (the underlying poset of) $\mathbf{BL}_{k,n}$ and computes its cardinality. We remark that the underlying order structure of any BL-algebra, in particular of $\mathbf{BL}_{k,n}$, is a distributive lattice, hence the ability of counting its elements is not completely trivial, since the general problem of determining the cardinality of the free n-generated distributive lattice is open since Dedekind posed in 1897 [Ded97]. An explicit recursive construction of $\mathbf{BL}_{k,n}$ is motivated by the problem of finding finite countermodels to BL-quasiequations (see Section 3.2.2), and by the problem of (approximating the) classification of locally finite subvarieties of BL-algebras.

For every integer $d \geq 1$, let $\mathbf{C}_d$ be the subalgebra of the MV-chain $[0,1]$ with subdomain $\{0, 1/d, \ldots, 1\}$. It is well known that the free n-generated MV$_k$-algebra, in symbols $\mathbf{MV}_{k,n}$, is isomorphic to the direct product of a finite number of subalgebras of the MV-chain $\mathbf{C}_k$ in (3.3), namely,

$$\mathbf{MV}_{k,n} = \prod_{d|k} \mathbf{C}_d^{\mathbf{d}_n},$$

where $\mathbf{d}_{k,n}$ is the multiplicity of factor $\mathbf{C}_d$ in the direct product representation of $\mathbf{MV}_{k,n}$, as computed in [CDM99, Theorem 6.8.1], that is, letting D be the set of coatoms in the lattice of divisors of d,

$$\mathbf{d}_n = (d+1)^n + \sum_{\emptyset \neq X \subseteq D} (-1)^{|X|} (\gcd(X) + 1)^n.$$

Adopting the previous notation, for every $k \geq 2$ and $n \geq 1$, we represent $\mathbf{MV}_{k,n}$ by the poset with cover graph,

$$\sum_{d|k} \mathbf{d}_n \cdot \mathbf{d}$$

We stipulate that for every $k \geq 2$, the poset representation of $\mathbf{MV}_{k,0}$ is the poset with cover graph,

$$\mathbf{1}$$

In the sequel, with a slight abuse of notation, we write $\mathbf{MV}_{k,n}$ for the cover graph of $\mathbf{MV}_{k,n}$.

Example 98. *Let $k = 4$. The poset representation of $\mathbf{MV}_{4,1}$ is given by the cover graph,*

$$\mathbf{1} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{4} \quad \mathbf{4}$$

and the poset representation of $\mathbf{MV}_{4,2}$ is given by the cover graph,

$$\mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{2} \quad \mathbf{2} \quad \mathbf{2} \quad \mathbf{2} \quad 16 \cdot \mathbf{4}$$

where $16 = \mathbf{4}_2$.

Fix $k \geq 2$, let P be any poset over the multiset K defined above, and let A be a maximal antichain in P. A map l from A to $\{0, \ldots, k\}$ is a maximal *labelled* antichain in P if, for every $\mathbf{d} \in A$,

$$l(\mathbf{d}) \in \begin{cases} \{0, \ldots, d\} & \text{if } \mathbf{d} \text{ is maximal in } P \\ \{0, \ldots, d-1\} & \text{otherwise} \end{cases}$$

By induction on $n \geq 1$, we shall compute a poset $S_{k,n}$ over K, as follows. Stipulate that $P_{k,0}$ is the poset with cover graph $\mathbf{1}$, and that the number of maximal labelled antichains in $P_{k,0}$, in symbols $|A(P_{k,0})|$, is equal to 1.

For the base case, let $n = 1$. The poset $S_{k,1}$ is defined as follows. Let $P_{k,1}$ be the poset with cover graph $\mathbf{P}_{k,1} = \mathbf{MV}_{k,1} - \mathbf{MV}_{k,0}$, that is,

$$\mathbf{P}_{k,1} = (\mathbf{1}_1 - 1) \cdot \mathbf{1} + \sum_{d|k, d \geq 2} \mathbf{d}_1 \cdot \mathbf{d}$$

Let $|A(P_{k,1})|$ denote the number of maximal labelled antichains in $P_{k,1}$. We have,

$$|A(P_{k,1})| = (1+1)^{\mathbf{1}_1 - 1} \cdot \prod_{d|k, d \geq 2} (d+1)^{\mathbf{d}_1}.$$

We let $S_{k,1}$ be the poset with cover graph,

$$\mathbf{S}_{k,1} = \begin{pmatrix} \mathbf{P}_{k,1} \\ \times \\ \mathbf{1} \end{pmatrix} + \mathbf{P}_{k,1}$$

where $|A(S_{k,1})| = (1 + |A(P_{k,1})|) \cdot |A(P_{k,1})|$. The base case is settled.

Example 99. *Let $k = 4$. Then, $P_{4,1}$ is the poset with cover graph,*

$$\mathbf{1} \quad \mathbf{2} \quad \mathbf{4} \quad \mathbf{4}$$

where $|A(P_{k,1})| = 2 \cdot 3 \cdot 5^2 = 150$. So, $S_{4,1}$ is the poset with cover graph,

$$\begin{pmatrix} \mathbf{1} \quad \mathbf{2} \quad \mathbf{4} \quad \mathbf{4} \\ \times \\ \mathbf{1} \end{pmatrix} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{4} \quad \mathbf{4}$$

where $|A(S_{4,1})| = 150 \cdot (1 + 150) = 22650$.

For the inductive step, let $n \geq 2$ and suppose that $P_{k,0}, \ldots, P_{k,n-1}$, along with $|A(P_{k,0})|, \ldots, |A(P_{k,n-1})|$, have already been computed. Notice that, letting,

$$\mathbf{d}'_n = d^n + \sum_{\emptyset \neq X \subseteq D} (-1)^{|X|} \gcd(X)^n,$$

where D is the set of coatoms in the lattice of divisors of d, we have that, for every $1 \leq i \leq n$,

$$\mathbf{MV}_{k,i} + \sum_{j=1}^{i} (-1)^j \binom{n}{i-j} \cdot \mathbf{MV}_{k,i-j} = \sum_{d|k} \mathbf{d}'_i \cdot \mathbf{d}.$$

We let $P_{k,n}$ be the poset with cover graph,

$$\mathbf{P}_{k,n} = \sum_{i=1}^{n} \binom{n}{i} \cdot \begin{pmatrix} P_{k,n-i} \\ \times \\ \sum_{d|k} \mathbf{d}'_i \cdot \mathbf{d} \end{pmatrix}$$

The number of maximal labelled antichains in $P_{k,n}$ is exactly,

$$|A(P_{k,n})| = \prod_{i=1}^{n} \prod_{d|k} (d + |A(P_{k,n-i})|)^{\binom{n}{i} \mathbf{d}'_i}.$$

We let $S_{k,n}$ be the poset with cover graph,

$$\mathbf{S}_{k,n} = \begin{pmatrix} \mathbf{P}_{k,n} \\ \times \\ \mathbf{1} \end{pmatrix} + \mathbf{P}_{k,n}$$

where $|A(S_{k,n})| = (1+|A(P_{k,n})|)\cdot|A(P_{k,n})|$. The inductive step is settled.

Example 100. *Let $k = 4$. On the basis of $P_{4,1}$ and $|A(P_{4,1})|$, we compute $S_{4,2}$ and $|A(S_{4,2})|$. First, $P_{4,2}$ is the poset with cover graph,*

$$\binom{2}{1} \cdot \begin{pmatrix} \mathbf{P}_{4,1} \\ \times \\ \sum_{d|4} \mathbf{d}'_1 \cdot \mathbf{d} \end{pmatrix} + \binom{2}{2} \cdot \begin{pmatrix} \mathbf{P}_{4,0} \\ \times \\ \sum_{d|4} \mathbf{d}'_2 \cdot \mathbf{d} \end{pmatrix}$$

that is,

$$2 \cdot \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{4} \\ & \times & & \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{4} \end{pmatrix} + 1 \cdot \begin{pmatrix} & & & \mathbf{1} & & & \\ & & & \times & & & \\ \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{4} & \underset{12}{\cdots} & \mathbf{4} \end{pmatrix}$$

that is,

$$2 \cdot \begin{pmatrix} \mathbf{1} & \mathbf{2} & 2\cdot\mathbf{4} \\ & \times & \\ \mathbf{1} & \mathbf{2} & 2\cdot\mathbf{4} \end{pmatrix} \quad \mathbf{1} \quad 3\cdot\mathbf{2} \quad 12\cdot\mathbf{4}$$

Here,

$$\begin{aligned}
|A(P_{4,2})| &= \prod_{i=1,2} \prod_{d=1,2,4} (d + |A(P_{4,2-i})|)^{\binom{2}{i}\mathbf{d}'_i} \\
&= \prod_{d=1,2,4} (d + |A(P_{4,1})|)^{2\mathbf{d}'_1} \cdot \prod_{d=1,2,4} (d + |A(P_{4,0})|)^{\mathbf{d}'_2} \\
&= \prod_{d=1,2,4} (d + 150)^{2\mathbf{d}'_1} \cdot \prod_{d=1,2,4} (d+1)^{\mathbf{d}'_2} \\
&= 151^{2\cdot\mathbf{1}'_1} \cdot 152^{2\cdot\mathbf{2}'_1} \cdot 154^{2\cdot\mathbf{4}'_1} \cdot 2^{\mathbf{1}'_2} \cdot 3^{\mathbf{2}'_2} \cdot 5^{\mathbf{4}'_2} \\
&= 151^{2\cdot1} \cdot 152^{2\cdot1} \cdot 154^{2\cdot2} \cdot 2^{1} \cdot 3^{3} \cdot 5^{12} \\
&= 39062295914931135 \cdot 10^{11}
\end{aligned}$$

So, $S_{4,2}$ is the poset with cover graph,

$$\left(\begin{array}{ccc} 2\cdot\begin{pmatrix}\mathbf{1} & \mathbf{2} & 2\cdot\mathbf{4}\\ & \times & \\ \mathbf{1} & \mathbf{2} & 2\cdot\mathbf{4}\end{pmatrix} & & \mathbf{1} \quad 3\cdot\mathbf{2} \quad 12\cdot\mathbf{4} \\ & \times & \\ & \mathbf{1} & \end{array}\right) + 2\cdot\begin{pmatrix}\mathbf{1} & \mathbf{2} & 2\cdot\mathbf{4}\\ & \times & \\ \mathbf{1} & \mathbf{2} & 2\cdot\mathbf{4}\end{pmatrix} \quad \mathbf{1} \quad 3\cdot\mathbf{2} \quad 12\cdot\mathbf{4}$$

where,

$$\begin{aligned}|A(S_{4,2})| &= (1 + |A(P_{4,2})|)\cdot|A(P_{4,2})|\\ &= 152586296214564563720863179277884795914931135\cdot 10^{11}.\end{aligned}$$

It is easy to realize that, if k is a prime number, the previous construction trivializes. Indeed, for $n = 1, 2, \ldots,$ the above $P_{k,n}$ reduces to the poset with cover graph,

$$\sum_{i=1}^{n}\binom{n}{i}\cdot\begin{pmatrix} & P_{k,n-i} \\ & \times \\ \mathbf{1} & (k^i - 1)\cdot\mathbf{k}\end{pmatrix}$$

with,

$$|A(P_{k,n})| = \prod_{i=1}^{n}(1 + |A(P_{k,n-i})|)^{\binom{n}{i}}\cdot(k + |A(P_{k,n-i})|)^{\binom{n}{i}(k^i-1)}.$$

On this basis, $S_{k,n}$, along with $|A(S_{k,n})|$, are defined as above.

It turns out that the algebra of maximal labelled antichains $A(S_{k,n})$ in $S_{k,n}$, with suitably defined operations, is a BL-algebra, and is in fact isomorphic to the free n-generated BL_k-algebra $\mathbf{BL}_{k,n}$.

Definition 101. *The algebra,*

$$\mathbf{F}_{k,n} = (A(S_{k,n}), \vee, \wedge, \odot, \to, \top, \bot),$$

is the algebra of type $(2, 2, 2, 2, 0, 0)$, where constants and operations are realized as follows.

The constant $\bot$ is realized by sending each element in the antichain over the minimal elements of $S_{k,n}$ to 0. The constant $\top$ is realized by sending each element in the antichain over the maximal elements of $S_{k,n}$, say the element $\mathbf{d}$, to d.

The operations are defined chainwise, as follows. Let l and l' be maximal labelled antichains in $A(S_{k,n})$, with underlying antichain A and A' in $S_{k,n}$. We let $l \vee l'$ be the maximal labelled antichain in $A(S_{k,n})$ having underlying antichain $A \vee A'$ such that, for every maximal chain C in $S_{k,n}$,

$$l \vee l'((A \vee A') \cap C) = \begin{cases} \max(l(A \cap C), l'(A' \cap C)) & \text{if } A \cap C = A' \cap C \\ l(A \cap C) & \text{if } A \cap C > A' \cap C \\ l'(A' \cap C) & \text{otherwise} \end{cases}$$

We define $l \wedge l'$ analogously. We let $l \odot l'$ be the maximal labelled antichain in $A(S_{k,n})$ having underlying antichain $A \wedge A'$ such that, for every maximal chain C in $S_{k,n}$, say with $\mathbf{c}$ as maximal element,

$$l \odot l'((A \wedge A') \cap C) = \begin{cases} \max(l(A \cap C) + l'(A' \cap C) - d, \mathbf{c}) & \text{if } A \cap C = A' \cap C = \mathbf{d} \\ l'(A' \cap C) & \text{if } A \cap C > A' \cap C \\ l(A \cap C) & \text{otherwise} \end{cases}$$

We let $l \to l'$ be the maximal labelled antichain in $A(S_{k,n})$ having underlying antichain $A \to A'$ such that, for every maximal chain C in $S_{k,n}$, say with $\mathbf{c}$ as maximal element,

$$(A \to A') \cap C = \begin{cases} \mathbf{c} & \text{if } A \cap C < A' \cap C \text{ or, } A \cap C = A' \cap C \text{ and } l(A \cap C) \leq l'(A' \cap C) \\ A' \cap C & \text{otherwise} \end{cases}$$

and,

$$l \to l'((A \to A') \cap C) = \begin{cases} \min(l'(A' \cap C) + d - l(A \cap C), \mathbf{c}) & \text{if } A \cap C = A' \cap C = \mathbf{d} \\ l'(A' \cap C) & \text{if } A \cap C > A' \cap C \\ \mathbf{c} & \text{otherwise} \end{cases}$$

For every $\circ \in \{\vee, \wedge, \odot, \to, \top, \bot\}$, we let $\circ^{\mathbf{F}_{k,n}}$ denote the realization of $\circ$ in $\mathbf{F}_{k,n}$.

Theorem 102. *$\mathbf{S}_{k,n}$ is isomorphic to the free n-generated BL_k-algebra $\mathbf{BL}_{k,n}$.*

Proof (Sketch). It is easy to check that $\mathbf{F}_{k,n}$ is a BL-algebra. Moreover, it is possible to define an isomorphism of BL-algebras from $F_n|_{A_{n,k}}$ to $A(S_{n,k})$. $\square$

3.2.2 Tight Countermodels

In this section, we shall consider the problem of deciding the validity of BL-equations: given a BL-equation of the form $t = \top$, [4] built upon variables $X_1, \dots, X_n$, we are to decide whether or not $t^{\mathbf{A}}(\mathbf{a}) = \top^{\mathbf{A}}$ for every BL-algebra $\mathbf{A}$ and every assignment $\mathbf{a} \in A^n$.

By the representation result of Aglianó and Montagna, stated in Theorem 13, $t = \top$ is valid in all BL-algebras if and only if there not exists a *countermodel* to $t = \top$ in the algebra $[0, n+1]$, that is, there not exists an assignment $\mathbf{b} \in [0, n+1]^n$ such that,

$$t^{[0,n+1]}(\mathbf{b}) < \top^{[0,n+1]}. \tag{3.4}$$

As we already mentioned in the introduction, the problem of deciding BL-equations (and BL-quasiequations) is in coNP. In fact, it is possible to prove that a BL-equation $t = \top$ has a countermodel in $[0, n+1]$ if and only if $t = \top$ has a countermodel $\mathbf{b} \in [0, n+1]^n$ such that $\mathbf{b}$ is a point with rational coordinates of denominator upper bounded in $O(\exp(n^c))$, for some $c \geq 1$. Therefore, a *nondeterministic* approach consists in guessing a rational point $\mathbf{b}$ having a small denominator (relative to the size of the input), and in checking that inequality (3.4) holds with respect to $\mathbf{b}$.

In the rest of this section, exploiting the explicit description of BL-functions given in Chapter 2, we approach the problem of finding *tight* countermodels to BL-equations, that is, countermodels to BL-equations that are as small as possible, in a formally defensible sense. We expect that, with minor modifications, the sketched approach generalizes to the quasiequational case. [5]

Let,

$$b(n, l) = \lfloor (l/n)^n \rfloor, \tag{3.5}$$

where $n, l \geq 1$ are integers. In [AG02], the following result is shown.

[4] W.l.o.g., indeed a BL-equation of the form $r = s$, with r and s arbitrary terms, holds in all BL-algebras if and only if $(r \to s) \odot (s \to r) = \top$ holds in all BL-algebras.

[5] A BL-quasiequation is a tuple $(t_1 = \top, \dots, t_k = \top, t = \top)$ of BL-equations, built upon variables $X_1, \dots, X_n$, and we are to decide if $r_i^{\mathbf{A}}(\mathbf{a}) = s_i^{\mathbf{A}}(\mathbf{a})$ for all $i \in [k]$ implies $r^{\mathbf{A}}(\mathbf{a}) = s^{\mathbf{A}}(\mathbf{a})$, for every BL-algebra $\mathbf{A}$ and every assignment $\mathbf{a} \in A^n$.

Theorem 103. *Let t be a term over l occurrences of n distinct variables. The BL-equation $t = \top$ is not valid if and only if there exist a positive integer $k \leq b(n,l)$ and a variable assignment $\mathbf{a} \in (A_{n,k})^n$ such that $t^{\mathbf{A}_{n,k}}(\mathbf{a}) < \top^{\mathbf{A}_{n,k}}$, where $\mathbf{A}_{n,k}$ is as in Definition 94.*

In [Agu06], Aguzzoli proved that, as regards to MV-equations, the bound (3.5) is asymptotically tight. Roughly, for every fixed integer k significantly lower than $b(n,l)$, there exists a term t with l occurrences of n distinct variables, where l is chosen sufficiently large, such that t succeeds over every rational point in $[0,1]^n$ with denominator less than k and fails over a rational point in $[0,1]^n$ with denominator between k and $b(n,l)$. Aguzzoli remarks that an explicit functional representation of the free n-generated BL-algebra, in the spirit of [Mon00], is necessary to generalize tightness results to the case of BL-equations. In this vein, we discuss below an approach to obtain tightness results on countermodels to BL-equations.

The initial observation is that the statement of Theorem 103 is sufficient to establish the coNP membership of BL-equations, but does not guarantee the optimality of the corresponding *deterministic* algorithm. Precisely, the statement of Theorem 103 ensures that a deterministic search for a countermodel to a BL-equation $t = \top$, specified as above, is complete only if its search space includes all the rational points in $[0, n+1]^n$ having denominator $\leq b(n,l)$. In light of the explicit descriptions of BL-functions given in Theorem 82, we can shrink such search space preserving the completeness of the procedure. As a strengthening, it is possible to prove that the resulting search space is *optimal*, in the sense that any further shrink breaks the completeness of the countermodel search procedure.

Let $n \geq 1$ and let $\mathbf{p} = (p_1, \ldots, p_n) \in [0, n+1)^n$ be an integer point. We let,

$$C(\mathbf{p}) = \{\mathbf{b} \mid p_1 \leq b_1 < p_1 + 1, \ldots, p_n \leq b_n < p_n + 1\} \subseteq [0, n+1]^n,$$

denote the cell having $\mathbf{p}$ as origin. The search space of the decision algorithm suggested by Theorem 103 contains all the $(n+1)^n$ distinct cells in $[0, n+1]^n$. As a first refinement of Theorem 103, we shall minimize the number of cells that are to be checked to accomplish a complete countermodel search. In particular, it turns out that a number of cells asymptotically equivalent to $n!$, thus growing asymptotically slower than n^n, suffices.

For every $n \geq 0$, the nth *Fubini number* (or *ordered Bell number*), that is, the number of ordered partitions of a set of n elements into nonempty subsets, is equal to,

$$F(n) = \sum_{k=0}^{n} k! \cdot S(n,k),$$

where $S(n,k)$ denotes the *Stirling set number*, that is, the number of partitions of a set of n elements into k nonempty subsets. It is known [Wil90] that,

$$\lim_{n\to\infty} \frac{F(n)}{\frac{n!}{2\cdot(\ln 2)^{n+1}}} = 1,$$

thus,

$$\lim_{n\to\infty} \frac{F(n)}{(n+1)^n} = \lim_{n\to\infty} \frac{\frac{n!}{2\cdot(\ln 2)^{n+1}}}{(n+1)^n} = 0,$$

that is, $(n+1)^n$ grows asymptotically faster than $F(n)$, which in fact grows asymptotically as $n!$.

Example 104. *We have* $2 \cdot F(4) = 150 < 625 = 4^5$, $2 \cdot F(6) = 1,082 < 7,776 = 5^6$, $2 \cdot F(7) = 9,366 < 117,649 = 6^7$, $2 \cdot F(8) = 94,586 < 2,097,152 = 7^8$, $2 \cdot F(9) = 1,091,670 < 43,046,721 = 8^9$.

It turns out that t has a countermodel in $[0,n+1]^n$ if and only if t has a countermodel inside a subset of $2 \cdot F(n)$ suitably chosen cells in $[0,n+1]^n$. Precisely, for $j \in [F(n)]$, let P_j be the jth ordered partition of $[n]$, say into the k nonempty blocks $B_1 < \cdots < B_k$. We let the points $\mathbf{p}_j = (p_1,\ldots,p_n)$ and $\mathbf{q}_j = (q_1,\ldots,q_n)$ in $[0,n+1]^n$ be defined by putting,

$$p_l = i-1,$$
$$q_l = i,$$

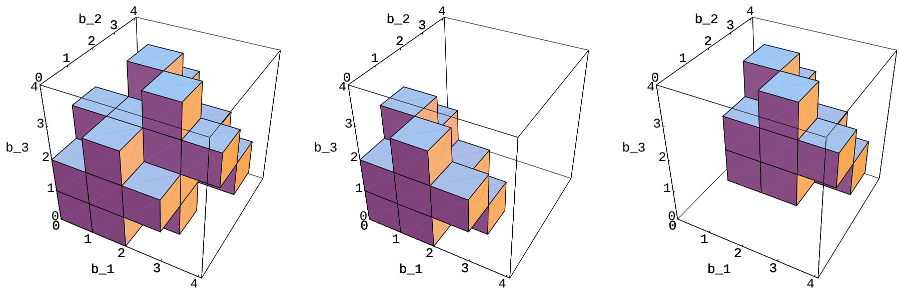
for every $i \in [k]$ and every $l \in B_i$.

Claim 105. *Let t be a term over n distinct variables. The BL-equation $t = \top$ is not valid if and only if there exists,*

$$\mathbf{b} \in \bigcup_{j\in[F(n)]} C(\mathbf{p}_j) \cup C(\mathbf{q}_j),$$

such that $t^{[0,n+1]}(\mathbf{b}) < \top^{[0,n+1]}$.

The previous claim allows to shrink the search space for deciding BL-equations to a number of cells, $2 \cdot F(n)$, which is significantly lower than the total number of cells in $[0, n+1]^n$. Moreover, let C be a cell of the form $C(\mathbf{p}_j)$ or $C(\mathbf{q}_j)$, for $j \in [F(n)]$. As an application of Theorem 82, there exists a term t over n variables that succeeds over $[0, n+1]^n \setminus C$, but fails inside C. Therefore, with respect to the completeness of a countermodel search algorithm, Claim 105 characterizes a minimal set of *unavoidable* cells in $[0, n+1]^n$.



(a) (b) (c)

Figure 3.2: The case $n = 3$ (see also Example 106). (a) The unavoidable cells in $[0, 4]^3$. (b) The unavoidable cells outside $[1, 4]^3$. (c) The unavoidable cells inside $[1, 4]^3$.

As a second refinement of Theorem 103, for each unavoidable cell C, we shall minimize the largest denominator of the rational points inside C that a complete countermodel search has to check. For every $j \in [F(n)]$, let P_j be the jth ordered partition of $[n]$ in some arbitrary fixed total ordering of the ordered partitions of $[n]$, let M_j be the maximal block in P_j, and let m_j be the block covered by M_j in P_j. For every rational number $a \in \mathbb{Q}$, let $\mathrm{den}(a) \geq 1$ be the denominator of the reduced form of a. We put,

$$D_j = \left\{ (b_i)_{i \in [n]} \in C(\mathbf{p}_j) \cap \mathbb{Q}^n \;\middle|\; \mathrm{den}(b_i) \leq \begin{cases} b(l, |m_j|) & \text{if } i \in m_j \\ b(l, |M_j|) & \text{if } i \in M_j \\ 1 & \text{otherwise} \end{cases} \right\},$$

$$E_j = \left\{ (b_i)_{i\in[n]} \in C(\mathbf{q}_j) \cap \mathbb{Q}^n \;\middle|\; \mathrm{den}(b_i) \leq \begin{cases} b(l, |m_j|) & \text{if } i \in m_j \\ b(l, |M_j|) & \text{if } i \in M_j \\ 1 & \text{otherwise} \end{cases} \right\}.$$

Example 106. *Let $n = 3$ and let t be a term over l occurrences of variables X_1, X_2, X_3. We have the following $P_1, \ldots, P_{13} = P_{F(3)}$ ordered partitions of $[3]$: P_1 is $\{1,2,3\}$, P_2 is $\{1\} < \{2,3\}$, P_3 is $\{2,3\} < \{1\}$, P_4 is $\{1,2\} < \{3\}$, P_5 is $\{3\} < \{1,2\}$, P_6 is $\{1,3\} < \{2\}$, P_7 is $\{2\} < \{1,3\}$, P_8 is $\{1\} < \{2\} < \{3\}$, P_9 is $\{1\} < \{3\} < \{2\}$, P_{10} is $\{2\} < \{1\} < \{3\}$, P_{11} is $\{2\} < \{3\} < \{1\}$, P_{12} is $\{3\} < \{1\} < \{2\}$, P_{13} is $\{3\} < \{2\} < \{1\}$. Each ordered partition corresponds to a pair $\mathbf{p}_j$ and $\mathbf{q}_j$ of integer points in $[0,3]^2$, precisely: $\mathbf{p}_1 = (0,0,0)$ and $\mathbf{q}_1 = (1,1,1)$; $\mathbf{p}_2 = (0,1,1)$ and $\mathbf{q}_2 = (1,2,2)$; $\mathbf{p}_3 = (1,0,0)$ and $\mathbf{q}_3 = (2,1,1)$; $\mathbf{p}_4 = (0,1,1)$ and $\mathbf{q}_4 = (1,1,2)$; $\mathbf{p}_5 = (1,1,0)$ and $\mathbf{q}_5 = (2,2,1)$; $\mathbf{p}_6 = (0,1,0)$ and $\mathbf{q}_6 = (1,2,1)$; $\mathbf{p}_7 = (1,0,1)$ and $\mathbf{q}_7 = (2,1,2)$; $\mathbf{p}_8 = (0,1,2)$ and $\mathbf{q}_8 = (1,2,3)$; $\mathbf{p}_9 = (0,2,1)$ and $\mathbf{q}_9 = (1,3,2)$; $\mathbf{p}_{10} = (1,0,2)$ and $\mathbf{q}_{10} = (2,1,3)$; $\mathbf{p}_{11} = (2,0,1)$ and $\mathbf{q}_{11} = (3,1,2)$; $\mathbf{p}_{12} = (1,2,0)$ and $\mathbf{q}_{12} = (2,3,1)$; $\mathbf{p}_{13} = (2,1,0)$ and $\mathbf{q}_{13} = (3,2,1)$. Eventually, we sample D_j for $j = 1,2,3,8$:*

$$D_1 = \{\mathbf{b} \in C(\mathbf{p}_1) \cap \mathbb{Q}^n \mid \mathrm{den}(b_1), \mathrm{den}(b_2), \mathrm{den}(b_3) \leq b(l,3)\},$$
$$D_2 = \{\mathbf{b} \in C(\mathbf{p}_2) \cap \mathbb{Q}^n \mid \mathrm{den}(b_1) \leq b(l,1), \mathrm{den}(b_2), \mathrm{den}(b_3) \leq b(l,2)\},$$
$$D_3 = \{\mathbf{b} \in C(\mathbf{p}_3) \cap \mathbb{Q}^n \mid \mathrm{den}(b_2), \mathrm{den}(b_3) \leq b(l,2), \mathrm{den}(b_1) \leq b(l,1)\},$$
$$D_8 = \{\mathbf{b} \in C(\mathbf{p}_8) \cap \mathbb{Q}^n \mid \mathrm{den}(b_1) \leq 1, \mathrm{den}(b_2) \leq b(l,1), \mathrm{den}(b_3) \leq b(l,1)\}.$$

Claim 107. *Let t be a term with l occurrences of n distinct variables. The BL-equation $t = \top$ is not valid if and only if there exists,*

$$\mathbf{b} \in \bigcup_{j\in[F(n)]} D_j \cup E_j,$$

such that $t^{[0,n+1]}(\mathbf{b}) < \top^{[0,n+1]}$.

Example 108. *Continuing Example 106, let $l = 15$, so that $b(3,15) = 125$, $b(2,15) = 56$ and $b(1,15) = 15$. As a rough estimation, the number of points to check by Claim 107 is,*

$$\leq 2\left(\sum_{k=1}^{125} k^3 + 3\left(\sum_{k=1}^{56}\sum_{h=1}^{15} k^2 h\right) + 3\left(\sum_{k=1}^{56}\sum_{h=1}^{15} k h^2\right) + 6\left(\sum_{k=1}^{15} 2k^2\right)\right)$$
$$= 179{,}218{,}770,$$

whereas the number of points to check by Theorem 103 is $\leq \sum_{k=1}^{125}(4k+1)^3 = 4,000,720,625$.

For every $j \in [F(n)]$, every rational point $\mathbf{b}$ inside either D_j or E_j, and every coordinate $i \in [n]$, the upper bound on the denominator $\mathrm{den}(b_i)$ is *tight*, in the following sense. Suppose to admit a significantly lower upper bound k on $\mathrm{den}(b_i)$ in D_j (the case of E_j is similar), for some $j \in [F(n)]$ and $i \in [n]$. Then, by combining constructions in Theorem 82 and in [Agu06], it is possible to construct a term t over a sufficiently large number l of occurrences of variables in $\{X_l \mid l \in M_j\}$ if $i \in M_j$, or in $\{X_l \mid l \in m_j\}$ if $i \in m_j$, such that t succeeds over each unavoidable cell distinct from D_j, t succeeds over each rational point in D_j of denominator less than k on coordinate i, but t fails over a rational point in D_j with denominator between k and $\mathrm{den}(b_i)$ on coordinate i.

Therefore, roughly, Claim 107 suggests a countermodel search which is *optimal* in the sense that it checks only unavoidable cells, and inside each unavoidable cell, only unavoidable rational points, where unavoidability stems from the fact that checking either less cells, or less points inside a cell, breaks the completeness of the procedure.

3.2.3 Deductive Interpolation

It is known that Basic logic has the *deductive interpolation* property: for $I, K \subseteq [n]$, if r is a term over variables $\{X_i \mid i \in I\}$, t is a term over variables $\{X_k \mid k \in K\}$, and $r \vdash_{BL} t$, then there exists a *deductive interpolant* of r and t, that is, term s over variables $\{X_j \mid j \in I \cap K\}$ such that $r \vdash_{BL} s$ and $s \vdash_{BL} t$ [Mon06]. [6]

In [Mon06], Montagna raised the problem of providing an effective construction of deductive interpolants in Basic logic. In this section, we discuss a geometrical approach to the problem, enlightened by the representation of the Lindenbaum-Tarski algebra of Basic logic in terms of BL-functions presented in Chapter 2.

[6] If a variety of commutative residuated lattices forms the equivalent algebraic semantics of a propositional logic, as BL-algebras do with respect to Basic logic, then the deductive interpolation property of the logic is equivalent to the *amalgamation* property of the variety [GO06].

Let $n \geq 1$, let $I, K \subseteq [n]$ with $I \cup K = [n]$, and let r and t be terms in L_n, with $r \in L_I$ and $t \in L_K$, such that $r \vdash_{BL} t$. For every n-ary BL-function $f\colon [0, n+1]^n \to [0, n+1]$ in F_n, let,

$$\mathbf{1}_f = \{\mathbf{b} \in [0, n+1]^n \mid f(\mathbf{b}) = n+1\},$$

denote the *oneset* of f; for every $u \in L_n$, we write in short $\mathbf{1}_u$ instead of $\mathbf{1}_{u^{[0,n+1]}}$. Note that,

$$\mathbf{1}_u = \{\mathbf{b} \in [0, n+1]^n \mid u^{[0,n+1]}(\mathbf{b}) = \top^{[0,n+1]}\}.$$

As we discussed in the introduction to Chapter 2, the Lindenbaum-Tarski algebra of the n-variate fragment of Basic logic is isomorphic to the free n-generated BL-algebra, which has been explicitly described in terms of BL-functions in Theorem 82. Therefore, $r \vdash_{BL} t$ if and only if the BL-functions $r^{[0,n+1]}\colon [0, n+1]^n \to [0, n+1]$ and $t^{[0,n+1]}\colon [0, n+1]^n \to [0, n+1]$, respectively essentially at most $|I|$-ary over the coordinates indexed in I and essentially at most $|K|$-ary over the coordinates indexed in K, satisfy the relation,

$$\mathbf{1}_r \subseteq \mathbf{1}_t.$$

In this setting, the logical problem of constructing a deductive interpolant s to r and t is equivalent to the following geometrical problem: given explicit descriptions (say, the implementing systems) of the above BL-functions $r^{[0,n+1]}$ and $t^{[0,n+1]}$ in F_n, compute an explicit description (say, the implementing system) of a BL-function $f\colon [0, n+1]^n \to [0, n+1]$ in F_n, essentially at most $|I \cap K|$-ary over the coordinates indexed in $I \cap K$, such that f satisfies the relation,

$$\mathbf{1}_r \subseteq \mathbf{1}_f \subseteq \mathbf{1}_t. \tag{3.6}$$

Indeed, upon availability of the implementing system of f, the explicit construction of Lemma 81 furnishes a term s in L_n such that $s^{[0,n+1]} = f$, so that in particular,

$$\mathbf{1}_r \subseteq \mathbf{1}_s \subseteq \mathbf{1}_t;$$

moreover, we can suppose that s contains only variables indexed in $K \cap I$: for otherwise, for every $j \notin K \cap I$, replace every occurrence of variable X_j in s with $\perp$, and notice that the function computed by the resulting term coincides with $s^{[0,n+1]}$.

Therefore, we reduced to the problem of specifying a function $f \in F_n$ satisfying (3.6), given the specifications of the above functions $r^{[0,n+1]}$ and $t^{[0,n+1]}$ in F_n. In the rest of this section, we sketch a solution method to this problem.

The intuition underlying our solution is the following. By the explicit descriptions of BL-functions, we know that, for every function f in F_n, there exists a finite family $P_1, \ldots, P_q$ of rational polyhedra in $[0, n+1]^n$, such that,

$$\mathbf{1}_f = \bigcup_{i \in [q]} \text{relint } P_i.$$

Moreover, $\mathbf{1}_f$ satisfies the following constraints. Let $B_1 < \cdots < B_k$ be an ordered partition of $[n]$ into $k \geq 1$ nonempty blocks, and let,

$$C^{B_1 < \cdots < B_k}_{0 < \cdots < k-1} \text{ and } C^{B_1 < \cdots < B_k}_{1 < \cdots < k},$$

be the unavoidable cells, respectively outside and inside $[1, n+1]^n$, corresponding to $B_1 < \cdots < B_k$, as in Definition 84.

Constraint 1: If $\mathbf{b} = (b_i)_{i \in [n]} \in \mathbf{1}_f \cap C^{B_1 < \cdots < B_k}_{0 < \cdots < k-1}$, then,

$$\bigcup_{C} \{(a_i)_{i \in [n]} \in C \mid a_i - \lfloor a_i \rfloor = b_i - \lfloor b_i \rfloor\} \subseteq \mathbf{1}_f,$$

where C ranges over the cells outside $[1, n+1]^n$ corresponding to the ordered partition $B_1 < \cdots < B_k$.

Constraint 2: If $\mathbf{b} = (b_i)_{i \in [n]} \in \mathbf{1}_f \cap C^{B_1 < \cdots < B_k}_{1 < \cdots < k}$, then,

$$\bigcup_{C} \{(a_i)_{i \in [n]} \in C \mid a_i - \lfloor a_i \rfloor = b_i - \lfloor b_i \rfloor\} \subseteq \mathbf{1}_f,$$

where C ranges over the cells inside $[1, n+1]^n$ corresponding to the ordered partition $B_1 < \cdots < B_k$.

In particular, if $f \in F_n$ is essentially at most $|I \cap K|$-ary over the coordinates indexed in $I \cap K$, then $\mathbf{1}_f$ is *encoded* at most by the coordinates indexed in $I \cap K$, that is,

$$\mathbf{1}_f = \bigcup_{(b_i)_{i \in [n]} \in \mathbf{1}_f} \{(a_i)_{i \in [n]} \mid a_i = b_i \text{ if } i \in I \cap K, \text{otherwise } 0 \leq a_i \leq n+1\}.$$

The idea of the construction is to attain a set $S' \subseteq [0, n+1]^n$, encoded by the coordinates indexed in $I \cap K$, such that on the one hand,

$$\mathbf{1}_r \subseteq S' \subseteq \mathbf{1}_t,$$

and on the other hand, S' satisfies Constraint 1 and Constraint 2. The latter property turns out to be equivalent to the possibility of specifying a function f in F_n, essentially at most $|I \cap K|$-ary over the coordinates indexed in $I \cap K$, such that $S' = \mathbf{1}_f$. By Lemma 81, this is sufficient to construct a deductive interpolant to r and t in Basic logic.

We provide some additional details on the construction. It is easy to observe that, since $r^{[0,n+1]}$ is essentially at most $|I|$-ary over the coordinates indexed in I, $t^{[0,n+1]}$ is essentially at most $|K|$-ary over the coordinates indexed in K, and $\mathbf{1}_r \subseteq \mathbf{1}_t$, the following fact holds.

Claim 109 (Initialization). *Let $S \subseteq [0, n+1]^n$ be defined by,*

$$S = \bigcup_{(b_i)_{i \in [n]} \in \mathbf{1}_r} \{(a_i)_{i \in [n]} \mid a_i = b_i \text{ if } i \in I \cap K, \text{ otherwise } 0 \leq a_i \leq n+1\}.$$

Then,

$$\mathbf{1}_r \subseteq S \subseteq \mathbf{1}_t.$$

By construction, S is the smallest set that is encoded by the coordinates indexed in $I \cap K$ and includes $\mathbf{1}_r$. Therefore a natural question is whether or not it is possible to specify a function f in F_n such that $\mathbf{1}_f = S$. Note that the restriction of any such f to S is essentially at most $|I \cap K|$-ary over the coordinates indexed in $I \cap K$. It turns out that the answer to this question is, in general, negative. Nevertheless, it is possible to extend the set S to a set $S' \subseteq \mathbf{1}_t$ that suits the scope in the following sense.

Claim 110 (Normalization). *Let S be as in Claim 109 and let $S' \subseteq [0, n+1]^n$ be the smallest superset of S satisfying Constraints 1 and 2. Then,*

$$\mathbf{1}_r \subseteq S' \subseteq \mathbf{1}_t,$$

and S' is encoded by the coordinates indexed in $I \cap K$.

We remark that, appealing to the decomposition of $\mathbf{1}_r$ and $\mathbf{1}_t$ as finite unions of rational polyhedra, it is possible to compute effectively a specification of the set S'. Then, the construction of normal forms in Lemma 81 gives the conclusion.

Claim 111 (Specification)**.** *Let S' be as in Claim 110. Then, it is possible to specify a function f in F_n, essentially at most $|I \cap K|$-ary over the coordinates indexed in $I \cap K$, such that,*

$$\mathbf{1}_f = S'.$$

Claim 112 (Construction)**.** *Let f be specified as in Claim 111. Then, it is possible to construct a term $s \in L_{I \cap K}$, such that,*

$$s^{[0,n+1]} = f.$$

The formal proof of the previous facts is left as a future work. In the rest of this section, we sample the general construction in the case $n = 3$. Consider the following instance of the constructive deductive interpolation problem of Basic logic. The input is a pair of terms r and t in L_3, over variables X_1, X_2 and X_2, X_3 respectively, such that,

$$\mathbf{1}_r = \{\mathbf{b} \in [0,4]^3 \mid r^{[0,4]}(\mathbf{b}) = 4\} \subseteq \{\mathbf{b} \in [0,4]^3 \mid t^{[0,4]}(\mathbf{b}) = 4\} = \mathbf{1}_t.$$

The expected output is a term s in L_3, over the variable X_2, such that $\mathbf{1}_s = \{\mathbf{b} \in [0,4]^3 \mid s^{[0,4]}(\mathbf{b}) = 4\}$ satisfies,

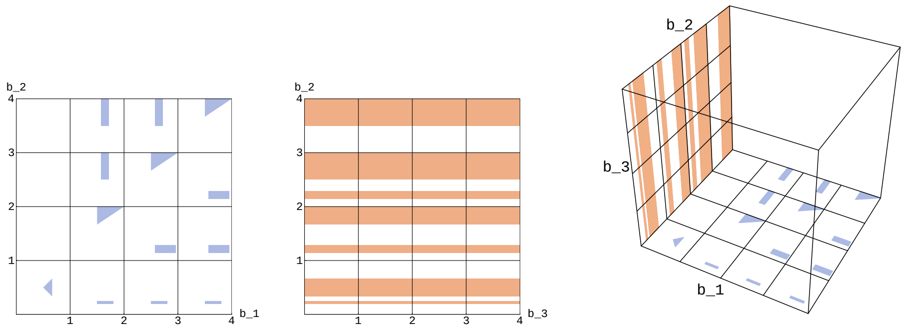
$$\mathbf{1}_r \subseteq \mathbf{1}_s \subseteq \mathbf{1}_t.$$

To solve the problem, it is sufficient to specify a ternary BL-function $f\colon [0,4]^3 \to [0,4]$, essentially unary over the second coordinate, such that $\mathbf{1}_r \subseteq \mathbf{1}_f \subseteq \mathbf{1}_t$. The output term s, w.l.o.g. over variable X_2, such that $s^{[0,4]} = f$, is then given by the construction in Lemma 81 (in this case, Lemma 46 is sufficient).

By the explicit description of BL-functions, we know that $\mathbf{1}_r$ can be displayed as the union of a finite number of rational polyhedra in $[0,4]^3$, satisfying Constraint 1 and Constraint 2. For concreteness, suppose that $\mathbf{1}_r$ is as in Figure 3.3(a). Since $r^{[0,4]}$ is essentially at most binary on the first and second coordinate, $t^{[0,4]}$ is essentially at most binary on the second and third coordinate, and $\mathbf{1}_r \subseteq \mathbf{1}_t$, for every point $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbf{1}_r$, the set $\{(a_1, b_2, a_3) \mid 0 \leq a_1, a_3 \leq 4\}$ is included in $\mathbf{1}_t$. Hence letting,

$$S = \bigcup_{(b_1,b_2,b_3) \in \mathbf{1}_r} \{(a_1, b_2, a_3) \mid 0 \leq a_1, a_3 \leq 4\}, \tag{3.7}$$

we have $\mathbf{1}_r \subseteq S \subseteq \mathbf{1}_t$. Compare Figure 3.3(b)-(c).



(a) (b) (c)

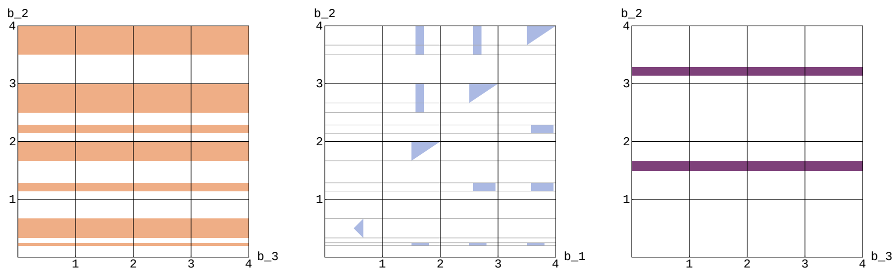
Figure 3.3: Let $r \in L_3$ over variables X_1 and X_2 be such that figure (a) shows the projection $\{(b_1, b_2, 0) \mid (b_1, b_2, b_3) \in \mathbf{1}_r\}$ of $\mathbf{1}_r$. Note that $(b_1, b_2, 0) \in \mathbf{1}_r$ implies $\{(b_1, b_2, a_3) \mid 0 \leq a_3 \leq 4\} \subseteq \mathbf{1}_r$, because $r^{[0,4]}$ is essentially at most binary over the first and second coordinate. Note also that $\mathbf{1}_r$ is the union of a finite number of rational polyhedra in $[0, 4]^3$, that satisfy Constraint 1 and Constraint 2. Figure (b) shows the projection $\{(0, b_2, b_3) \mid (b_1, b_2, b_3) \in S\}$ of the set S defined in (3.7). For every $(b_1, b_2, b_3) \in \mathbf{1}_r$, we have $\{(a_1, b_2, a_3) \mid 0 \leq a_1, a_3 \leq 4\} \subseteq \mathbf{1}_t$, so that the set S in (3.7) is contained in $\mathbf{1}_t$. Figure (c) displaces the projections above in $[0, 4]^3$.

It is easy to realize that there not exists a function f in F_3 such that $\mathbf{1}_f = S$. Indeed, suppose that there exists $f \in F_3$ such that $S = \mathbf{1}_f$. Then, for instance, by Constraint 1, if $\mathbf{b} = (b_1, b_2, b_3) \in S$ and $\mathbf{b} \in C_{0<3}^{\{1\}<\{2,3\}}$, then,

$$\left\{ (a_i)_{i\in[n]} \in \bigcup_{j_2 \in [3]} C_{0<j_2}^{\{1\}<\{2,3\}} \;\middle|\; a_i - \lfloor a_i \rfloor = b_i - \lfloor b_i \rfloor \right\} \subseteq S.$$

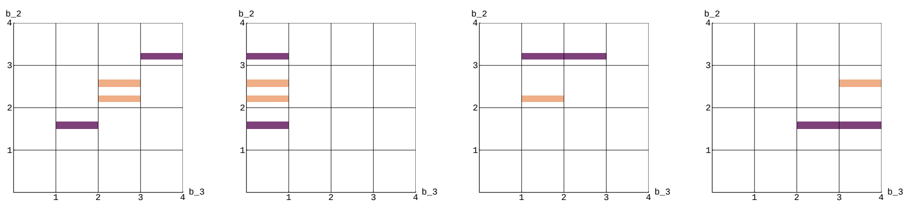
But, in the example under consideration, by inspection of Figure 3.4, we have for instance $(0, 3 + 1/2, 3 + 1/2) \in S$ and $(0, 1 + 1/2, 1 + 1/2) \notin S$. Thus, there not exists $f \in F_3$ such that $S = \mathbf{1}_f$.

The next step consists in the *normalization* of S. We extend S to the smallest superset $S' \supseteq S$ that satisfies Constraints 1 and 2. In the example under consideration, we normalize S applying iteratively Constraint 1, as shown in Figure 3.5. The resulting set S' is depicted in Figure 3.6(a).



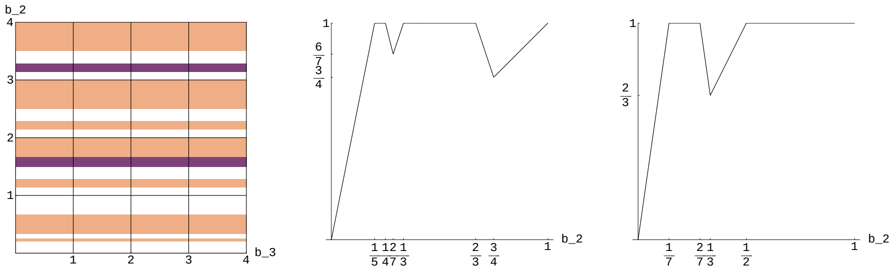
(a) (b) (c)

Figure 3.4: The set S defined in (3.7) is constructed by projecting the set $\mathbf{1}_r$ onto the first coordinate. Compare (a) and (b). In general, there not exists $f \in F_3$ such that $\mathbf{1}_f = S$, because S violates either Constraint 1 or Constraint 2. In the example under consideration, the points highlighted in (c) should be in $S = \mathbf{1}_f$ by Constraint 1, but they are not.



(a) (b) (c) (d)

Figure 3.5: The normalization of the set S, defined in (3.7), consists in extending S to a superset $S' \supseteq S$, by adding to S the minimal set of points such that the resulting S' satisfies both Constraint 1 and Constraint 2. In figures (a)-(d), the light color marks points $(0, b_2, b_3) \in [0,4]^3$ that are in S, and the dark color marks points $(0, b_2, b_3) \in [0,4]^3$ that are not in S. It is easy to realize that, applying Constraint 1 to the light points, the dark points have to be added to S in order to obtain a normalized $S' \supseteq S$. For instance, in (a), the light points $\mathbf{b} = (b_1, b_2, b_3) \in C_{0<2}^{\{1\}<\{2,3\}}$, which are in S, ask for adding the dark points in $\{(a_i)_{i\in[n]} \in C_{0<1}^{\{1\}<\{2,3\}} \cup C_{0<3}^{\{1\}<\{2,3\}} \mid a_i - \lfloor a_i \rfloor = b_i - \lfloor b_i \rfloor\}$ to S.



(a) S'. (b) g_1. (c) g_2.

Figure 3.6: (a) depicts the projection $\{(0, b_2, b_3) \mid (b_1, b_2, b_3) \in S'\}$ of the set S' that results from the normalization of S. (b) and (c) depict a pair of unary McNaughton functions, g_1 and g_2. By Theorem 28, let t_1 and t_2 be the terms, w.l.o.g. in $L^+_{\{2\}}$, such that $t_1^{[0,1]} = g_1$ and $t_2^{[0,1]} = g_2$. By Lemma 81, the term $s = (\neg\neg t_1) \wedge (\neg\neg t_2 \to t_2)$ is such that $\mathbf{1}_f = S'$, and contains only variable X_2. Therefore, s is a deductive interpolant of r and t in Basic logic.

The normalization step guarantees that $\mathbf{1}_r \subseteq S' \subseteq \mathbf{1}_t$: the former inclusion is trivial; the latter follows from the fact that $S \subseteq \mathbf{1}_t$ and $\mathbf{1}_t$ satisfies Constraints 1 and 2. Given S', we are in the position to specify a function $f \in F_3$, essentially at most unary on the second coordinate, such that $\mathbf{1}_f = S'$. In the example under consideration, the function $f \in F_3$ is specified in terms of the pair of unary McNaughton functions g_1 and g_2 plotted in Figure 3.6(b)-(c). An appeal to Lemma 81 concludes the construction.

Appendix

We collect in this appendix a number of technical proofs.

Proofs of Claims in Lemma 71 on Page 74

To prove Claim 72 on Page 75 and Claim 73 on Page 76, we introduce a bunch of terminology and notation.

Consider the partition of $[0, n+1]^n$ into $(n+1)^n$ blocks, whose ith block $B_{\mathbf{i}}$, indexed by the ith element $\mathbf{i} = (i_1, \ldots, i_n)$ in the lexicographical order on $\{0, \ldots, n\}^n$, is defined as follows:

$$B_{\mathbf{i}} = \{(b_1, \ldots, b_n) \in [0, n+1]^n \mid \mathbf{i}_j \leq b_j \lhd \mathbf{i}_j + 1 \text{ for all } j \in [n]\},$$

where $\lhd$ is for $<$ if $i_j < n$ and for $\leq$ if $i_j = n$. We let $(B_{\mathbf{i}})_j$ denote the jth component of $\mathbf{i}$, that is, $(B_{\mathbf{i}})_j = i_j$. Hence the equation,

$$(B_{\mathbf{i}})_j = k,$$

states that every $\mathbf{b} \in B_{\mathbf{i}}$ is such that $k \leq b_j \lhd k + 1$, with $\lhd$ settled as above. The map $\mathrm{enc}\colon L_n \to L_n^{(n+1)^n}$ is such that, for every $t \in L_n$, $\mathrm{enc}(t) = (t_1, \ldots, t_{(n+1)^n})$ if and only if for every $i \in [(n+1)^n]$ and every $\mathbf{b} \in B_{\mathbf{i}}$,

$$t^{[n+1]}(\mathbf{b}) = t_i^{[n+1]}(\mathbf{b}).$$

We call $\mathrm{enc}(t)$ the *encoding* of the term t. We let $\mathrm{enc}(t)_i$ denote the ith component of $\mathrm{enc}(t)$, that is, $\mathrm{enc}(t)_i = t_i$.

Proof of Claim 72 on Page 75

Proof. We appeal repeatedly to Definition 3 and Fact 63 without explicit mention.

(i) We have to prove that the term $t_{(1,i)}$ given in equation (2.20) isolates X_i over $\{\mathbf{b} \mid b_i < 1\}$. Let $k \in [(n+1)^n]$. By definition, $\mathrm{enc}(X_i)_k = X_i$. Then, $\mathrm{enc}(\neg\neg X_i)_k = X_i$ if $(B_{\mathbf{k}})_i = 0$ and $\mathrm{enc}(\neg\neg X_i)_k = \top$ if $(B_{\mathbf{k}})_i > 1$. That is, $t_{(1,i)}$ isolates X_i over $\{\mathbf{b} \mid b_i < 1\}$.

(ii) We have to prove that the term $t_{(2,i)}$ given in equation (2.21) isolates X_i over $\{\mathbf{b} \mid 1 \le b_i\}$. Let $k \in [(n+1)^n]$. By (i), $\mathrm{enc}(t_{(1,i)})_k = X_i$ if $(B_{\mathbf{k}})_i = 0$ and $\mathrm{enc}(t_{(1,i)})_k = \top$ otherwise. Then, $\mathrm{enc}(t_{(2,i)})_k = \top$ if $\mathrm{enc}(t_{(1,i)})_k = X_i$, and $\mathrm{enc}(t_{(2,i)})_k = X_i$ if $\mathrm{enc}(t_{(1,i)})_k = \top$. So, Then, $\mathrm{enc}(t_{(2,i)})_k = X_i$ if $(B_{\mathbf{k}})_i > 0$ and $\mathrm{enc}(t_{(2,i)})_k = \top$ otherwise. That is, $t_{(2,i)}$ isolates X_i over $\{\mathbf{b} \mid 1 \le b_i\}$.

(iii) We have to prove that the term $t_{(3,i,j)}$ given in equation (2.22) isolates X_i over $\{\mathbf{b} \mid 1 \le b_i, b_j\}$. Let $k \in [(n+1)^n]$. By (i), $\mathrm{enc}(t_{(1,j)})_k = X_j$ if $(B_{\mathbf{k}})_j = 0$ and $\mathrm{enc}(t_{(1,j)})_k = \top$ otherwise. By (ii), $\mathrm{enc}(t_{(2,i)})_k = \top$ if $(B_{\mathbf{k}})_i = 0$ and $\mathrm{enc}(t_{(2,i)})_k = X_i$ otherwise. Then, $\mathrm{enc}(t_{(3,i,j)})_k$ is as follows: if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j = 0$, then $\mathrm{enc}(t_{(3,i,j)})_k = \top$; if $(B_{\mathbf{k}})_i = 0$ and $(B_{\mathbf{k}})_j > 0$, then $\mathrm{enc}(t_{(3,i,j)})_k = \top$; if $(B_{\mathbf{k}})_i > 0$ and $(B_{\mathbf{k}})_j = 0$, then $\mathrm{enc}(t_{(3,i,j)})_k = \top$; if $(B_{\mathbf{k}})_i, (B_{\mathbf{k}})_j > 0$, then $\mathrm{enc}(t_{(3,i,j)})_k = X_i$. Hence, $\mathrm{enc}(t_{(3,i,j)})_k = X_i$ if $(B_{\mathbf{k}})_i, (B_{\mathbf{k}})_j > 0$, and $\mathrm{enc}(t_{(3,i,j)})_k = \top$ otherwise, that is, $t_{(3,i,j)}$ isolates X_i over $\{\mathbf{b} \mid 1 \le b_i, b_j\}$.

(iv) We have to prove that the term $t_{(4,i,j)}$ given in equation (2.23) isolates $X_i \vee X_j$ over $\{\mathbf{b} \mid 1 \le \lfloor b_i \rfloor = \lfloor b_j \rfloor\}$. Let $k \in [(n+1)^n]$. By (ii), $\mathrm{enc}(t_{(2,i)})_k = \top$ if $(B_{\mathbf{k}})_i = 0$ and $\mathrm{enc}(t_{(2,i)})_k = X_i$ otherwise, and $\mathrm{enc}(t_{(2,j)})_k = \top$ if $(B_{\mathbf{k}})_j = 0$ and $\mathrm{enc}(t_{(2,j)})_k = X_j$ otherwise. So, for $s_1 = ((t_{(2,i)} \to t_{(2,j)}) \to t_{(2,j)})$, $\mathrm{enc}(s_1)$ is as follows: if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j = 0$, then $\mathrm{enc}(s_1)_k = \top$ (via $(\top \to \top) \to \top$); if $(B_{\mathbf{k}})_i = 0$ and $(B_{\mathbf{k}})_j > 0$, then $\mathrm{enc}(s_1)_k = \top$ (via $(\top \to X_j) \to X_j$); if $(B_{\mathbf{k}})_i > 0$ and $(B_{\mathbf{k}})_j = 0$, then $\mathrm{enc}(s_1)_k = \top$ (via $(X_i \to \top) \to \top$); if $(B_{\mathbf{k}})_i, (B_{\mathbf{k}})_j > 0$, there are three cases (via $(X_i \to X_j) \to X_j$): if $(B_{\mathbf{k}})_i < (B_{\mathbf{k}})_j$, then $\mathrm{enc}(s_1)_k = X_j$; if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j$, then $\mathrm{enc}(s_1)_k = X_i \vee X_j$; if $(B_{\mathbf{k}})_i > (B_{\mathbf{k}})_j$, then $\mathrm{enc}(s_1)_k = \top$. Hence, $\mathrm{enc}(s_1)_k = X_i \vee X_j$ if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j > 0$, $\mathrm{enc}(s_1)_k = X_j$ if $0 < (B_{\mathbf{k}})_i < (B_{\mathbf{k}})_j$, $\mathrm{enc}(s_1)_k = \top$ if $(B_{\mathbf{k}})_i > (B_{\mathbf{k}})_j > 0$, and $\mathrm{enc}(s_1)_k = \top$ otherwise. Similarly, for $s_2 = ((t_{(2,j)} \to t_{(2,i)}) \to t_{(2,i)})$, $\mathrm{enc}(s_2)_k = X_i \vee X_j$ if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j > 0$, $\mathrm{enc}(s_2)_k = \top$ if $0 < (B_{\mathbf{k}})_i < (B_{\mathbf{k}})_j$, $\mathrm{enc}(s_2)_k = X_i$ if $(B_{\mathbf{k}})_i > (B_{\mathbf{k}})_j > 0$, and $\mathrm{enc}(s_2)_k = \top$ otherwise. Therefore, $\mathrm{enc}(\theta_{4,i,j})_k = X_i \vee X_j$ if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j > 0$ and $\mathrm{enc}(\theta_{4,i,j})_k = \top$ otherwise, that is, $\theta_{4,i,j}$ isolates $X_i \vee X_j$

over $\{\mathbf{b} \mid 1 \leq \lfloor b_i \rfloor = \lfloor b_j \rfloor\}$.

(v) We have to prove that the term $t_{(5,i,j)}$ given in equation (2.24) isolates X_i over $\{\mathbf{b} \mid 1 \leq \lfloor b_j \rfloor < \lfloor b_i \rfloor\}$. Let $k \in [(n+1)^n]$. We have: $\mathrm{enc}((X_i \to X_j) \to X_j)_k = X_i \vee X_j$ if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j$, $\mathrm{enc}((X_i \to X_j) \to X_j)_k = \top$ if $(B_{\mathbf{k}})_i > (B_{\mathbf{k}})_j$ and $\mathrm{enc}((X_i \to X_j) \to X_j)_k = X_j$ if $(B_{\mathbf{k}})_i < (B_{\mathbf{k}})_j$; $\mathrm{enc}(t_{(2,i)} \vee t_{(2,j)})_k = X_i \vee X_j$ if $(B_{\mathbf{k}})_i = (B_{\mathbf{k}})_j > 0$ and $\mathrm{enc}(t_{(2,i)} \vee t_{(2,j)})_k = \top$ otherwise. Hence, $\mathrm{enc}(t_{(5,i,j)})_k = X_i$ if $(B_{\mathbf{k}})_i > (B_{\mathbf{k}})_j > 0$, and $\mathrm{enc}(t_{(5,i,j)})_k = \top$ otherwise, that is, $t_{(5,i,j)}$ isolates X_i over $\{\mathbf{b} \mid 1 \leq \lfloor b_j \rfloor < \lfloor b_i \rfloor\}$.

(vi) We have to prove that the term $t_{(6,i,j)}$ given in equation (2.25) isolates X_j over $\{\mathbf{b} \mid 0 \leq \lfloor b_j \rfloor \leq \lfloor b_i \rfloor\}$. Let $k \in [(n+1)^n]$. By (iii) and (v), for all k, $\mathrm{enc}(t_{(5,j,i)} \to t_{(3,j,i)})_k = X_j$ if $(B_{\mathbf{k}})_j = (B_{\mathbf{k}})_i > 0$ or $(B_{\mathbf{k}})_j > (B_{\mathbf{k}})_i > 0$, and $\mathrm{enc}(t_{(5,j,i)} \to t_{(3,j,i)})_k = \top$ otherwise. Hence, by (i), for all k, $\mathrm{enc}(t_{(6,i,j)})_k = X_j$ if $(B_{\mathbf{k}})_j = (B_{\mathbf{k}})_i > 0$ or $(B_{\mathbf{k}})_j > (B_{\mathbf{k}})_i > 0$ or $(B_{\mathbf{k}})_j = 0$, and $\mathrm{enc}(t_{(6,i,j)})_k = \top$ otherwise, that is, $t_{(6,i,j)}$ isolates X_j over $\{\mathbf{b} \mid 0 \leq \lfloor b_j \rfloor \leq \lfloor b_i \rfloor\}$.

The claim is proved. $\square$

Proof of Claim 73 on Page 76

Proof. By hypothesis, $\mathbf{a} \in [0,1]^n$ is such that $i \in \mathrm{par}(\mathbf{a})$ and $j \in \mathrm{par}(\mathbf{a})' = [n] \setminus \mathrm{par}(\mathbf{a})$. We appeal repeatedly to Claim 72, Definition 3 and Fact 63 without explicit mention. For $n = 2$, Lemma 58(i) settles the claim. So, we assume $n \geq 3$. We split the proof of the claim in two parts.

Part 1: We prove that the term,

$$r_{(i,\mathbf{a})} = \bigvee_{j' \in \mathrm{par}(\mathbf{a})'} t_{(1,j')} \vee \bigvee_{i' \in \mathrm{par}(\mathbf{a}) \setminus \{i\}} t_{(3,i,i')},$$

with the stipulation that $t_{(2,i)}$ substitutes $\bigvee_{i' \in \mathrm{par}(\mathbf{a}) \setminus \{i\}} t_{(3,i,i')}$ if $\mathrm{par}(\mathbf{a}) = \{i\}$, isolates X_i over,

$$\begin{aligned} D_{(i,\mathbf{a})} &= \{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))\} \\ &= \{\mathbf{b} \mid b_{j'} < 1 \leq b_{i'} \text{ for all } (j', i') \in \mathrm{par}(\mathbf{a})' \times \mathrm{par}(\mathbf{a})\}. \end{aligned}$$

First suppose that $\mathrm{par}(\mathbf{a}) = \{i\}$, so that $\mathrm{par}(\mathbf{a})' = \{j, j', \dots\}$. We have $r_{(i,\mathbf{a})} = t_{(1,j)} \vee t_{(1,j')} \vee \dots \vee t_{(2,i)}$. Let $k \in [(n+1)^n]$. Let $B_{\mathbf{k}} \in D_{(i,\mathbf{a})}$, that is, $0 = (B_{\mathbf{k}})_j, (B_{\mathbf{k}})_{j'}, \dots < 1 \leq (B_{\mathbf{k}})_i$. Hence, $\mathrm{enc}(t_{(1,j)})_k = X_j$, $\mathrm{enc}(t_{(1,j')})_k = X_{j'}$, …, $\mathrm{enc}(t_{(2,i)})_k = X_i$. But then, $\mathrm{enc}(r_{(i,\mathbf{a})})_k = X_i$. Otherwise, let $B_{\mathbf{k}} \notin D_{(i,\mathbf{a})}$, that is, either $1 \leq (B_{\mathbf{k}})_j$, or $1 \leq (B_{\mathbf{k}})_{j'}$, …, or $(B_{\mathbf{k}})_i = 0$. Hence, either $\mathrm{enc}(t_{(1,j)})_k = \top$, or $\mathrm{enc}(t_{(1,j')})_k = \top$, …, or $\mathrm{enc}(t_{(2,i)})_k = \top$. But then, $\mathrm{enc}(r_{(i,\mathbf{a})})_k = \top$. Noticing that $B_{\mathbf{k}}$ lies entirely inside or entirely outside $D_{(i,\mathbf{a})}$, we conclude that if $\mathrm{par}(\mathbf{a}) = \{i\}$, then $r_{(i,\mathbf{a})}$ isolates X_i over $D_{(i,\mathbf{a})}$.

Next suppose that $\mathrm{par}(\mathbf{a}) = \{i, i', \dots\}$ and $\mathrm{par}(\mathbf{a})' = \{j, j', \dots\}$. We have $r_{(i,\mathbf{a})} = t_{(1,j)} \vee t_{(1,j')} \vee \dots \vee t_{(3,i,i')} \vee \dots$. Let $k \in [(n+1)^n]$. Let $B_{\mathbf{k}} \in D_{(i,\mathbf{a})}$, that is, $0 = (B_{\mathbf{k}})_j, (B_{\mathbf{k}})_{j'}, \dots < 1 \leq (B_{\mathbf{k}})_i, (B_{\mathbf{k}})_{i'}, \dots$, so that, $\mathrm{enc}(t_{(1,j)})_k = X_j$, $\mathrm{enc}(t_{(1,j')})_k = X_{j'}$, …, $\mathrm{enc}(t_{(3,i,i')})_k = X_i$, …. But then, $\mathrm{enc}(r_{(i,\mathbf{a})})_k = X_i$. Otherwise, let $B_{\mathbf{k}} \notin D_{(i,\mathbf{a})}$, that is, either $1 \leq (B_{\mathbf{k}})_j$, or $1 \leq (B_{\mathbf{k}})_{j'}$, …, or $(B_{\mathbf{k}})_i = 0$, or $(B_{\mathbf{k}})_{i'} = 0$, …. Hence, either $\mathrm{enc}(t_{(1,j)})_k = \top$, or $\mathrm{enc}(t_{(1,j')})_k = \top$, …, or $\mathrm{enc}(t_{(3,i,i')})_k = \top$, …. But then, $\mathrm{enc}(r_{(i,\mathbf{a})})_k = \top$. Noticing that $B_{\mathbf{k}}$ lies entirely inside or entirely outside $D_{(i,\mathbf{a})}$, we conclude that if $\mathrm{par}(\mathbf{a}) = \{i, i', \dots\}$ and $\mathrm{par}(\mathbf{a})' = \{j, j', \dots\}$, then $r_{(i,\mathbf{a})}$ isolates X_i over $D_{(i,\mathbf{a})}$.

The case $\mathrm{par}(\mathbf{a}) = \{i, i', \dots\}$ and $\mathrm{par}(\mathbf{a})' = \{j\}$ is similar.

Part 2: We prove that the term,

$$s_{(i,\mathbf{a})} = t_{(5,i,j)} \vee \bigvee_{j' < j'' \in \mathrm{par}(\mathbf{a})'} t_{(4,j',j'')} \vee \bigvee_{i' \in \mathrm{par}(\mathbf{a}) \setminus \{i\}} (t_{(6,j,i')} \to t_{(5,i,j)}),$$

isolates X_i over,

$$\begin{aligned} E_{(i,\mathbf{a})} &= \{\mathbf{b} \in [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))\} \\ &= \bigcup_{m \in [n]} \{\mathbf{b} \mid m \leq b_{j'} < m+1 \leq b_{i'} \text{ for all } (j', i') \in \mathrm{par}(\mathbf{a})' \times \mathrm{par}(\mathbf{a})\}. \end{aligned}$$

First suppose that $\mathrm{par}(\mathbf{a}) = \{i, i', \dots\}$ and $\mathrm{par}(\mathbf{a})' = \{j, j', j'', \dots\}$. We have $s_{(i,\mathbf{a})} = t_{(5,i,j)} \vee t_{(4,j',j'')} \vee \dots \vee (t_{(6,j,i')} \to t_{(5,i,j)}) \vee \dots$. Let $k \in [(n+1)^n]$. Let $B_{\mathbf{k}} \in E_{(i,\mathbf{a})}$, that is, there exists $1 \leq m \leq n$ such that $m = (B_{\mathbf{k}})_j, (B_{\mathbf{k}})_{j'}, (B_{\mathbf{k}})_{j''}, \dots < m+1 \leq (B_{\mathbf{k}})_i, (B_{\mathbf{k}})_{i'}, \dots$, so that, $\mathrm{enc}(t_{(5,i,j)})_k = X_i$, $\mathrm{enc}(t_{(4,j',j'')})_k = X_{j'} \vee X_{j''}$, …, $\mathrm{enc}(t_{(6,j,i')})_k = \top$, …. Hence, $\mathrm{enc}(X_i \vee (X_{j'} \vee X_{j''}) \vee \dots \vee (\top \to X_i) \vee \dots)_k = \mathrm{enc}(s_{(i,\mathbf{a})})_k = X_i$. Otherwise, let $B_{\mathbf{k}} \notin E_{(i,\mathbf{a})}$. Let $(B_{\mathbf{k}})_j = m$. If $m = 0$, then $\mathrm{enc}(t_{(5,i,j)})_k = \top$. Otherwise, $1 \leq m \leq n+1$. Since we are assuming $B_{\mathbf{k}} \notin E_{(i,\mathbf{a})}$, either

$(B_{\mathbf{k}})_{j'} \neq m$, or $(B_{\mathbf{k}})_{j''} \neq m$, ..., or $(B_{\mathbf{k}})_i < m+1$, or $(B_{\mathbf{k}})_{i'} < m+1$, ... If $1 < |\{\lfloor j \rfloor, \lfloor j' \rfloor, \lfloor j'' \rfloor, \dots\}|$, then if w.l.o.g. $\lfloor j' \rfloor \neq \lfloor j'' \rfloor$, we have $\mathrm{enc}(t_{(4,j',j'')})_k = \top$. Otherwise, $\{m\} = \{\lfloor j \rfloor, \lfloor j' \rfloor, \lfloor j'' \rfloor, \dots\}$. But then, either $(B_{\mathbf{k}})_i < m+1$, or $(B_{\mathbf{k}})_{i'} < m+1$, ..., say w.l.o.g. $(B_{\mathbf{k}})_{i'} < m+1$, and we have $\mathrm{enc}(t_{(6,j,i')})_k = X_{i'}$. But we also have, $\mathrm{enc}(t_{(5,i,j)})_k = \top$, so that $\mathrm{enc}(t_{(6,j,i')} \to t_{(5,i,j)})_k = \top$. Hence, in all cases, $\mathrm{enc}(s_{(i,\mathbf{a})})_k = \top$. Noticing that $B_{\mathbf{k}}$ lies entirely inside or entirely outside $E_{(i,\mathbf{a})}$, we conclude that if $\mathrm{par}(\mathbf{a}) = \{i, i', \dots\}$ and $\mathrm{par}(\mathbf{a})' = \{j, j_1', \dots\}$, then $s_{(i,\mathbf{a})}$ isolates X_i over $E_{(i,\mathbf{a})}$.

The cases where either $\mathrm{par}(\mathbf{a}) = \{i\}$ or (exclusively) $\mathrm{par}(\mathbf{a})' = \{j\}$ are subsumed by the previous argument. Indeed, if $\mathrm{par}(\mathbf{a}) = \{i\}$, we have $s_{(i,\mathbf{a})} = t_{(5,i,j)} \vee t_{(4,j',j'')} \vee \dots$, and if $\mathrm{par}(\mathbf{a})' = \{j\}$, we have $s_{(i,\mathbf{a})} = t_{(5,i,j)} \vee (t_{(6,j,i')} \to t_{(5,i,j)}) \vee \dots$.

The claim is proved. $\square$

Proof of Claim 74 on Page 77

Proof. We have to prove that the term $r_{(i,\mathbf{u})}$ in equation (2.28) isolates X_i over $D_{(i,\mathbf{u})}$, and the term $s_{(i,\mathbf{u})}$ in equation (2.29) isolates X_i over $E_{(i,\mathbf{u})}$. We repeatedly apply Definitions 31 and 32 without explicit mention.

For the first statement, if $\mathbf{b} \in [1, n+1]^n$, then $r_{(i,\mathbf{a})}(\mathbf{b}) = \top^{[0,n+1]}$ by Lemma 71(i), so that $r^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}$ by Definition 3, as required. Otherwise, let $\mathbf{b} \notin [1, n+1]^n$ such that $\mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$. Note that $\mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \subseteq \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$, since we settled $\mathbf{a} = \mathbf{u}$. If $\mathbf{b} \notin \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, then by construction,

$$t^{[0,n+1]}_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}}(\mathbf{b}) < 1.$$

Thus, since $1 \leq r_{(i,\mathbf{a})}(\mathbf{b})$ by Lemma 71(i), we have that $r^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}$ by Definition 3, as required. Otherwise, if $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, then by construction,

$$t^{[0,n+1]}_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}}(\mathbf{b}) = \top^{[0,n+1]},$$

so that $r^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = X_i^{[0,n+1]}$ by Definition 3, as required.

For the second statement, if $\mathbf{b} \notin [1, n+1]^n$, then $s_{(i,\mathbf{a})}(\mathbf{b}) = \top^{[0,n+1]}$ by Lemma 71(i), so that $s^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}$ by Definition 3, as required. Otherwise, let $\mathbf{b} \in [1, n+1]^n$ such that $\mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$. Note that $\mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \subseteq \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$, since we settled $\mathbf{a} = \mathbf{u}$. If $\mathbf{b} \notin \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, then by construction,

$$m \leq t^{[0,n+1]}_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}}(\mathbf{b}) < m+1,$$

where $m = \min\{\lfloor b_i \rfloor \mid i \in [n]\}$. But, by Lemma 71(i), $m \leq s_{(i,\mathbf{a})}(\mathbf{b})$. Thus, $s^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}$ by Definition 3, as required. Otherwise, if $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, then by construction,

$$t^{[0,n+1]}_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}}(\mathbf{b}) = \top^{[0,n+1]},$$

so that $s^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = X^{[0,n+1]}_i$ by Definition 3, as required.

The claim is proved. □

Proof of Claim 75 on Page 77

Proof. We have to prove that the term $r_{(i,\mathbf{u})}$ in equation (2.30) isolates X_i over $D_{(i,\mathbf{u})}$, and the term $s_{(i,\mathbf{u})}$ in equation (2.31) isolates X_i over $E_{(i,\mathbf{u})}$. We repeatedly apply Definitions 31 and 32 without explicit mention.

As regards to $r_{(i,\mathbf{u})}$, let $s = t_{\{1\backslash j_1,\dots,m-n\backslash j_{m-n}\}} \to r_{(i,\mathbf{a})}$, and let $s' = \bigwedge_{j=1}^{k} r_{(i,\mathbf{u}_j)}$. Reasoning along the lines of Claim 74, we observe that by construction s isolates X_i over $\{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(F)\}$. By the induction hypothesis, for all $j \in [k]$, $r_{(i,\mathbf{u}_j)}$ isolates X_i over $\{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}_j))\}$. Note that,

$$\{\mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathrm{realm}(\mathrm{sibl}(\mathbf{u}_1)), \dots, \mathrm{realm}(\mathrm{sibl}(\mathbf{u}_k))\},$$

form a partition of $\mathrm{realm}(F)$. Now, if $\mathbf{b} \in [1, n+1]^n$ or $\mathbf{b} \notin \mathrm{realm}(F)$, then $s^{[0,n+1]}(\mathbf{b}) = \top^{[0,n+1]}$ and $r^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}$, as required. Otherwise, let $\mathbf{b} \notin [1, n+1]^n \cap \mathrm{realm}(F)$. If $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}_j))$ for some $j \in [k]$, then $s'^{[0,n+1]}(\mathbf{b}) = X^{[0,n+1]}_i$, so that since $s^{[0,n+1]}(\mathbf{b}) = X^{[0,n+1]}_i(\mathbf{b})$, we have $r^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}$ by Definition 3, as required. Otherwise, if $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, then $s'^{[0,n+1]}(\mathbf{b}) = \top^{[0,n+1]}$, so that since $s^{[0,n+1]}(\mathbf{b}) = X^{[0,n+1]}_i(\mathbf{b})$, we have $r^{[0,n+1]}_{(i,\mathbf{u})}(\mathbf{b}) = X^{[0,n+1]}_i$ by Definition 3, as required. Hence, $r_{(i,\mathbf{u})}$ isolates X_i over $D_{(i,\mathbf{u})}$. As regards to $s_{(i,\mathbf{u})}$, a similar argument shows that $s_{(i,\mathbf{u})}$ isolates X_i over $E_{(i,\mathbf{u})}$.

The claim is proved. □

Proof of Claim 76 on Page 78

Proof. Let $i \in [n]$. For every $\mathbf{a} \in [0,1]^n$, we let $\mathrm{par}(\mathbf{a})' = [n] \setminus \mathrm{par}(\mathbf{a})$. We split the proof of the claim in two parts. We repeatedly apply Definitions 31 and 32 without explicit mention.

Part 1: We have to prove that the term $v_i = \neg\neg X_i$ isolates X_i over,

$$J_i = \{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a})), \mathbf{a} \in [0,1]^n, i \in \mathrm{par}(\mathbf{a})'\}.$$

By Claim 72(i), $\neg\neg X_i$ isolates X_i over $H_i = \{\mathbf{b} \mid b_i < 1\}$. Hence, it is sufficient to prove that $J_i = H_i$. If $\mathbf{b} \in J_i$, then $b_j < 1$ for every $j \in \mathrm{par}(\mathbf{a})'$, hence $\mathbf{b} \in H_i$. Conversely, let $\mathbf{b} \in H_i$ (so that $\mathbf{b} \notin [1, n+1]^n$), and let $\mathbf{a} = \mathrm{controller}(\mathbf{b})$. Then, $i \notin \mathrm{par}(\mathbf{a})$ and $\mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$. Hence, $\mathbf{b} \in J_i$.

Part 2: We have to prove that the term,

$$w_i = \left(\bigwedge_{\mathbf{a} \in A} (r_{(i,\mathbf{a})} \wedge s_{(i,\mathbf{a})}) \right) \to (\neg\neg X_i \to X_i),$$

where $A = \{\mathbf{a} \in [0,1]^n \mid i \in \mathrm{par}(\mathbf{a})\}$, and $r_{(i,\mathbf{a})}$ and $s_{(i,\mathbf{a})}$ are as in Lemma 71(i), isolates X_i over,

$$K_i = \{\mathbf{b} \in [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a})), \mathbf{a} \in [0,1]^n, i \in \mathrm{par}(\mathbf{a})'\}.$$

By Claim 72(ii), $\neg\neg X_i \to X_i$ isolates X_i over $H_i = \{\mathbf{b} \mid 1 \leq b_i\}$. Clearly, $K_i \subseteq H_i$. Let $\mathbf{b} = (b_1, \ldots, b_n) \in H_i$, so that $1 \leq b_i$. If $\mathbf{b} \notin K_i$, then either there exists $\mathbf{a} \in [0,1]^n$ such that $\mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$ and $i \in \mathrm{par}(\mathbf{a})$, or else $\mathbf{b} \notin [1, n+1]^n$, so that there exists $j \neq i$ such that $b_j < 1$, and again there exists $\mathbf{a} \in [0,1]^n$ such that $\mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$ and $i \in \mathrm{par}(\mathbf{a})$. Hence, by Lemma 71(i), $r_{(i,\mathbf{a})}^{[0,n+1]}(\mathbf{b}) = X_i^{[0,n+1]}(\mathbf{b})$ or $s_{(i,\mathbf{a})}^{[0,n+1]}(\mathbf{b}) = X_i^{[0,n+1]}(\mathbf{b})$. Therefore, if $\mathbf{b} \notin K_i$, then $w_i^{[0,n+1]}(\mathbf{b}) = \top^{[0,n+1]}$, as required. Otherwise, if $\mathbf{b} \in K_i$, then there not exists $\mathbf{a} \in A$ such that $\mathbf{b} \in \mathrm{realm}(\mathrm{neigh}(\mathbf{a}))$ and $i \in \mathrm{par}(\mathbf{a})$, and then, by Lemma 71(i), we have that $r_{(i,\mathbf{a})}^{[0,n+1]}(\mathbf{b}) = s_{(i,\mathbf{a})}^{[0,n+1]}(\mathbf{b}) = \top^{[0,n+1]}$ for every $\mathbf{a} \in A$. Therefore, if $\mathbf{b} \in K_i$, then $w_i^{[0,n+1]}(\mathbf{b}) = X_i$, as required.

The claim is proved. $\square$

Proofs of Claims in Lemma 77 on Page 78

We collect below the proofs of the claims in Lemma 77.

Proof of Claim 78 on Page 79

Proof. We prove that $\hat{t}$ satisfies the claim, that is, $\hat{t}$ isolates t over $\{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})), \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^n \text{ or } t^{[0,1]}(\mathbf{u}) < 1\}$.

Let $\mathbf{b} \in [0, n+1]^n$. If $\mathbf{b} \in [1, n+1]^n$, then since $v_1^{[0,n+1]}(\mathbf{b}) = \cdots = v_n^{[0,n+1]}(\mathbf{b}) = \top^{[0,n+1]}$ by Lemma 71(iii), and $t \in L_n^+$ by hypothesis, we have that $\hat{t}(\mathbf{b}) = \top^{[0,n+1]}$, as required. Otherwise, if $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ for some $\mathbf{u} \in \tilde{U}$ such that $\mathbf{u} \in [0,1)^n$, then since $v_1^{[0,n+1]}(\mathbf{b}) = X_1^{[0,n+1]}(\mathbf{b})$, $\ldots$, $v_n^{[0,n+1]}(\mathbf{b}) = X_n^{[0,n+1]}(\mathbf{b})$ by Lemma 71(iii), we have that $\hat{t}(\mathbf{b}) = t^{[0,n+1]}(\mathbf{b})$, as required. Otherwise, let $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$ for some $\mathbf{u} \in \tilde{U}$ such that $\emptyset \subset \mathrm{par}(\mathbf{u}) \subset [n]$. We distinguish two cases. As a first case, suppose that $t^{[0,1]}(\mathbf{u}) < 1$. By the third clause of equation (2.18) in Lemma 68, $t^{[0,n+1]}(\mathbf{b}) = t^{[0,1]}(\mathrm{controller}(\mathbf{b})) + j$, where in this case, since $\mathbf{b} \notin [1, n+1]^n$ we have $j = 0$. Hence,

$$\begin{aligned}
\hat{t}^{[0,n+1]}(\mathbf{b}) &= t^{[0,n+1]}_{\{1\backslash v_1, \ldots, n\backslash v_n\}}(\mathbf{b}) \\
&= t^{[0,n+1]}_{\{k\backslash\top \,\mid\, k\in\mathrm{par}(\mathbf{u})\}\cup\{k\backslash k \,\mid\, k\notin\mathrm{par}(\mathbf{u})\}}(\mathbf{b}) && \text{L. 71(iii)} \\
&= t^{[0,1]}_{\{k\backslash\top \,\mid\, k\in\mathrm{par}(\mathbf{u})\}\cup\{k\backslash k \,\mid\, k\notin\mathrm{par}(\mathbf{u})\}}(\mathrm{controller}(\mathbf{b})) + j \\
&= t^{[0,n+1]}(\mathbf{b}),
\end{aligned}$$

as required. As a second case, suppose that $t^{[0,1]}(\mathbf{u}) = 1$. By the fourth clause of equation (2.18) in Lemma 68, there exists a 1-reproducing term $\bar{t} \in L_{\mathrm{par}(\mathbf{u})}$ such that $t^{[0,n+1]}(\mathbf{b}) = \bar{t}^{[0,n+1]}(\mathbf{b})$. Hence,

$$\begin{aligned}
\hat{t}^{[0,n+1]}(\mathbf{b}) &= t^{[0,n+1]}_{\{1\backslash v_1, \ldots, n\backslash v_n\}}(\mathbf{b}) \\
&= t^{[0,n+1]}_{\{k\backslash\top \,\mid\, k\in\mathrm{par}(\mathbf{u})\}\cup\{k\backslash k \,\mid\, k\notin\mathrm{par}(\mathbf{u})\}}(\mathbf{b}) && \text{Lemma 71(iii)} \\
&= \bar{t}^{[0,n+1]}_{\{j\backslash\top \,\mid\, j\in\mathrm{par}(\mathbf{u})\}\cup\{j\backslash j \,\mid\, j\notin\mathrm{par}(\mathbf{u})\}}(\mathbf{b}) \\
&= \top^{[0,n+1]},
\end{aligned}$$

as required (the last equality holds because $\bar{t}$ is 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$). Hence, $\hat{t}$ satisfies the claim.

A similar argument shows that $\check{t}$ satisfies the claim. □

Proof of Claim 79 on Page 79

Proof. We prove that $\hat{t}$ isolates t over the union of $[1, n+1]^n$ and $\{\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u})) \mid \mathbf{u} \in \tilde{U}, \mathbf{u} \in [0,1)^n \text{ or } t^{[0,1]}(\mathbf{u}) < 1\}$. Since $t = \neg t'$ with $t' \in L_n^+$ by hypothesis, we can apply Claim 78 to t'. Let $\mathbf{b} \in [0, n+1]^n$. If $\mathbf{b} \in [1, n+1]^n$, then,

$$\begin{aligned}
\hat{t}^{[0,n+1]}(\mathbf{b}) &= (\neg t')^{[0,n+1]}_{\{1\backslash v_1,\ldots,n\backslash v_n\}}(\mathbf{b}) \\
&= \neg^{[0,n+1]}(t'^{[0,n+1]}_{\{1\backslash v_1,\ldots,n\backslash v_n\}}(\mathbf{b})) \\
&= \neg^{[0,n+1]}\top^{[0,n+1]} && \text{Claim 78} \\
&= 0 \\
&= t^{[0,n+1]}(\mathbf{b}), && \text{Lemma 68}
\end{aligned}$$

as required. Otherwise, let $\mathbf{b} \notin [1, n+1]^n$, and suppose that $\mathbf{u} \in \tilde{U}$ is such that $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$. If $\mathbf{u} \notin [0,1)^n$ and $t^{[0,1]}(\mathbf{u}) = 1$, then $t'^{[0,1]}(\mathbf{u}) = 0$ and,

$$\begin{aligned}
\hat{t}^{[0,n+1]}(\mathbf{b}) &= (\neg t')^{[0,n+1]}_{\{1\backslash v_1,\ldots,n\backslash v_n\}}(\mathbf{b}) \\
&= \neg^{[0,n+1]}(t'^{[0,n+1]}_{\{1\backslash v_1,\ldots,n\backslash v_n\}}(\mathbf{b})) \\
&= \neg^{[0,n+1]}(t'^{[0,n+1]}(\mathbf{b})) && \text{Lemma 71(iii)} \\
&= \neg^{[0,n+1]}(t'^{[0,1]}(\mathrm{controller}(\mathbf{b})) + j) && \text{Lemma 68 with } j = 0 \\
&= \neg^{[0,n+1]}(t'^{[0,1]}(\mathbf{u})) \\
&= \top^{[0,n+1]},
\end{aligned}$$

as required. Otherwise, if $\mathbf{u} \in [0,1)^n$ or $t^{[0,1]}(\mathbf{u}) < 1$, then,

$$\begin{aligned}
\hat{t}^{[0,n+1]}(\mathbf{b}) &= (\neg t')^{[0,n+1]}_{\{1\backslash v_1,\ldots,n\backslash v_n\}}(\mathbf{b}) \\
&= \neg^{[0,n+1]}(t'^{[0,n+1]}_{\{1\backslash v_1,\ldots,n\backslash v_n\}}(\mathbf{b})) \\
&= \neg^{[0,n+1]}(t'^{[0,n+1]}(\mathbf{b})) && \text{Claim 78} \\
&= (\neg t')^{[0,n+1]}(\mathbf{b}) \\
&= t^{[0,n+1]}(\mathbf{b}), && \text{Lemma 68}
\end{aligned}$$

as required. Hence, $\hat{t}$ satisfies the claim. $\square$

Proof of Claim 80 on Page 79

Proof. We prove that $\dot{t}_{\mathbf{u}}$ isolates t over $\{\mathbf{b} \notin [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\}$. The argument for proving that $\ddot{t}_{\mathbf{u}}$ isolates t over $\{\mathbf{b} \in [1, n+1]^n \mid \mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))\}$ is similar.

Let $\mathbf{b} \in [0, n+1]^n$. If either $\mathbf{b} \in [1, n+1]^n$ or $\mathbf{b} \notin \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, then since $r^{[0,n+1]}_{(i_1,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}, \ldots, r^{[0,n+1]}_{(i_m,\mathbf{u})}(\mathbf{b}) = \top^{[0,n+1]}$ by Lemma 71(ii), and t is 1-reproducing in $L_{\mathrm{par}(\mathbf{u})}$ by hypothesis, we have that $\dot{t}_{\mathbf{u}}(\mathbf{b}) = \top^{[0,n+1]}$, as required. If $\mathbf{b} \notin [1, n+1]^n$ and $\mathbf{b} \in \mathrm{realm}(\mathrm{sibl}(\mathbf{u}))$, then since $r^{[0,n+1]}_{(i_1,\mathbf{u})}(\mathbf{b}) = X_{i_1}(\mathbf{b})^{[0,n+1]}, \ldots, r^{[0,n+1]}_{(i_m,\mathbf{u})}(\mathbf{b}) = X_{i_m}(\mathbf{b})^{[0,n+1]}$ by Lemma 71(ii), we have that,

$$t^{[0,n+1]}_{\{i_1 \backslash r_{(i_1,\mathbf{u})}, \ldots, i_m \backslash r_{(i_m,\mathbf{u})}\}}(\mathbf{b}) = t^{[0,n+1]}_{\{i_1 \backslash i_1, \ldots, i_m \backslash i_m\}}(\mathbf{b}) = t^{[0,n+1]}(\mathbf{b}),$$

as required. Hence, $\dot{t}_{\mathbf{u}}$ satisfies the claim. $\square$

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