BL-Functions and Free BL-Algebra

Simone Bova
bova@unisi.it
www.mat.unisi.it/~bova

Department of Mathematics and Computer Science
University of Siena (Italy)

December 19, 2008
Ph.D. Thesis Defense

Outline

Outline

Motivation

Basic Logic Truthfunctions

Background

Contribution

Conclusion

Fuzzy Logics

Fuzzy logics are propositional logics over $\odot, \rightarrow, \bot$ st:

- variables $X_1, X_2, \ldots$ are interpreted over $[0, 1]$;
- $\top = \bot \rightarrow \bot$ and $\bot$ are interpreted over 1 and 0;
- $\odot$ and $\rightarrow$ are interpreted over binary functions on $[0, 1]$;
- $\neg \cdot = \cdot \rightarrow \bot$.

Fuzzy conjunction and implication *must* maintain:

- the behavior of Boolean counterparts over $\{0, 1\}^2$;
- *intuitive* properties of Boolean counterparts over $[0, 1]^2$;
- validity (and power) of modus ponens.

Continuous Triangular Norms

Definition (Continuous Triangular Norm, Residuum)

A *(continuous) triangular norm* $*$ is a continuous binary function on $[0, 1]$ that is associative, commutative, monotone ($x \leqslant y$ implies $x * z \leqslant y * z$) and has 1 as unit ($x * 1 = x$). Given a continuous triangular norm $*$, its *residuum* is the binary function $\rightarrow_*$ on $[0, 1]$ defined by $x \rightarrow_* y = max\{z : x * z \leqslant y\}$.

Triangular norms and their residua provide suitable interpretations for fuzzy conjunction and implication.

Triangular Tautologies

$t, s, r, \ldots$ terms over $\odot, \rightarrow, \perp$ and variables $X_1, X_2, \ldots$.

Definition (Triangular Tautology)

t is a *triangular tautology* iff t evaluates identically to 1 for every assignment of the variables in $[0, 1]$ and every interpretation of $\odot$ over a triangular norm, $*$, and of $\rightarrow$ over the corresponding residuum, $\rightarrow_*$.

Hájek's Basic Logic

Hájek's *Basic logic*:

(BL1) $(r \to s) \to ((s \to t) \to (r \to t))$

(BL2) $(r \odot s) \to r$

(BL3) $(r \odot (r \to s)) \to (s \odot (s \to r))$

(BL4) $(r \to (s \to t)) \to ((r \odot s) \to t)$

(BL5) $((r \odot s) \to t) \to (r \to (s \to t))$

(BL6) $((r \to s) \to t) \to (((s \to r) \to t) \to t)$

(BL7) $\bot \to r$

(MP) $r, r \to s \vdash_{BL} s$

Theorem (Cignoli et al., 2000)

$\vdash_{BL} t$ iff t is a triangular tautology.

BL-Algebras

$x \wedge y = x \odot (x \to y)$ aka *divisibility*,
$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
$\neg x = x \to \bot$,
$\top = \neg \bot$.

Definition (BL-Algebras)

A *BL-algebra* is an algebra $(A, \odot, \to, \bot)$ of type $(2, 2, 0)$ st:

(*i*) $(A, \odot, \top)$ is a commutative monoid;
(*ii*) $(A, \vee, \wedge, \top, \bot)$ is a bounded lattice;
(*iii*) *residuation* holds, ie $x \odot y \leqslant z$ iff $y \leqslant x \to z$;
(*iv*) *prelinearity* holds, ie $(x \to y) \vee (y \to x) = \top$.

Free BL-Algebra and Basic Logic Truthfunctions

BL-algebras form the equivalent algebraic semantics of Basic logic.

The free n-generated BL-algebra is isomorphic to the Lindenbaum-Tarski algebra of the n-variate fragment of Basic logic.

An explicit description of the free n-generated BL-algebra is equivalent to an explicit description of the *truthfunctions* of the n-variate fragment of Basic logic.

Outline

Generic BL-Algebra

Definition (Generic BL-Algebra)

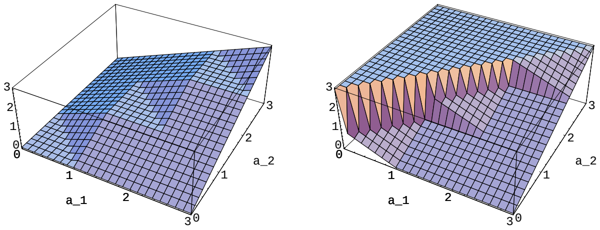
$(n+1)[0,1] = ([0,n+1], \odot^{(n+1)[0,1]}, \rightarrow^{(n+1)[0,1]}, \bot^{(n+1)[0,1]})$ is the algebra of type $(2,2,0)$ such that $\bot^{(n+1)[0,1]} = 0$, and for all $a_1, a_2 \in [0, n+1]$,

$$a_1 \odot^{(n+1)[0,1]} a_2 = \begin{cases} \min\{a_1, a_2\} & \text{if } \lfloor a_1 \rfloor \neq \lfloor a_2 \rfloor \\ (a_1 - j \odot^{[0,1]_{WH}} a_2 - j) + j & \text{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = j \end{cases}$$

$$a_1 \rightarrow^{(n+1)[0,1]} a_2 = \begin{cases} n+1 & \text{if } a_1 \leqslant a_2 \\ a_2 & \text{if } \lfloor a_2 \rfloor < \lfloor a_1 \rfloor \\ (a_1 - j \rightarrow^{[0,1]_{WH}} a_2 - j) + j & \text{if } \lfloor a_1 \rfloor = \lfloor a_2 \rfloor = j \end{cases}$$

where $a_1 \odot^{[0,1]_{WH}} a_2 = \max\{0, a_1 + a_2 - 1\}$,
and $a_1 \rightarrow^{[0,1]_{WH}} a_2 = \min\{1, a_2 + 1 - a_1\}$.

Generic BL-Algebra | *Case* $n = 2$



(a) $a_1 \odot^{3[0,1]} a_2$. (b) $a_1 \rightarrow^{3[0,1]} a_2$.

Figure: $3[0,1] = ([0,3], \odot^{3[0,1]}, \rightarrow^{3[0,1]}), \perp^{3[0,1]} = 0)$.

Generic BL-Algebra

Theorem (Aglianó and Montagna, 2003)

$(n+1)[0,1]$ *generates the variety generated by all the n-generated BL-algebras.*

Let F_n be the smallest set of n-ary functions from $[0, n+1]^{[n]}$ to $[0, n+1]$ that contains the n-ary projection functions $x_1, \ldots, x_n$, the n-ary constant function 0, and is closed under pointwise application of $\odot^{(n+1)[0,1]}$ and $\to^{(n+1)[0,1]}$.

Corollary

The free n-generated BL-algebra is isomorphic to the set F_n*, equipped with the operations of* $(n+1)[0,1]$ *defined pointwise.*

McNaughton Functions

Definition (McNaughton Function)

n positive integer, $I \subseteq [n] = \{1, \dots, n\}$. A continuous $|I|$-ary function $f\colon [0,1]^I \to [0,1]$ is a *McNaughton function* iff there are $|I|$-ary linear polynomials with integer coefficients $p_1, \dots, p_k\colon \mathbb{R}^I \to \mathbb{R}$ st, for every $\mathbf{a}$ in $[0,1]^I$, $f(\mathbf{a}) = p_j(\mathbf{a})$ for some $j \in [k]$. f is said *positive* if $f(\{1\}^I) = 1$, and *negative* ow.

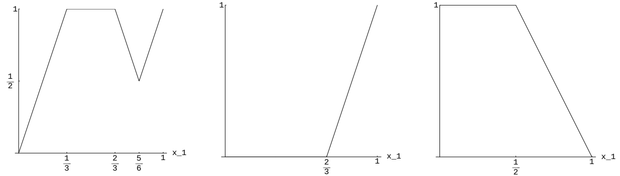
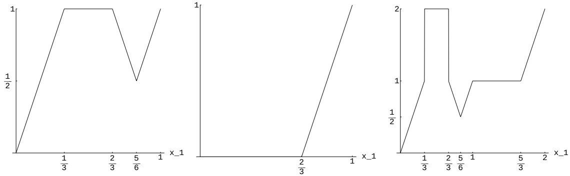


Figure: Unary McNaughton functions from $[0,1]^{\{1\}} \to [0,1]$.

Free BL-Algebra over One Generator

$f\colon [0,2]^{\{1\}} \rightarrow [0,2] \in F_1$ have two forms.

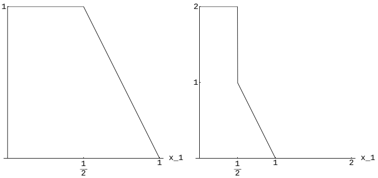


(a) (b) (c)

Figure: Form 1: (a), (b) positive McNaughton functions; (c) in F_1.

Free BL-Algebra over One Generator

$f\colon [0,2]^{\{1\}} \to [0,2] \in F_1$ have two forms.

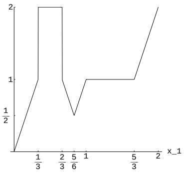


(a) (b)

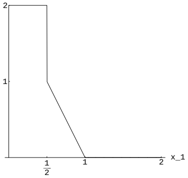
Figure: Form 2: (a) negative McNaughton function; (b) in F_1.

Free BL-Algebra over One Generator

$f\colon [0,2]^{\{1\}} \to [0,2] \in F_1$ have two forms.



(a) Form 1.



(b) Form 2.

Figure: F_1.

References

P. Aglianó and F. Montagna.
Varieties of BL-Algebras I: General Properties.
Journal of Pure and Applied Algebra, 181:105–129, 2003.

S. Aguzzoli and B. Gerla.
Normal Forms for the One-Variable Fragment of Hájek's Basic Logic.
Proceedings of ISMVL'05, 284–289, 2005.

M. Baaz, P. Hájek, F. Montagna, and H. Veith.
Complexity of t-Tautologies.
Annals of Pure and Applied Logic, 113:3–11, 2002.

R. Cignoli, F. Esteva, L. Godo, and A. Torrens.
Basic Fuzzy Logic is the Logic of Continuous t-Norms and their Residua.
Soft Computing, 4(2):106–112, 2000.

F. Montagna
The Free BL-Algebra on One Generator.
Neural Network World, 5:837–844, 2000.

Outline

BL_n-Tree

n positive integer, $[n] = \{1, \ldots, n\}$.

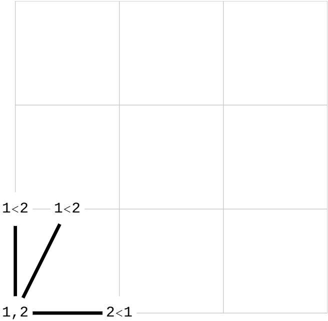
Definition (Fubini Partition)

A *Fubini partition* of $[n]$ is an ordered partition of $[n]$ into nonempty subsets.

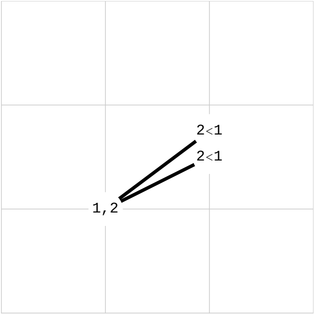
Definition (BL_n-Tree)

A *BL_n-tree* is a rooted tree $T = (V, E)$ st V is a finite multiset of Fubini partitions of $[n]$, V contains exactly one copy of $([n])$, $([n])$ is the root of the tree, and $(u, v) \in E$ only if the last block of u is equal to the union of the last two blocks of v.

BL_n-Tree | Case $n = 2$



(a) $T = (V, E)$.



(b) $T' = (V', E')$.

Figure: BL_2-trees.

Labeled BL_n-Tree

Definition (Rational Polyhedron)

A k-dimensional *(rational) simplex* S in $[0,1]^I$ ($k \leqslant |I|$) is the convex hull of a set of $k+1$ affinely independent [1] *(rational) vertices* $v_1, \ldots, v_{k+1}$ in $[0,1]^I$. A *(rational) simplicial complex* C in $[0,1]^I$ is a collection of (rational) simplexes in $[0,1]^I$ st every face of a simplex of C is in C, and the intersection of any two simplexes of C is a face of each of them. A *(rational) polyhedron* in $[0,1]^I$ is the underlying space of a (rational) simplicial complex in $[0,1]^I$, ie, the union of the (rational) simplexes in a simplicial complex in $[0,1]^I$.

[1] No hyperplane of dimension m contains more than $m+1$ of them.

Labeled BL_n-Tree

Definition (Labeled BL_n-Tree)

A *labeled BL_n-tree*, $l(T)$, is a BL_n-tree $T = (V, E)$ with $V \cup E$ labeled as follows. Let $\cdots B < B' = v \in V$ with successors $\{u_j = \cdots C < C'\}_{j \in [k]}$, so $B' = C \cup C'$. [2]

(*i*) $l(v)$ McNaughton function over $[0,1]^{B'}$, positive if $B' \neq [n]$.

(*ii*) $l(v, u_1), \ldots, l(v, u_k)$ are st:

- (*ii.i*) $l(v, u_j)$ is the relint [3] of a rational polyhedron in $[0,1]^C$.
- (*ii.ii*) $\{l(v, u_j) \times \{1\}^{C'} \mid j \in [k]\}$ partitions the 1-set of $l(v)$ in: $[0,1]^{B'}$ minus the union of $[0,1)^{B'}$ and $\{1\}^{B'}$.

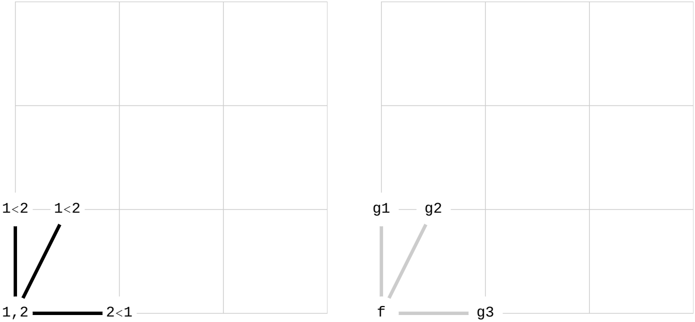
$l(T)$ is *positive* if $l([n])$ is positive (*negative*, ow).

[2] Display $([n])$ by $(\emptyset, [n])$.

[3] Relative interior.

Labeled BL_n-Tree | Case $n = 2$

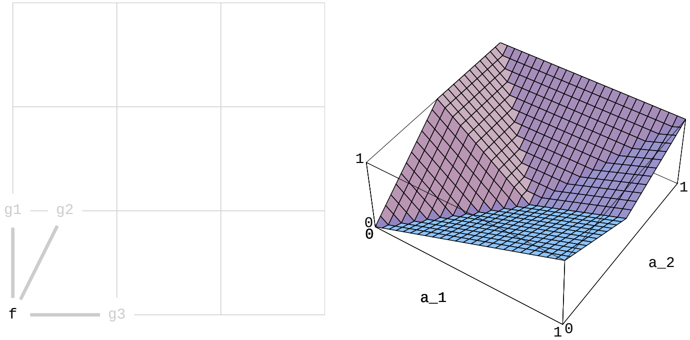
Sampling a labeled BL_2-tree, $l(T)$.



(a) $T = (V, E)$. (b) $l(V)$.

Figure: l sends V to McNaughton functions f, g_1, g_2, g_3.

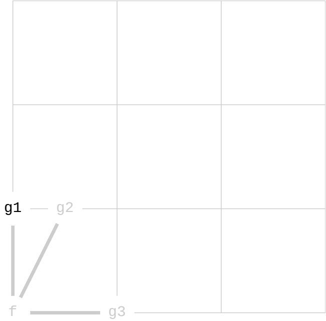
Labeled BL_n-Tree | *Case $n = 2$*



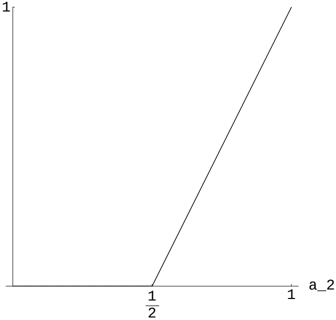
(a) $l(\{1,2\}) = f$. (b) $f\colon [0,1]^{\{1,2\}} \to [0,1]$.

Figure: f is positive, thus $l(T)$ is positive.

Labeled BL_n-Tree | Case $n = 2$



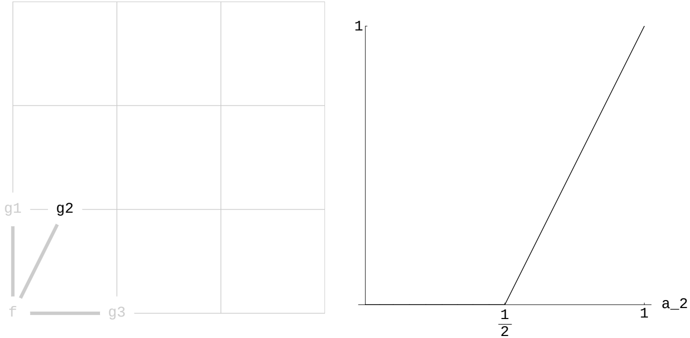
(a) $l(\{1\},\{2\}) = g_1$.



(b) $g_1 : [0,1]^{\{2\}} \to [0,1]$.

Figure: g_1 is positive.

Labeled BL_n-Tree | Case $n = 2$

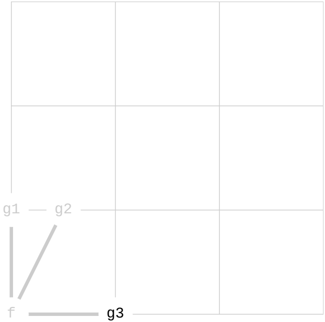


(a) $l(\{1\},\{2\}) = g_2$.

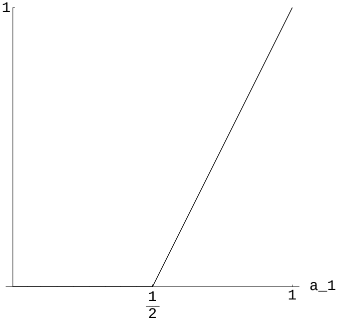
(b) $g_2 \colon [0,1]^{\{2\}} \to [0,1]$.

Figure: g_2 is positive.

Labeled BL_n-Tree | Case $n = 2$



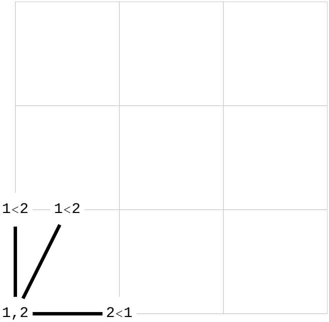
(a) $l(\{2\},\{1\}) = g_3$.



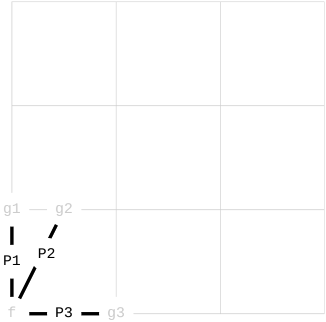
(b) $g_3\colon [0,1]^{\{1\}} \to [0,1]$.

Figure: g_3 is positive.

Labeled BL_n-Tree | Case $n = 2$



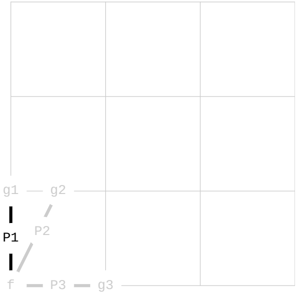
(a) $T = (V, E)$.



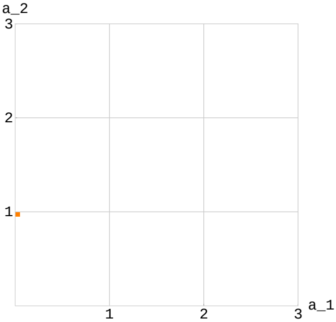
(b) $l(E)$.

Figure: l bijects btw E and a *suitable* set of rational polyhedra relints.

Labeled BL_n-Tree | *Case $n = 2$*



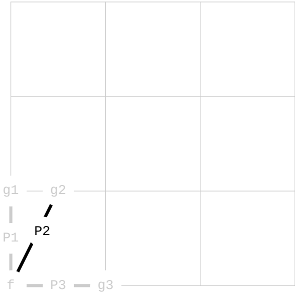
(a) $P_1 \subseteq [0,1]^{\{1\}}$.



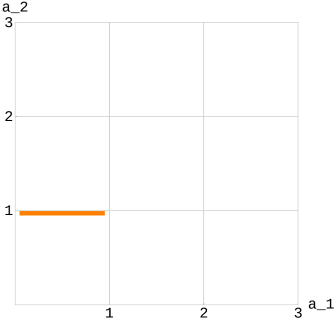
(b) $P_1 \times \{1\}^{\{2\}}$.

Figure: $P_1 = \{(a_1) \mid a_1 = 0\}$. $f|_{P_1 \times \{1\}^{\{2\}}} = 1$.

Labeled BL_n-Tree | *Case* $n = 2$



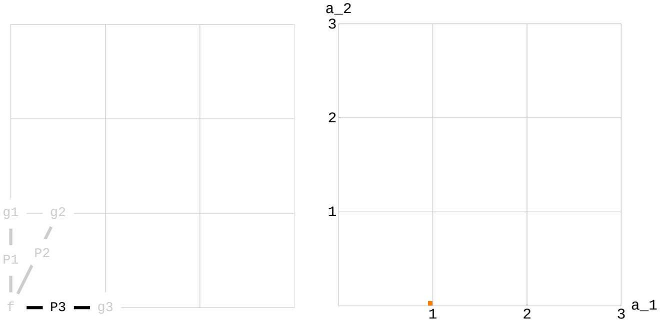
(a) $P_2 \subseteq [0,1]^{\{1\}}$.



(b) $P_2 \times \{1\}^{\{2\}}$.

Figure: $P_2 = \{(a_1) \mid 0 < a_1 < 1\}$. $f|_{P_2 \times \{1\}^{\{2\}}} = 1$.

Labeled BL_n*-Tree* | *Case* $n = 2$



(a) $P_3 \subseteq [0,1]^{\{2\}}$. (b) $P_3 \times \{1\}^{\{1\}}$.

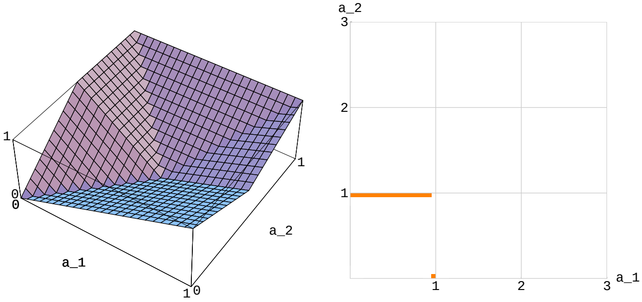
Figure: $P_3 = \{(a_2) \mid a_2 = 0\}$. $f|_{P_3 \times \{1\}^{\{1\}}} = 1$.

Labeled BL_n-Tree | Case $n = 2$

Suitability of $\{P_1, P_2, P_3\}$:

(*i*) ■ $= \{\mathbf{a} = (a_1, a_2) \mid f(\mathbf{a}) = 1\} \in [0,1]^{\{1,2\}} \setminus \left([0,1)^{\{1,2\}} \cup \{1\}^{\{1,2\}}\right)$;

(*ii*) $P_1 \times \{1\}^{\{2\}}, P_2 \times \{1\}^{\{2\}}, P_3 \times \{1\}^{\{1\}}$ form a partition of ■.



(a) $f\colon [0,1]^{\{1,2\}} \to [0,1]$.

(b) Part of f's 1-set.

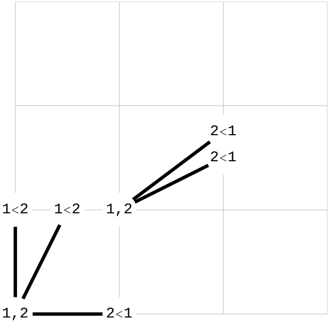
BL_n-Encoding

Definition (BL_n-Encoding)

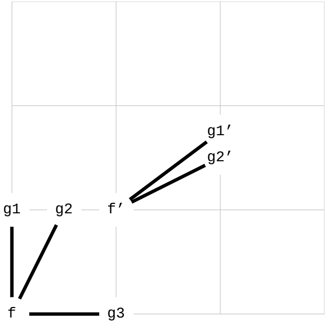
A *BL_n-encoding* is a pair $(l(T), l'(T'))$ of of labeled BL_n-trees st $l'(T')$ is positive if $l(T)$ is positive.

BL_n-Encoding | Case $n = 2$

Sampling a BL_2-encoding, $(l(T), l'(T'))$.



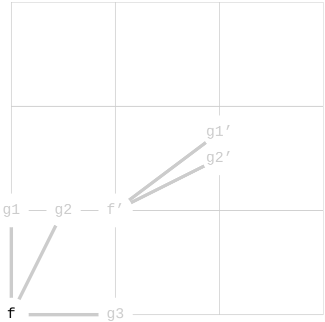
(a) $T = (V, E), T' = (V', E')$.



(b) $l(V), l'(V')$.

Figure: $l(V)$ and $l'(V')$ range over McNaughton functions.

BL_n-Encoding | Case $n = 2$



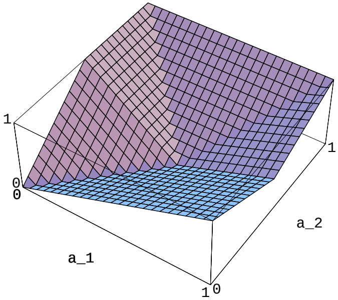
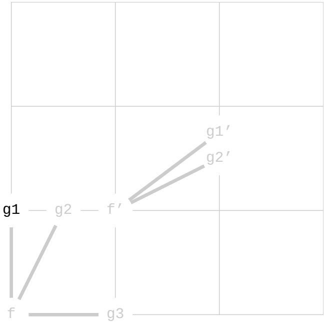


Figure: $l(V)$ specification, $f: [0,1]^{\{1,2\}} \to [0,1]$.

BL_n-Encoding | Case $n = 2$



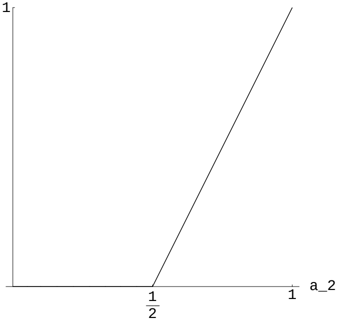


Figure: $l(V)$ specification, $g_1 : [0,1]^{\{2\}} \rightarrow [0,1]$.

BL_n-*Encoding* | *Case* $n = 2$

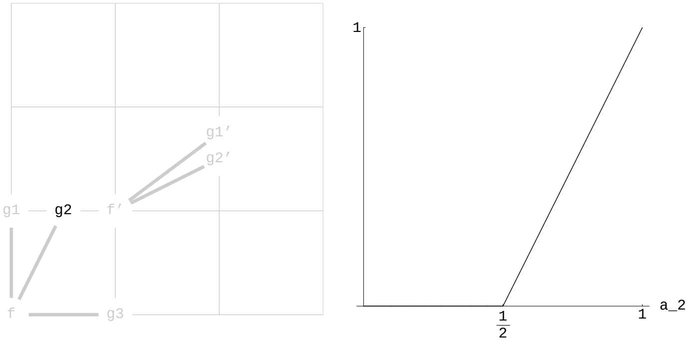


Figure: $l(V)$ specification, $g_2\colon [0,1]^{\{2\}} \to [0,1]$.

BL_n-Encoding | Case $n = 2$

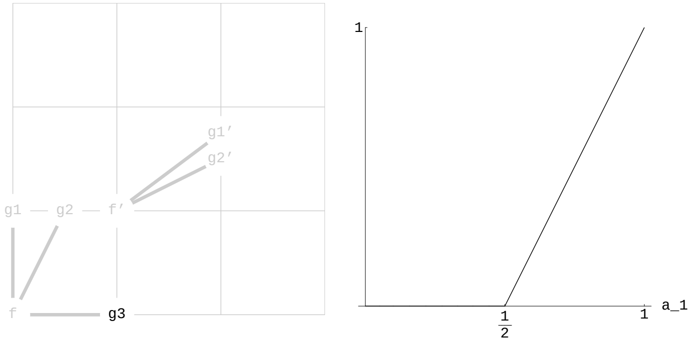


Figure: $l(V)$ specification, $g_3 \colon [0,1]^{\{1\}} \to [0,1]$.

BL_n-Encoding | Case $n = 2$

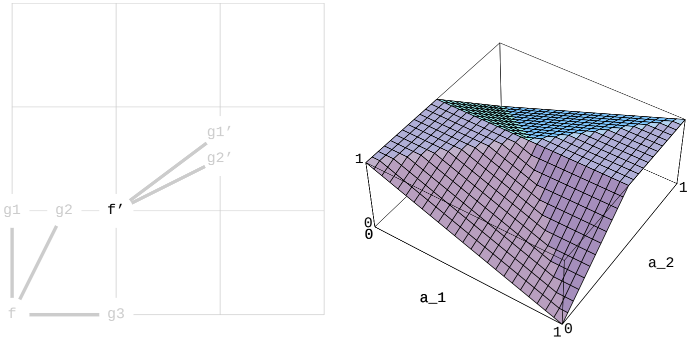


Figure: $l'(V')$ specification, $f' \colon [0,1]^{\{1,2\}} \to [0,1]$.

BL_n-Encoding | Case $n = 2$

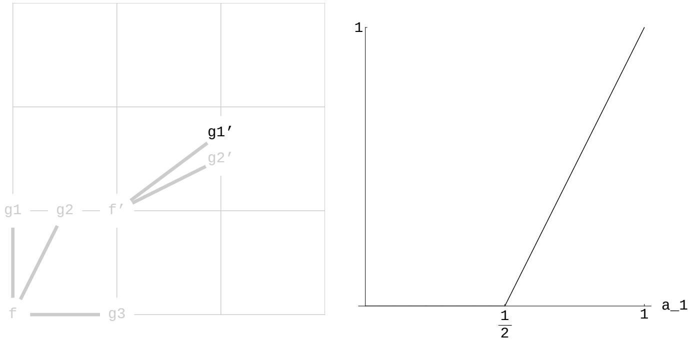
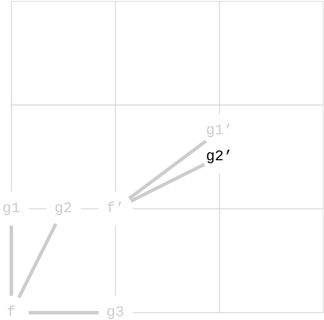


Figure: $l'(V')$ specification, $g_1' \colon [0,1]^{\{1\}} \to [0,1]$.

BL_n-Encoding | Case $n = 2$



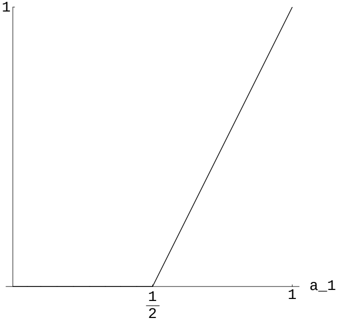
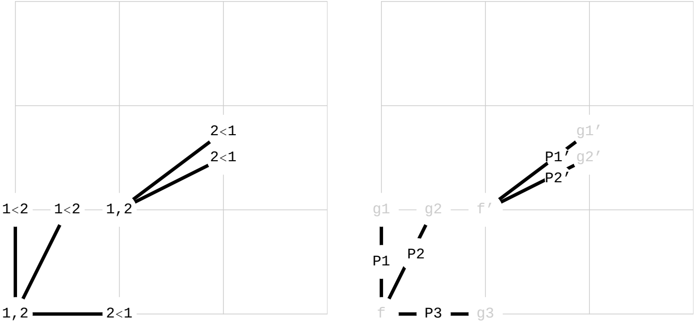


Figure: $l'(V')$ specification, $g_2' \colon [0,1]^{\{1\}} \to [0,1]$.

BL_n-Encoding | Case $n = 2$



(a) $T = (V, E), T' = (V', E')$.

(b) $l(E), l'(E')$.

Figure: $l(E)$ and $l'(E')$ range over suitable rational polyhedra relints.

BL_n-Encoding | Case $n = 2$

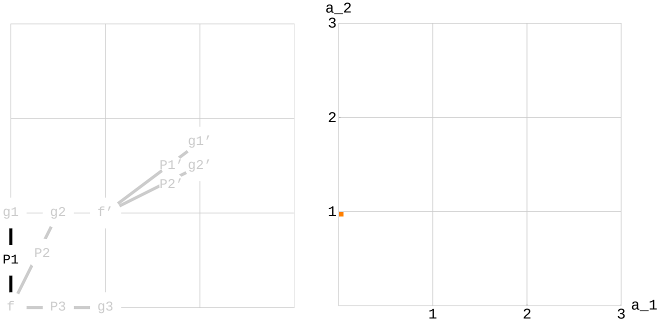


Figure: $l(E)$ specification, $P_1 = \{(a_1) \mid a_1 = 0\} \subseteq [0,1]^{\{1\}}$.

BL_n-Encoding | Case $n = 2$

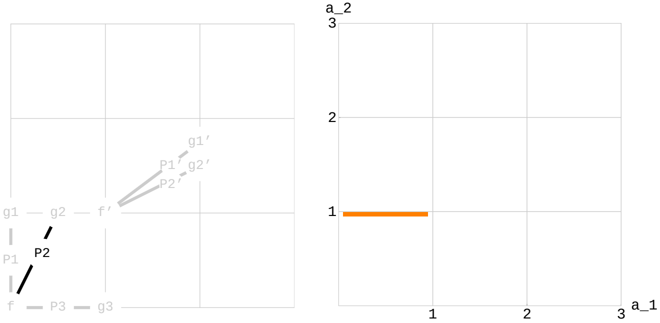


Figure: $l(E)$ specification, $P_2 = \{(a_1) \mid 0 < a_1 < 1\} \subseteq [0,1]^{\{1\}}$.

BL_n-Encoding | Case $n = 2$

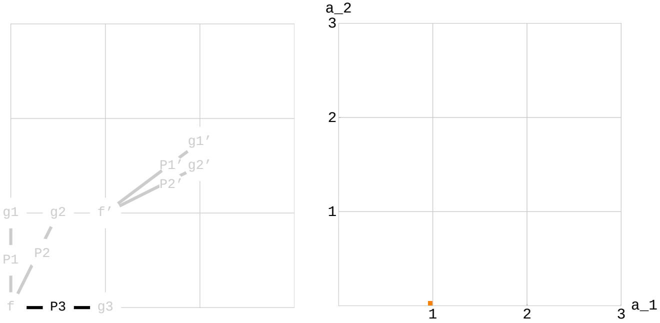
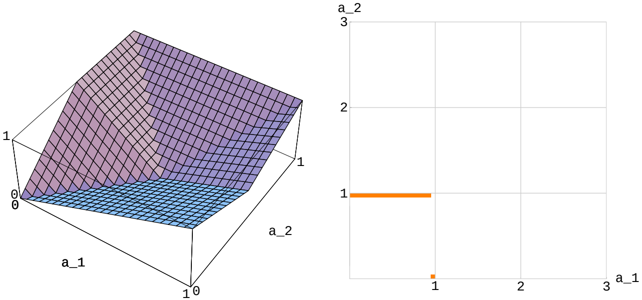


Figure: $l(E)$ specification, $P_3 = \{(a_1) \mid a_1 = 0\} \subseteq [0,1]^{\{2\}}$.

BL_n-Encoding | Case $n = 2$



(a) f. (b) Part of f's 1-set.

Figure: Suitability of $\{P_1, P_2, P_3\}$.

BL_n-*Encoding* | *Case* $n = 2$

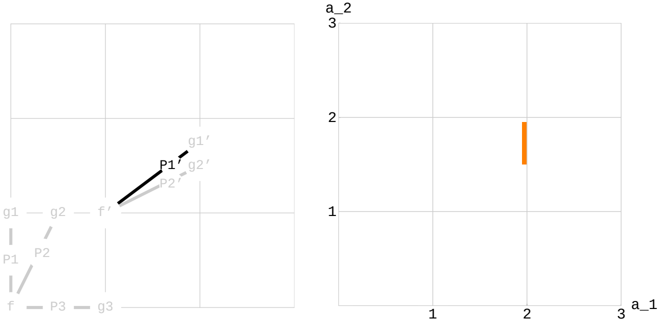


Figure: $l(E)$ specification, $P_1' = \{(a_2) \mid 1/2 < a_2 < 1\} \subseteq [0,1]^{\{2\}}$.

BL_n-*Encoding* | *Case* $n = 2$

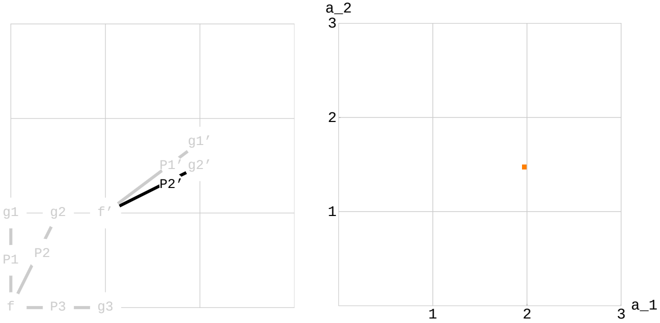
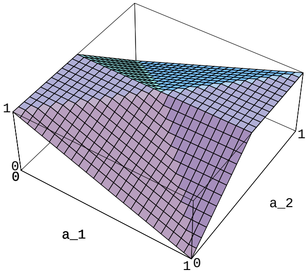
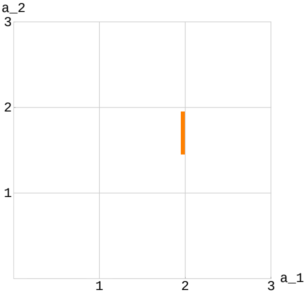


Figure: $l(E)$ specification, $P_2' = \{(a_2) \mid a_2 = 1/2\} \subseteq [0,1]^{\{2\}}$.

BL_n-*Encoding* | *Case* $n = 2$



(a) f'.



(b) Part of f'''s 1-set.

Figure: Suitability of $\{P'_1, P'_2\}$.

Realize

Algorithm Realize

Input BL_n-encoding $(l(T), l'(T'))$ and $\mathbf{a} \in [0, n+1]^{[n]}$

Output $b \in [0, n+1]$

Realize receives in input $(l(T), l'(T'))$ and $\mathbf{a}$ and:

1. uniquely determines a McNaughton function f in $l(V) \cup l'(V')$, which is *responsible* for $\mathbf{a}$;
2. outputs a value b which is a function of f and $\mathbf{a}$ only.

Proposition (Realization)

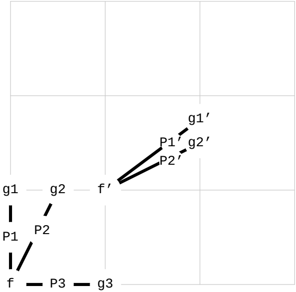
For any BL_n-encoding e, Realize$(e, \cdot)$ is a function, denoted by $f_e \colon [0, n+1]^{[n]} \to [0, n+1]$.

REALIZE | *Pseudocode*

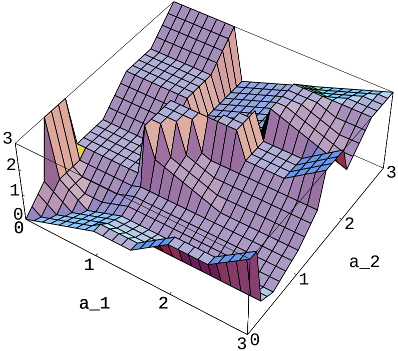
REALIZE$((l(T), l'(T')), \mathbf{a} = (a_1, \ldots, a_n) \in [0, n+1]^{[n]})$
if $\lfloor a_i \rfloor = 0$ for some $i \in [n]$
  $k(S) \leftarrow l(T)$
else
  if $l'(T')$ is negative
    output 0
  endif
  $k(S) \leftarrow l'(T')$
endif
$Q = B_1 < \cdots < B_m \leftarrow$ the Fubini partition corresponding to $\mathbf{a}$ ($m \geqslant 2$)
$i \leftarrow 1$
$c \leftarrow$ the root of S
while $i \leqslant m$
  $U \leftarrow$ the successors of c that are copies of $B_1 < \cdots < B_i < B_{i+1} \cup \cdots \cup B_m$
  $\mathbf{a}' \leftarrow$ the point in $[0,1]^{B_i}$ such that $a'_j = a_j - \lfloor a_j \rfloor$ for all $j \in B_i$
  if exists $u \in U$ such that $\mathbf{a}' \in k(c, u)$
    $i \leftarrow i + 1, c \leftarrow u$
  else
    $\mathbf{a}'' \leftarrow$ the point in $[0,1]^{B_i \cup \cdots \cup B_m}$ such that $a''_j = a_j - \lfloor a_j \rfloor$ for all $j \in B_i \cup \cdots \cup B_m$
    if $k(c)(\mathbf{a}'') < 1$ **and** $\mathbf{a}'' \neq \{n+1\}^{B_i \cup \cdots \cup B_m}$
      output $k(c)(\mathbf{a}'') + \lfloor a_j \rfloor$ where j is any index in $B_i \cup \cdots \cup B_m$
    else
      output $n + 1$
    endif
  endif
endwhile

Realize | *Case* $n = 2$

Sampling Realize on the BL_2-encoding $((l(T), l'(T')))$.



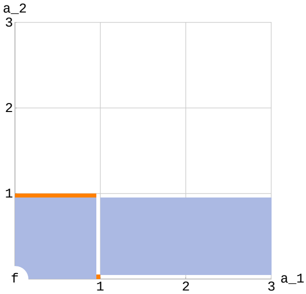
(a) $(l(T), l'(T'))$



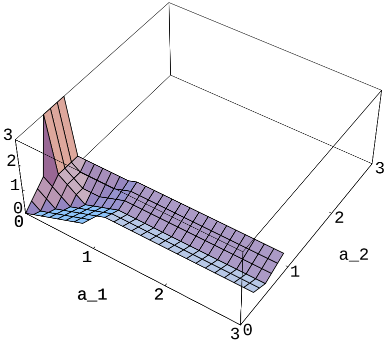
(b) $\textsc{Realize}((l(T), l'(T')), \cdot)$.

Figure: $\textsc{Realize}((l(T), l'(T')), \cdot) \colon [0,3]^{\{1,2\}} \to [0,3]$.

REALIZE | *Case* $n = 2$



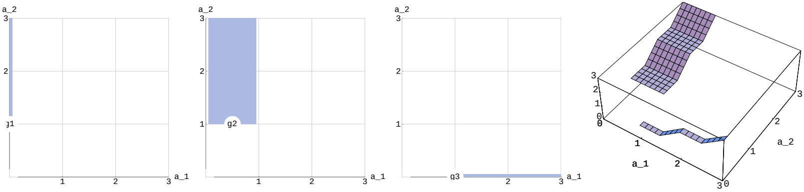
(a) f's responsibilities.



(b)

Figure: (b) REALIZE$((l(T), l'(T')), \cdot)$ restricted to f's responsibilities.

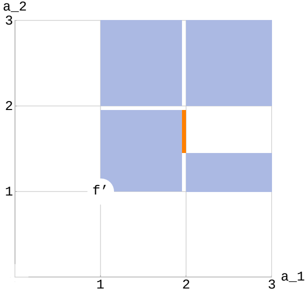
REALIZE | *Case* $n = 2$



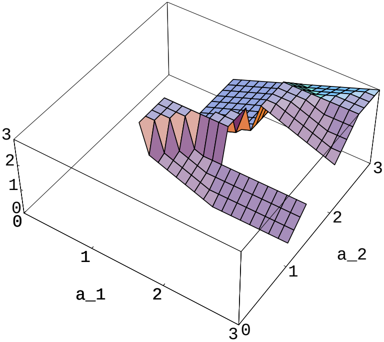
(a) g_1's resp. (b) g_2's resp. (c) g_3's resp. (d)

Figure: (d) REALIZE$((l(T), l'(T')), \cdot)$ restricted to $\bigcup_{i \in [3]} g_i$'s resp..

REALIZE | *Case* $n = 2$



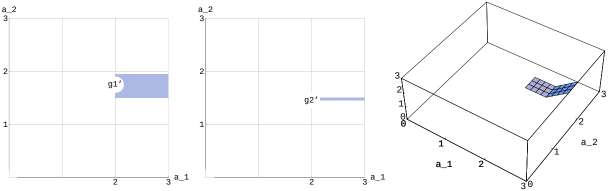
(a) f''s responsibilities.



(b)

Figure: (b) REALIZE$((l(T), l'(T')), \cdot)$ restricted to f''s responsibilities.

REALIZE | *Case* $n = 2$



(a) g_1''s resp. (b) g_2''s resp. (c)

Figure: (d) REALIZE$((l(T), l'(T')), \cdot)$ restricted to $\bigcup_{i \in [2]} g_i''$s resp..

BL_n-Functions

Definition (BL_n-Function)

A function $g\colon [0, n+1]^{[n]} \to [0, n+1]$ is an n-ary BL-function iff $g = f_e$ for some BL_n-encoding e.

Functional Representation

Theorem (Functional Representation)

The set of n-ary BL-functions equipped with pointwise defined operations $\odot^{(n+1)[0,1]}$ and $\rightarrow^{(n+1)[0,1]}$ is isomorphic to the free n-generated BL-algebra.

$t^{(n+1)[0,1]}$ denotes the n-ary term operation of $(n+1)[0,1]$ corresponding to term t.

(t, t') iff $t^{(n+1)[0,1]} = t'^{(n+1)[0,1]}$ is an equivalence over terms, whose representatives are n-ary BL-functions.

Functional Representation | Proof

Lemma (Closure)

The class of n-ary BL-functions contains all the n-ary term operations of $(n+1)[0,1]$.

Lemma (Normal Form)

Let f be any n-ary BL-function. There is an algorithm $T(\cdot)$ that takes in input (an encoding of) f and returns in output a term $T(f)$ over $(\odot, \to, \perp)$ and $\{X_i \mid i \in [n]\}$ st $T(f)^{(n+1)[0,1]} = f$.

(e, e') iff $f_e = f_{e'}$ is an equivalence over BL_n-encodings, whose blocks have cardinality > 1, so canonicity lacks.

Outline

Summary and Future Work

We gave a concrete representation of the free BL-algebra over finitely many generators (ie, Basic logic truthfunctions) in terms of a suitable class of real functions.

The problem was open, even for the case of two generators.

This result encourages the development of further work on locally finite subvarieties of BL-algebras, tight countermodels to BL-quasiequations, and constructive amalgamation property.

Publications

S. Aguzzoli and S. Bova.
The Free n-Generated BL-Algebra.
Accepted to *Annals of Pure and Applied Logic*.

S. Aguzzoli, S. Bova, and V. Marra.
Applications of Finite Duality to Locally Finite Varieties of BL-Algebras.
To appear in *Proceedings of LFCS'09*, 2009.

S. Bova.
BL-Functions and Free BL-Algebra.
Ph.D. Thesis, University of Siena, 2008.

S. Bova and F. Montagna.
Proof Search in Hájek's Basic Logic.
ACM Transactions on Computational Logic, 9(3), 2008.

S. Bova and F. Montagna.
The Consequence Relation in the Logic of Commutative GBL-Algebras is PSPACE-complete.
Accepted to *Theoretical Computer Science*.

Thank you!