Distributed Nonmonotonic Multi-Context Systems*

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Abstract

We present a distributed algorithm for computing equilibria of heterogeneous nonmonotonic multi-context systems (MCS). The algorithm can be parametrized to compute only partial equilibria, which can be used for reasoning tasks like query answering or satisfiability checking that need only partial information and not whole belief states. Furthermore, caching is employed to cut redundant solver calls. As a showcase, we instantiate the MCS framework with answer set program contexts. To characterize equilibria of such MCS, we develop notions of loop formulas that enable reductions to the classical satisfiability problem (SAT). Notably, loop formulas for bridge rules between contexts and for the local contexts can be combined to a uniform encoding of an MCS into a (distributed) SAT instance. As a consequence, we can use SAT solvers for belief set building. We demonstrate this approach by an experimental prototype implementation, which uses an off-the-shelf SAT solver.

Introduction

In the last years, there has been increasing interest in systems comprising multiple knowledge bases. The rise of distributed systems and the World Wide Web fostered this development, and to date, several formalisms are available that accommodate multiple, possibly distributed knowledge bases. One of them are Multi-Context Systems (MCS), which consist of several theories (the contexts) that are interlinked with bridge rules that allow to add knowledge into a context depending on knowledge in other contexts. For instance, the bridge rule $a \leftarrow (2 : b)$ of a context C_1 means that C_1 should conclude a if context C_2 believes b. MCS have applications in various areas, such as argumentation, data integration, or multi-agent systems. There, each context may model the beliefs of an agent while the bridge rules model an agent's perception of the environment, i.e., other contexts. Data integration from different knowledge sources is in general a driving force for MCS applications. One possibility is the example given in (Eiter et al. 2010) for finding explanations of inconsistencies in MCS. Another real life application is project costs and time management, which typically includes many constraints like working laws, holiday restrictions, working time per week, etc. Here, regulations are kept in different knowledge bases like an ontology of personal costs, some rules that can compute the work amount for work packages, personal timekeeping, central administration data, local preferences, and so on. Bridge rules may act to build up a system whose models describe consistent states of such a management system. Moreover, multi-context systems allow to model bidirectional input and output for external knowledge sources that goes beyond the import interface of HEX-programs (Eiter et al. 2005).

Roughly, MCS can be divided into monotonic and nonmonotonic MCS. Examples of the former kind are (McCarthy 1993; Giunchiglia and Serafini 1994; Ghidini and Giunchiglia 2001) (see also (Serafini and Bouquet 2004) for a comparison of these approaches), while (Brewka, Roelofsen, and Serafini 2007; Brewka and Eiter 2007) are nonmonotonic MCS. The general MCS framework of Brewka and Eiter (2007) is of special interest, as it generalizes previous approaches in contextual reasoning and allows for *heterogeneous and nonmonotonic* MCS, i.e., a system may have different, possibly nonmonotonic logics in its contexts (thus furthering heterogeneity), and bridge rules may use default negation (to deal, e.g., with incomplete information). Hence, nonmonotonic MCS interlinking monotonic context logics are possible.

Although virtually all formalizations of MCS are inherently targeted for distributed, autonomous systems, no distributed algorithms for MCS exist. Closest to one is an algorithm for checking satisfiability of homogeneous, monotonic MCS in (Roelofsen, Serafini, and Cimatti 2004), where a centralized control accesses contexts in parallel. The lack of distributed algorithms is due to several obstacles: (i) the semantic abstraction of contexts to belief sets hinders interference with the local evaluation processes in contexts; (ii) information hiding and security aspects disable access to the context theories themselves, merely interfaces to the belief sets are provided; (iii) the complete system topology might be unknown to a context, which hinders decomposed evaluation; and (iv) the bridge rules of two contexts may refer to each other, thus creating cyclic systems that must be handled with care.

Motivated by these defiances, we contribute the following.

- As a stepping stone, we introduce partial equilibria for Brewka and Eiter-style MCS, which enable us to build

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equilibria incrementally in a distributed setting.

- We present a truly distributed and modular meta-algorithm, DMCS, for partial equilibria building, which copes with problems (i)–(iv) above. It is distributed as no shared memory is needed across the contexts, and modular as it computes the partial equilibria starting from a context.
- To incorporate bridge rules into the local evaluations, we define *loop formulas*, inspired by similar compilations of logic programs into SAT theories (Lin and Zhao 2004). They may be added to belief sets abstractly represented as SAT theories to obtain a uniform encoding. We pursue this in particular for MCS whose contexts are answer set programs; notably, cycles in the MCS can be disregarded and equilibria are computable by a simplified algorithm.
- We report on a DMCS prototype implementation for the setting above and show initial experimental results. To our knowledge, this is the first implementation of such MCS.

Preliminaries

We recall some basic notions of heterogeneous nonmonotonic multi-context systems (Brewka and Eiter 2007) and disjunctive logic programs under answer set semantics.

Multi-Context Systems

A *logic* is, viewed abstractly, a tuple $L = (\mathbf{KB}_L, \mathbf{BS}_L, \mathbf{ACC}_L)$, where

- $\mathbf{KB}_L$ is a set of well-formed knowledge bases, each being a set (of formulas),
- $\mathbf{BS}_L$ is a set of possible belief sets, each being a set (of formulas), and
- $\mathbf{ACC}_L\colon \mathbf{KB}_L \to 2^{\mathbf{BS}_L}$ assigns each $kb \in \mathbf{KB}_L$ a set of acceptable belief sets.

This covers many (non-)monotonic KR formalisms like description logics, default logic, answer set programs, etc.

For example, a (propositional) *ASP logic* L may be such that $\mathbf{KB}_L$ is the set of answer set programs over a (propositional) alphabet $\mathcal{A}$, the set of possible belief sets $\mathbf{BS}_L = 2^{\mathcal{A}}$ contains all subsets of atoms, and $\mathbf{ACC}_L$ assigns each $kb \in \mathbf{KB}_L$ the set of all its answer sets (more details about ASP follow below).

Definition 1 *A* multi-context system (MCS) $M = (C_1, \ldots, C_n)$ *consists of* contexts $C_i = (L_i, kb_i, br_i)$, $1 \leq i \leq n$, *where* $L_i = (\mathbf{KB}_i, \mathbf{BS}_i, \mathbf{ACC}_i)$ *is a logic,* $kb_i \in \mathbf{KB}_i$ *is a knowledge base, and* br_i *is a set of* L_i-bridge rules *of the form*

$$\begin{aligned} s \leftarrow {} & (c_1 : p_1), \ldots, (c_j : p_j), \\ & \text{not}\,(c_{j+1} : p_{j+1}), \ldots, \text{not}\,(c_m : p_m) \end{aligned} \qquad (1)$$

where $1 \leq c_k \leq n$, p_k *is an element of some belief set of* L_{c_k}, $1 \leq k \leq m$, *and* $kb \cup \{s\} \in \mathbf{KB}_i$ *for each* $kb \in \mathbf{KB}_i$.

Informally, bridge rules allow to modify the knowledge base by adding s, depending on the beliefs in other contexts.

The semantics of an MCS M is defined in terms of particular *belief states*, which are sequences $S = (S_1, \ldots, S_n)$ of belief sets $S_i \in \mathbf{BS}_i$. Intuitively, S_i should be a belief set of the knowledge base kb_i; however, also the bridge rules br_i must be respected. To this end, kb_i is augmented with the conclusions of all $r \in br_i$ that are applicable.

Formally, r of form (1) is *applicable in* S, if $p_i \in S_{c_i}$, for $1 \leq i \leq j$, and $p_k \notin S_{c_k}$, for $j+1 \leq k \leq m$. Let $app(R, S)$ denote the set of all bridge rules $r \in R$ that are applicable in S, and $head(r)$ the part s of any r of form (1).

Definition 2 *A belief state* $S = (S_1, \ldots, S_n)$ *of a multi-context system* M *is an* equilibrium *iff for all* $1 \leq i \leq n$, $S_i \in \mathbf{ACC}_i(kb_i \cup \{head(r) \mid r \in app(br_i, S)\})$.

Example 1 Let $M = (C_1, C_2, C_3, C_4)$ be an MCS such that all L_i are ASP logics, with alphabets $\mathcal{A}_1 = \{a\}$, $\mathcal{A}_2 = \{b\}$, $\mathcal{A}_3 = \{c, d, e\}$, $\mathcal{A}_4 = \{f, g\}$. Suppose

– $kb_1 = \emptyset$, $br_1 = \{a \leftarrow (2:b), (3:c)\}$;
– $kb_2 = \emptyset$, $br_2 = \{b \leftarrow (4:g)\}$;
– $kb_3 = \{c \leftarrow d;\ d \leftarrow c\}$, $br_3 = \{c \vee e \leftarrow \text{not}\,(4:f)\}$;
– $kb_4 = \{f \vee g \leftarrow\}$, $br_4 = \emptyset$.

One can check that $S = (\{a\}, \{b\}, \{c, d, \neg e\}, \{\neg f, g\})$ is an equilibrium of M.

In the rest of this paper, we assume that contexts C_i have finite belief sets S_i that are represented by truth assignments $v_{S_i}\colon \Sigma_i \to \{0, 1\}$ to a finite set Σ_i of propositional atoms such that $p \in S_i$ iff $v_{S_i}(p) = 1$ (as in Brewka and Eiter, 2007, such S_i may serve as kernels that correspond one-to-one to infinite belief sets). To improve readability of our examples, we will include $\neg p$ in belief sets S_i to highlight false atoms p. Furthermore, we assume that the Σ_i are pairwise disjoint and that $\Sigma = \bigcup_i \Sigma_i$.

Answer Set Programs

Let $\mathcal{A}$ be a finite alphabet of atomic propositions. A *disjunctive rule* r is of the form

$$a_1 \vee \cdots \vee a_k \leftarrow b_1, \ldots, b_m, \text{not } b_{m+1}, \ldots, \text{not } b_n\ , \qquad (2)$$

for $k \geq 0$ and $n \geq m \geq 0$, where $a_1, \ldots, a_k, b_1, \ldots, b_n$ are atoms from alphabet $\mathcal{A}$. We let $H(r) = \{a_1, \ldots, a_k\}$ and $B(r) = B^+(r) \cup B^-(r)$, where $B^+(r) = \{b_1, \ldots, b_m\}$ and $B^-(r) = \{b_{m+1}, \ldots, b_n\}$. An *answer set program* P is a finite set of rules r of form (2).

An *interpretation* for P is any subset $I \subseteq \mathcal{A}$. It *satisfies* a rule r, if $H(r) \cap I \neq \emptyset$ whenever $B^+(r) \subseteq I$ and $B^-(r) \cap I = \emptyset$. I is a *model* of P, if it satisfies each $r \in P$.

The *GL-reduct* (Gelfond and Lifschitz 1991) P^I of P relative to I is the program obtained from P by deleting (i) every rule $r \in P$ such that $B^-(r) \cap I \neq \emptyset$, and (ii) all not b_j, for $b_j \in B^-(r)$, from every remaining rule r.

An interpretation I of a program P is called an *answer set* of P iff I is a $\subseteq$-minimal model of P^I.

For example, the program $P = \{a \vee b \leftarrow; d \leftarrow a, \text{not}\, c; e \leftarrow b\}$ has two answer sets, viz. $\{a, d\}$ and $\{b, e\}$.

Generic Evaluation of Nonmonotonic MCS

The aim of this section is to provide a distributed algorithm for computing equilibria of an MCS. Taking a local stance, we consider a context C_k and compute those parts of (potential) equilibria of the system which contain coherent information from all contexts that are 'reachable' from C_k.

Let us start defining some concepts required. The notion of import closure formally captures what is 'reachable'.

Definition 3 (Import Closure) *Let $M = (C_1, \dots, C_n)$ be an MCS. The* import neighborhood of a context C_k *is the set*

$$In(k) = \{c_i \mid (c_i : p_i) \in B(r), r \in br_k\} \ .$$

Moreover, the import closure $IC(k)$ of C_k *is the smallest set S such that (i) $k \in S$ and (ii) for all $i \in S$, $In(i) \subseteq S$.*

Alternatively, we can constructively characterize

$$IC(k) = \{k\} \cup \bigcup_{j \geq 0} IC^j(k) \ ,$$

where $IC^0(k) = In(k)$, and $IC^{j+1}(k) = \bigcup_{i \in IC^j(k)} In(i)$. Note that the import closure of any context is finite, i.e., for an MCS $M = (C_1, \dots, C_n)$ and C_k from M, $|IC(k)| \leq n$.

Example 2 Consider M in Example 1. Then $In(1) = \{2, 3\}$, $In(2) = In(3) = \{4\}$, and $In(4) = \emptyset$; the import closure of C_1 is $IC(1) = \{1, 2, 3, 4\}$ (see Figure 1(a)).

Based on the import closure we define partial equilibria.

Definition 4 (Partial Belief States and Equilibria) *Let $M = (C_1, \dots, C_n)$ be an MCS, and let $\epsilon \notin \bigcup_{i=1}^n \mathbf{BS}_i$. A* partial belief state *of M is a sequence $S = (S_1, \dots, S_n)$, such that $S_i \in \mathbf{BS}_i \cup \{\epsilon\}$, for $1 \leq i \leq n$.*

A partial belief state $S = (S_1, \dots, S_n)$ of M is a partial equilibrium of M w.r.t. a context C_k *iff $i \in IC(k)$ implies $S_i \in \mathbf{ACC}_i(kb_i \cup \{head(r) \mid r \in app(br_i, S)\})$, and if $i \notin IC(k)$, then $S_i = \epsilon$, for all $1 \leq i \leq n$.*

As an aside, $IC(k)$ essentially defines a subsystem M' of M that is connected by bridge rules. We use partial equilibria of M instead of equilibria of M' to keep the original MCS M intact. Our view is similar to unnamed attributes in a relational database; essentially, we reference contexts in an MCS by position as in standard equilibria. Alternative representations of equilibria for subsystems are possible but would prohibit to easily talk about the initial M without additional mappings from M' to M. Thus, for the purpose of this paper it is more convenient to use the reference-by-position approach.

For combining partial belief states $S = (S_1, \dots, S_n)$ and $T = (T_1, \dots, T_n)$, we define their *join* $S \bowtie T$ as the partial belief state $(U_1, \dots, U_n)$ with

(i) $U_i = S_i$, if $T_i = \epsilon$ or $S_i = T_i$,

(ii) $U_i = T_i$, if $T_i \neq \epsilon$ and $S_i = \epsilon$,

for all $1 \leq i \leq n$. Note that $S \bowtie T$ is void, if some S_i, T_i are from $\mathbf{BS}_i$ but different. The *join* of two sets $\mathcal{S}$ and $\mathcal{T}$ of partial belief states is then naturally defined as $\mathcal{S} \bowtie \mathcal{T} = \{S \bowtie T \mid S \in \mathcal{S}, T \in \mathcal{T}\}$.

Example 3 Consider two sets of partial belief states:

$$\mathcal{S} = \{\, (\epsilon, \{b\}, \epsilon, \{\neg f, g\}) \,,\, (\epsilon, \{\neg b\}, \epsilon, \{f, \neg g\}) \,\} \text{ and}$$

$$\mathcal{T} = \left\{ \begin{array}{l} (\epsilon, \epsilon, \{\neg c, \neg d, e\}, \{\neg f, g\}), \\ (\epsilon, \epsilon, \{c, d, \neg e\}, \{\neg f, g\}), \\ (\epsilon, \epsilon, \{\neg c, \neg d, \neg e\}, \{f, \neg g\}) \end{array} \right\} \ .$$

Their join is given by

$$\mathcal{S} \bowtie \mathcal{T} = \left\{ \begin{array}{l} (\epsilon, \{b\}, \{\neg c, \neg d, e\}, \{\neg f, g\}), \\ (\epsilon, \{b\}, \{c, d, \neg e\}, \{\neg f, g\}), \\ (\epsilon, \{\neg b\}, \{\neg c, \neg d, \neg e\}, \{f, \neg g\}) \end{array} \right\} \ .$$

Distributed Algorithm

Given an MCS M and a starting context C_k, we aim at finding all partial equilibria of M w.r.t. C_k in a distributed way. To this end, we design an algorithm DMCS, whose instances run independently at each context node and communicate with other instances for exchanging sets of partial belief states. This provides a method for distributed model building, and the DMCS algorithm can be applied to any MCS such that appropriate solvers for the respective context logics are available. As a main feature of DMCS, it can also compute *projected partial equilibria*, i.e., partial equilibria projected to a relevant portion of the signature of the import closure of the starting context. This can be exploited for specific tasks like, e.g., local query answering or consistency checking. When computing projected partial equilibria, the information communicated between contexts is minimized, keeping communication cost low.

In the sequel, we present a basic version of the algorithm, abstracting from low-level implementation issues. Moreover, it is assumed that the topology of the overall MCS is not known at context nodes (at the end of this section, we discuss potential enhancements given topology information). The idea is as follows: starting from context C_k, we visit the import closure of C_k by expanding the import neighborhood at each context like in a depth-first search, maintaining the set of visited contexts in a set *hist*, until a leaf context is reached, or a cycle is detected (by noticing the presence of the current context in *hist*). A leaf context simply computes its local belief sets, transforms all belief sets into partial belief states, and returns this result to its parent (invoking context). In case of a cycle, the context detecting the cycle, say C_i, must also break it, by (i) guessing belief sets for the "export" interface of C_i, (ii) transforming the guesses into partial belief states, and (iii) returning them to the invoking context.

The results of intermediate contexts are partial belief states, which can be joined, i.e., consistently combined, with partial belief states from their neighbors; a context C_k returns its local belief sets, joined with the results from its neighbors, as final result.

For computing projected partial equilibria, the algorithm offers a parameter V, the *relevant interface*. Given a (partial) belief state S and set $V \subseteq \Sigma$ of variables, the *restriction of S to V*, denoted $S|_V$, is given by the (partial) belief state $S' = (S_1|_V, \dots, S_n|_V)$, where $S_i|_V = S_i \cap V$ if $S_i \neq \epsilon$, and $\epsilon|_V = \epsilon$; the restriction of a set of (partial) belief states $\mathcal{S}$ to V is $\mathcal{S}|_V = \{S|_V \mid S \in \mathcal{S}\}$.

Let $V(k) = \{p_i \mid (c_i : p_i) \in B(r), r \in br_k\}$ denote the *import interface* of context C_k. By $V^*(k) = \bigcup_{i \in IC(k)} V(i)$, the *recursive import interface* of C_k, we refer to the interface of the import closure of C_k.

Given a context C_k, we have two extremal cases: 1. $V = V^*(k)$ and 2. $V = \Sigma$. In Case 1, DMCS basically checks for consistency on the import closure of C_k by computing partial

Algorithm 1: $\mathsf{DMCS}(V, hist)$ at $C_k = (L_k, kb_k, br_k)$

```
    Input: $V$: relevant interface, $hist$: visited contexts
    Data: $c(k)$: static cache
    Output: set of accumulated partial belief states
(a) if $c(k)$ is not empty then return $c(k)$
    $\mathcal{S} := \emptyset$
(b) if $k \in hist$ then          // cyclic: guess local beliefs w.r.t. $V$
(c)     $\mathcal{S} := \mathsf{guess}(V, C_k)$
    else     // acyclic: collect neighbor beliefs and add local ones
        $\mathcal{T} := \{(\epsilon, \ldots, \epsilon)\}$ and $hist := hist \cup \{k\}$
(d)     foreach $i \in In(k)$ do
            if for some $T \in \mathcal{T}$, $T_i = \epsilon$ then
                $\mathcal{T} := \mathcal{T} \bowtie C_i.\mathsf{DMCS}(V, hist)$
(e)     foreach $T \in \mathcal{T}$ do $\mathcal{S} := \mathcal{S} \cup \mathsf{lsolve}(T)$
(f)     $c(k) := \mathcal{S}|_V$
    return $\mathcal{S}|_V$
```

Algorithm 2: $\mathsf{lsolve}(S)$ at $C_k = (L_k, kb_k, br_k)$

```
    Input: $S$: partial belief state
    Output: set of locally acceptable partial belief states
    $\mathbf{T} := \mathbf{ACC}_k(kb_k \cup \{head(r) \mid r \in app(br_k, S)\})$
    return $\{(S_1, \ldots, S_{k-1}, T_k, S_{k+1}, \ldots, S_n) \mid T_k \in \mathbf{T}\}$
```

equilibria projected to interface variables only. In Case 2, the algorithm computes partial equilibria w.r.t. C_k. Between these two, by providing a fixed interface V, problem-specific knowledge (such as query variables) and/or infrastructure information can be exploited to keep computations focused on relevant projections of partial belief states.

The projections of partial belief states are cached in every context such that re-computation and the recombination of belief states with local belief sets are kept at a minimum.

We assume that each context C_k has a background process (or daemon in Unix terminology) that waits for incoming requests of the form $(V, hist)$, upon which it starts the computation outlined in Algorithm 1. This process also serves the purpose of keeping the cache $c(k)$ persistent. We write $C_i.\mathsf{DMCS}(V, hist)$ to specify that we send $(V, hist)$ to the process at context C_i and wait for its return message.

Algorithm 1 uses the following primitives:

- function $\mathsf{lsolve}(S)$ (Algorithm 2): augments the knowledge base kb of the current context with the heads of bridge rules in br that are applicable w.r.t. partial belief state S, computes local belief sets using function $\mathbf{ACC}$, combines each local belief set with S, and returns the resulting set of partial belief states; and
- function $\mathsf{guess}(V, C_k)$: guesses all possible truth assignments for the relevant interface w.r.t. C_k, i.e., for $\Sigma_k \cap V$.[1]

DMCS proceeds in the following way:

[1] In order to relate variables to context signatures, V can either be a vector of sets, or variables in V are prefixed with context ids; for simplicity, we kept V as a set without further assumptions.

(a) check the cache for an appropriate partial belief state;

(b) check for a cycle;

(c) if a cycle is detected, then guess partial belief states of the relevant interface of the context running DMCS;

(d) if no cycle is detected, but import from neighbor contexts is needed, then request partial belief states from all neighbors and join them;

(e) compute local belief states given the imported partial belief states collected from neighbors;

(f) cache the current (projected) partial belief state.

The next examples illustrate evaluation runs of DMCS for finding all partial equilibria with different MCS. We start with an acyclic run.

Example 4 Reconsider M from Example 1. Suppose the user invokes $C_1.\mathsf{DMCS}(V, \emptyset)$, where $V = \{a, b, c, f, g\}$, to trigger the evaluation process. Next, C_1 forwards in (d) requests to C_2 and C_3, which both call C_4. When called for the first time, C_4 calculates in (e) its own belief sets and assembles the set of partial belief states

$$\mathcal{S}_4 = \{(\epsilon, \epsilon, \epsilon, \{f, \neg g\}),\ (\epsilon, \epsilon, \epsilon, \{\neg f, g\})\} \quad .$$

After caching $\mathcal{S}_4|_V$ in (f), C_4 returns $\mathcal{S}_4|_V$ to one of the contexts C_2, C_3 whose request arrived first. On second call, C_4 simply returns to the other context $\mathcal{S}_4|_V$ from the cache.

C_2 and C_3 next call lsolve (in (e)) two times each, which results in $\mathcal{S}_2 = \mathcal{S}$ resp. $\mathcal{S}_3 = \mathcal{T}$ with $\mathcal{S}, \mathcal{T}$ from Example 3.

C_1, after computing in (d) $\mathcal{S}_2|_V \bowtie \mathcal{S}_3|_V$ $(= (\mathcal{S} \bowtie \mathcal{T})|_V$ in Example 3), calls lsolve in (e) thrice to compute the final result:

$$\mathcal{S}_1|_V = \left\{ \begin{array}{l} (\{a\}, \{b\}, \{c\}, \{\neg f, g\}), \\ (\{\neg a\}, \{b\}, \{\neg c\}, \{\neg f, g\}), \\ (\{\neg a\}, \{\neg b\}, \{\neg c\}, \{f, \neg g\}) \end{array} \right\} \quad .$$

The next example illustrates the run of DMCS on a cyclic topology.

Example 5 Let $M = (C_1, C_2, C_3)$ be an MCS such that each L_i is an ASP logic, and

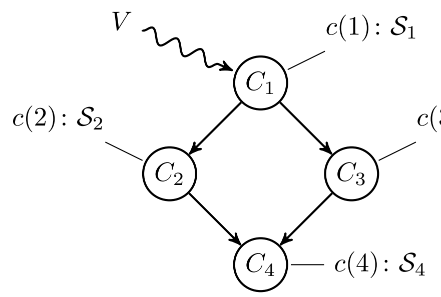
- $kb_1 = \emptyset$, $br_1 = \{a \leftarrow \mathrm{not}\,(2 : b)\}$;
- $kb_2 = \emptyset$, $br_2 = \{b \leftarrow (3 : c)\}$; and
- $kb_3 = \emptyset$, $br_3 = \{c \vee d \leftarrow \mathrm{not}\,(1 : a)\}$.

Figure 1(b) shows the cyclic topology of M. Suppose that the user sends a request to C_1 by calling $C_1.\mathsf{DMCS}(V, \emptyset)$ with $V = \{a, b, c\}$.

In step (d) of Algorithm 1, C_1 calls $C_2.\mathsf{DMCS}(V, \{1\})$, then context C_2 calls $C_3.\mathsf{DMCS}(V, \{1, 2\})$, thus C_3 invokes $C_1.\mathsf{DMCS}(V, \{1, 2, 3\})$. At this point, the instance of DMCS at C_1 detects a cycle in (b) and guesses the partial belief states

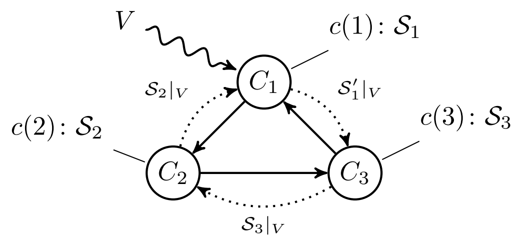
$$\mathcal{S}_1' = \{(\{a\}, \epsilon, \epsilon), (\{\neg a\}, \epsilon, \epsilon)\}$$

for $\Sigma_1 \cap V$. Then, following the dotted lines in Figure 1(b), the set $\mathcal{S}_1'|_V = \mathcal{S}_1'$ is the return value for the request from C_3,



V
C_1
$c(1)\colon \mathcal{S}_1$
$c(2)\colon \mathcal{S}_2$
C_2
C_3
$c(3)\colon \mathcal{S}_3$
C_4
$c(4)\colon \mathcal{S}_4$

(a) Topology of Example 1 (diamond)



V
$c(1)\colon \mathcal{S}_1$
$\mathcal{S}_2|_V$
C_1
$\mathcal{S}'_1|_V$
$c(2)\colon \mathcal{S}_2$
$c(3)\colon \mathcal{S}_3$
C_2
C_3
$\mathcal{S}_3|_V$

(b) Cyclic daisy chain topology of Example 5 (ring)

Figure 1: Acyclic and cyclic topologies

who joins it with $\mathcal{T}$ and then calls $\mathsf{lsolve}(T)$ for each $T \in \mathcal{T}$ in (e), resulting in

$$\mathcal{S}_3 = \left\{ \begin{array}{l} (\{\neg a\}, \epsilon, \{c, \neg d\}), \\ (\{\neg a\}, \epsilon, \{\neg c, d\}), \\ (\{a\}, \epsilon, \{\neg c, \neg d\}) \end{array} \right\} .$$

The next step of C_3 is to return $\mathcal{S}_3|_V$ back to C_2, which will proceed as C_3 before, joins $\mathcal{S}_3|_V$ with $\mathcal{T}$, and calls $\mathsf{lsolve}(T)$ for each $T \in \mathcal{T}$ in (e). The result is the set of belief states

$$\mathcal{S}_2 = \left\{ \begin{array}{l} (\{\neg a\}, \{b\}, \{c\}), \\ (\{\neg a\}, \{\neg b\}, \{c\}), \\ (\{a\}, \{\neg b\}, \{\neg c\}) \end{array} \right\} ,$$

which will be sent back to C_1 as $\mathcal{S}_2|_V$. Notice that belief state $(\{\neg a\}, \{\neg b\}, \{c\})$ is inconsistent with C_1, but will be eventually eliminated once C_1 evaluates $\mathcal{S}_2|_V$ with lsolve.

Next, C_1 will join $\mathcal{S}_2|_V$ with $\mathcal{T}$. In (e), C_1 uses the results from C_2 to call lsolve, and the union gives us

$$\mathcal{S}_1 = \{(\{\neg a\}, \{b\}, \{c\}),\ (\{a\}, \{\neg b\}, \{\neg c\})\} ,$$

which is also sent back to the user as final result.

Given an MCS $M = (C_1, \ldots, C_n)$ and a context C_k, using the recursive import interface of C_k, i.e., $V^*(k)$, as the relevant interface is a safe (lower) bound for the correctness of Algorithm 1. In what follows, let M, C_k, and $V^*(k)$ as above.

Theorem 1 *For all $V \supseteq V^*(k)$, $S' \in C_k.\mathsf{DMCS}(V, \emptyset)$ iff there exists a partial equilibrium S of M w.r.t. C_k such that $S' = S|_V$.*

We can compute partial equilibria at C_k if we use V_Σ.

Corollary 2 *S is a partial equilibrium of M w.r.t. C_k iff $S \in C_k.\mathsf{DMCS}(V_\Sigma, \emptyset)$.*

Under the assumption that M has a single root context C_1, i.e., such that $i \in IC(1)$ for all $2 \leq i \leq n$, DMCS computes equilibria. (Disconnected contexts in M can be always connected to a new root context using simple bridge rules.)

Corollary 3 *S is an equilibrium of the MCS M iff $S \in C_1.\mathsf{DMCS}(V_\Sigma, \emptyset)$ for a single root context C_1.*

An analysis of the algorithm yields the following upper bound on the communication activity.

Proposition 4 *In a run of DMCS, the number of messages exchanged between contexts C_i, where $i \in IC(k)$, is bounded by $2 \cdot |E(k)|$, where $E(k) = \{(i, c_j) \mid i \in IC(k), r \in br_i, (c_j : p_j) \in B(r)\}$.*

Discussion

Algorithm DMCS naturally proceeds "forward" in the import direction of context C_k. Thus, starting from there, it computes partial equilibria which cover C_k and contexts in its import closure. All other contexts will be ignored; in fact, they are unknown to all contexts in the closure. While partial equilibria may exist for C_k and its import closure, the whole MCS could have no equilibrium, because, e.g., (P1) contexts that access beliefs from C_k or its closure get inconsistent, or (P2) an isolated context or subsystem is inconsistent.

Enhancements of DMCS may deal with such situations: As for (P1), the context neighborhood may include both importing and supporting contexts. Intuitively, if C_i imports from C_j, then C_i must register to C_j. By carefully adapting DMCS, we can then solve (P1). However, (P2) remains; this needs knowledge about the global system topology.

A suitable assumption is the existence of a manager $\mathcal{M}$ that is reachable from every context C_i in the system, which can ask $\mathcal{M}$ whether some isolated inconsistent context or subsystem exists. If $\mathcal{M}$ affirms, C_i's DMCS simply returns $\emptyset$, eliminating all partial equilibria.

We can weaken the manager assumption by introducing *routers* (improving decentralization and information encapsulation). Instead of asking the manager, a context C_i queries an assigned router $\mathcal{R}$, which collects the necessary topology information for C_i or makes a cache look-up. The information exchange between C_i and $\mathcal{R}$ is flexible, depending on the system setting, and could contain contexts that import information from C_i, or isolated and inconsistent contexts.

A further advantage of topological information is that C_i can recognize cyclic and acyclic branches upfront, and the invocation order of the neighborhood can then be optimized, by starting with all acyclic branches before entering cyclic subsystems. The caching mechanism can be adapted for acyclic branches, as intermediate results are complete and the cache is meaningful even across different evaluation sessions.

In our setting, we are safe assuming that $V^*(k) \subseteq V$. But this is not needed if M resp. the import closure of C_k has no *join-contexts*, i.e., contexts which have at least two parents. If we have access to path information in M at each context, we could calculate V on the fly and change it accordingly during MCS traversal. In particular, for tree-shaped or ring topology, we can restrict V to the *local shared interface* between C_k and its import neighbors. In presence of join-contexts, V must be made "big enough," e.g., using path information (see Bairakdar et al., 2010). Furthermore, join-contexts may be eliminated by virtually splitting them, if orthogonal parts of the contexts are accessed. This way, scalability to many contexts can be achieved.

Loop Formulas for Multi-Context Systems

Algorithm DMCS incorporates in step (e) via lsolve the bridge rules br_k into the local knowledge base kb_k, given belief input from a belief state T, and then computes the belief sets; this is done for all $T \in \mathcal{T}$.

In certain settings, it is possible to compile br_k into kb_k, yielding some kb'_k, such that the belief sets of kb'_k are precisely the possible belief sets T_k in the return value of any lsolve(S); hence, the for-loop in step (e) can be replaced by a *single* join $\mathcal{S} := \mathcal{T} \bowtie \mathcal{B}_k|_V$, where $\mathcal{B}_k$ are the acceptable belief sets of kb'_k, properly converted to partial belief states.

For example, this is possible for classical logics L_k (assuming that contexts are not self-referential), or ASP logics. This is because there are well-known transformations of ASP programs P into equivalent classical theories $\phi(P)$, such that the answer sets of P are given by the classical models of $\phi(P)$, which hinge on *loop formulas* (Lin and Zhao 2004; Lee and Lifschitz 2003). Informally, such formulas ensure that cyclic rules like $a \leftarrow b$, $b \leftarrow a$ are satisfied, viewed as classical implications, by assigning a and b true only if there is *support* for the truth of a or b by some other rule.

In this section, we develop loop formulas for MCS's, by which bridge rules can be compiled into a local classical theory. In fact, we combine this with a loop formula transformation of ASP programs into classical theories; this enables us to obtain particular equilibria satisfying groundedness. Roughly, we adapt support such that also bridge rules have an effect on loops (e.g., $a \leftarrow (1 : c)$ on the loop above), but we distinguish *local support* and *bridge support*.

We assume that in $M = (C_1, \ldots, C_n)$, all logics L_i are ASP logics with $\Sigma_i = \mathcal{A}_i$ and $\Sigma = \mathcal{A}$. Furthermore, we assume that all heads of bridge rules are (disjunctive) facts (this is no loss of generality). This allows us to adapt disjunctive loop formulas to encode bridge rules as classical theories.

Let $\neg.A = \{\neg a \mid a \in A\}$ and, as usual, $\bigvee F = \bigvee_{f \in F} f$ and $\bigwedge F = \bigwedge_{f \in F} f$ (note that $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$).

For any ASP rule r, we then define

$$\kappa(r) = \bigwedge B^+(r) \wedge \bigwedge \neg.B^-(r) \supset \bigvee H(r) \ ,$$

and for any set R of ASP rules, $\kappa(R) = \bigwedge_{r \in R} \kappa(r)$.

The *support formula of a set $A \subseteq \mathcal{A}$ w.r.t. an ASP rule r* is

$$\varepsilon(A, r) = \bigwedge B^+(r) \wedge \bigwedge \neg.B^-(r) \wedge \bigwedge \neg.(H(r) \setminus A) \ ,$$

and w.r.t. any set R of ASP rules, $\varepsilon(A, R) = \bigvee_{r \in R} \varepsilon(A, r)$.[2]

[2]Ferraris, Lee, and Lifschitz (2006) call $\varepsilon(A, r)$ the *external* support formula, which is not entirely true in our setting.

To build support formulas w.r.t. a bridge rule r of form (1), we convert it to an ASP rule $\ell(r)$ by replacing $(c_k : p_k)$ with p_k, $1 \leq k \leq m$; for any bridge rule set R, we let $\ell(R) = \{\ell(r) \mid r \in R\}$.

We identify the *support rules* and the *external support rules* of a set of ASP rules R w.r.t. a set $A \subseteq \mathcal{A}$ as

$$SR(A, R) = \{r \in R \mid H(r) \cap A \neq \emptyset\} \text{ and}$$
$$ER(A, R) = \{r \in R \mid H(r) \cap A \neq \emptyset, B^+(r) \cap A = \emptyset\} \ ,$$

respectively (note the $SR(A, R)$ are not minimizing).

We next define necessary dependency relations. In a set of ASP rules R, we say that *a depends on b*, denoted $a \to b$, if $a \in H(r)$ and $b \in B^+(r)$ for some rule $r \in R$. The set of *dependencies in context C_i* is then the set of all pairs $a \to_i b$ such that $a \to b$ in $kb_i \cup \ell(br_i)$.

Based on this, we define the dependency graph of an MCS and loops for contexts and MCS's.

Definition 5 (Dependency Graph and Loops) *The* dependency graph of an MCS $M = (C_1, \ldots, C_n)$ *is the digraph* $G = (\mathcal{A}, \bigcup_{1 \leq i \leq n} \to_i)$.

A loop of C_i (resp., M) *is any set $\mathcal{L} \subseteq \mathcal{A}_i$ (resp., $\mathcal{L} \subseteq \mathcal{A}$) of atoms iff the subgraph of G induced by $\mathcal{L}$ is strongly connected.*

Note that each singleton $\{a\}$ is a loop.

Example 6 Consider $M = (C_1)$ and $M' = (C'_1)$, where

$$kb_1 = \left\{ \begin{array}{c} a \leftarrow b \\ b \leftarrow a \end{array} \right\}, br_1 = \left\{ \begin{array}{c} a \leftarrow (1 : b) \\ b \leftarrow (1 : a) \end{array} \right\},$$
$$kb'_1 = \left\{ \begin{array}{c} c \leftarrow d \\ d \leftarrow c \end{array} \right\}, br'_1 = \left\{ \begin{array}{c} c \leftarrow \text{not } (1 : d) \\ d \leftarrow \text{not } (1 : c) \end{array} \right\} \ .$$

The loops of C_1 are $\{a\}$, $\{b\}$, and $\{a, b\}$, and those of C'_1 are $\{c\}$, $\{d\}$, and $\{c, d\}$.

Next we define the local loop formula of a context.

Definition 6 (Local Loop Formulas) *Let $\mathcal{L}$ be a loop of context C_i. Then the* loop formula for $\mathcal{L}$ w.r.t. C_i *is*

$$\lambda(\mathcal{L}, C_i) = \left(\bigvee \mathcal{L}\right) \supset \varepsilon(\mathcal{L}, ER(\mathcal{L}, kb_i) \cup SR(\mathcal{L}, \ell(br_i))) \ .$$

Furthermore, the loop formula of context C_i *is the conjunction $\lambda(C_i) = \bigwedge_{\mathcal{L}} \lambda(\mathcal{L}, C_i)$ of all loops $\mathcal{L}$ of C_i.*

Example 7 Continuing Example 6, $\mathcal{L}_1 = \{a, b\}$ has the loop formula (i) $\lambda(\mathcal{L}_1, C_1) = a \vee b \supset b \vee a$. Indeed, both rules of br_1 are support rules of C_1 w.r.t. $\mathcal{L}_1$, which has no external support rules in kb_1; thus $\lambda(\mathcal{L}_1, C_1) = a \vee b \supset \varepsilon(\mathcal{L}_1, \ell(br_1))$, and $\varepsilon(\mathcal{L}_1, \ell(br_1)) = b \vee a$. Similarly, $\mathcal{L}'_1 = \{c, d\}$ has the loop formula (ii) $\lambda(\mathcal{L}'_1, C'_1) = c \vee d \supset \neg d \vee \neg c$.

Experts on loop formulas will notice that (i) is weaker than the loop formula $a \vee b \supset \bot$ of the program $\{a \leftarrow b;\ b \leftarrow a\}$, and admits $\{a, b\}$ as a model of the translation; this complies with the MCS semantics. Similarly, (ii) is weaker than $c \vee d \supset \bot$ but can eliminate the model $\{c, d\}$. Finally, we have $\lambda(\{a\}, C_1) = a \supset b$, $\lambda(\{b\}, C_1) = b \supset a$, $\lambda(\{c\}, C'_1) = c \supset d \vee \neg d$, and $\lambda(\{d\}, C'_1) = d \supset c \vee \neg c$.

We now can transform the whole MCS into a formula.

Definition 7 (MCS loop transformation) *Given the multi-context system $M = (C_1, \dots, C_n)$ with ASP logics, let*

$$\pi(C_i) = \lambda(C_i) \wedge \kappa(kb_i) \wedge \kappa(\ell(br_i)) \ \ , 1 \leq i \leq n, \ \textit{and}$$

$$\pi(M) = \bigwedge_{i=1}^{n} \pi(C_i) \ \ .$$

Here, $\pi(C_i)$ describes the belief sets of C_i, depending on valuations of the atoms in bridge rule bodies; $\pi(M)$ just aligns descriptions. The next result shows that this correctly characterize the equilibria of M.

Theorem 5 *The equilibria of any M with ASP logics correspond one-to-one to the models of the formula $\pi(M)$.*

Note that by this theorem, consistency of M (i.e., existence of some equilibrium) maps to a distributed SAT problem.

Example 8 For M and M' from Example 6, we have

$$\pi(M) = (b \supset a) \wedge (a \supset b) \wedge (a \vee b \supset a \vee b), \text{ and}$$

$$\begin{aligned}\pi(M') = {} & (c \supset d) \wedge (d \supset c) \wedge (\neg c \supset d) \wedge (\neg d \supset c) \wedge \\ & (c \supset d \vee \neg d) \wedge (d \supset c \vee \neg c) \wedge \\ & (c \vee d \supset \neg c \vee \neg d) \ \ .\end{aligned}$$

Clearly, $\pi(M)$ has the models $\emptyset$ and $\{a, b\}$, while it can be checked that $\pi(M')$ has no model.

Example 9 Let us reconsider M from Example 1. We have

$$\begin{aligned}\pi(C_1){:}\ & \kappa(C_1) = b \wedge c \supset a \ \text{ and } \ \lambda(C_1) = a \supset b \wedge c \\ \pi(C_2){:}\ & \kappa(C_2) = g \supset b \ \text{ and } \ \lambda(C_2) = b \supset g \\ \pi(C_3){:}\ & \kappa(C_3) = (d \supset c) \wedge (c \supset d) \wedge (\neg f \supset c \vee e) \text{ and} \\ & \lambda(C_3) = (c \supset d \vee (\neg e \wedge \neg f)) \wedge (d \supset c) \wedge \\ & \qquad (c \vee d \supset (\neg e \wedge \neg f)) \wedge (e \supset (\neg f \wedge \neg c)) \\ \pi(C_4){:}\ & \kappa(C_4) = f \vee g \text{ and } \lambda(C_4) = (f \supset \neg g) \wedge (g \supset \neg f).\end{aligned}$$

The formula $\pi(M) = \pi(C_1) \wedge \cdots \wedge \pi(C_4)$ has three models, namely $\{a, b, c, d, g\}$, $\{b, e, g\}$, and $\{f\}$. They correspond to the three projected equilibria of M shown in Example 4.

We can adapt the algorithm DMCS for π easily: after step (a), we insert $\mathcal{S} := \mathsf{lsolve}((\epsilon, \dots, \epsilon))$; moreover, we delete in step (c) the for-loop, and replace in step (e) the for-loop with $\mathcal{S} := \mathcal{S}|_V \bowtie \mathcal{T}$. Furthermore, we replace in M each context C_i with $C'_i = (L'_i, \pi(C_i)', br')$, where L'_i is propositional logic, $\pi(C_i)'$ is a renaming of $\pi(C_i)$ such that variables in different contexts are disjoint, and br' contains $\{a_i \leftarrow (j : a_j); \neg a_i \leftarrow (j : \neg a_j)\}$ for every renamed original atom a occurring in both $\pi(C_i)$ and $\pi(C_j)$, $i \neq j$. Applying then DMCS to the MCS M', we obtain the equilibria of M.

Theorem 5 may be generalized to contexts with extensions of ASP logics that have loop formula characterizations, like those in (Ferraris, Lee, and Lifschitz 2006; Janhunen et al. 2009). Furthermore, we have developed such a characterization for modular logic programs (Dao-Tran et al. 2009), which feature modules akin to imperative programs and have increased expressiveness.

Note that this loop formula characterization of equilibria may lead in the worst case to an exponential blow-up in the size of the MCS. This is not surprising, as standard loop formulas (Ferraris, Lee, and Lifschitz 2006; Lin and Zhao 2004; Lee and Lifschitz 2003) also face this situation, and Lifschitz and Razborov (2006) show that this unavoidable, under the widely believed assumption from computational complexity theory that polynomial time computations cannot be simulated with small propositional formulas. A remedy would be to encode bridge rules in answer set programs. Note however, that some ASP solvers like ASSAT rely internally on loop formulas and SAT solving techniques for model search; thus the expected performance gain from a short ASP encoding might not always surface in practice.

Loop Formulas for Grounded Equilibria

Equilibria lack groundedness in general, as cyclic bridge rules might be applied unfoundedly (e.g., $a \leftarrow (1 : b)$ and $b \leftarrow (1 : a)$ in Example 6). To overcome this, grounded equilibria were proposed in (Brewka and Eiter 2007) for certain MCS's, in which bridge rules intuitively act under ASP semantics.

Our transformation π can be adapted to capture grounded equilibria. We restrict here to *normal* ASP logics L_i, i.e., $\mathbf{KB}_i$ is the set of all normal (disjunction-free) ASP programs (this ensures a technical reducibility condition for L_i). Adapting Definitions 6 and 7, we define global loop formulas.

Definition 8 (Global Loop Formulas) *Let $\mathcal{L}$ be a loop of MCS $M = (C_1, \dots, C_n)$. The* loop formula for $\mathcal{L}$ w.r.t. M *is*

$$\lambda(\mathcal{L}, M) = \left(\bigvee \mathcal{L}\right) \supset \varepsilon(\mathcal{L}, \bigcup_{i=1}^{n} ER(\mathcal{L}, kb_i \cup \ell(br_i))) \ \ ,$$

and the loop formula of M *is the conjunction $\lambda(M) = \bigwedge_{\mathcal{L}} \lambda(\mathcal{L}, M)$ for all loops $\mathcal{L}$ of M. Furthermore, we let*

$$\pi_{\mathbf{GE}}(M) = \lambda(M) \wedge \bigwedge_{i=1}^{n} (\kappa(kb_i) \wedge \kappa(\ell(br_i))) \ \ .$$

We then can show that $\pi_{\mathbf{GE}}$ captures grounded equilibria.

Theorem 6 *The grounded equilibria of any M with normal ASP logics correspond one-to-one to the models of $\pi_{\mathbf{GE}}(M)$.*

Example 10 The MCS M in Example 6 has

$$\pi_{\mathbf{GE}}(M) = (a \supset b) \wedge (b \supset a) \wedge (a \vee b \supset \bot) \ \ ,$$

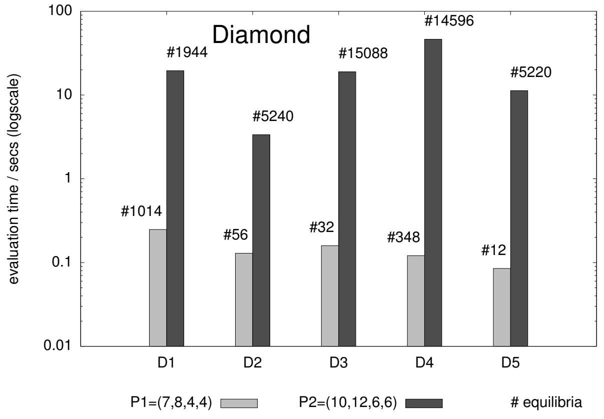
as the external support rules $ER(\{a, b\}, kb_1 \cup \ell(br_1)) = \emptyset$. The only model of $\pi_{\mathbf{GE}}(M)$ is $\emptyset$, which corresponds to the grounded equilibrium of M.

Note that DMCS cannot be run straight on $\pi_{\mathbf{GE}}(M)$, as the formulas in $\pi_{\mathbf{GE}}(M)$ are intermingled and prohibit a clear context separation. Intuitively, this is due to the encoded groundedness check. We can overcome this simply by extending M' for $\pi(M)$ above to $M'' = (C_0, C'_1, \dots, C'_n)$, where $C_0 = (L_0, \pi_{\mathbf{GE}}(M), br_0)$ has propositional logic L_0 and $br_0 = \bigcup_{i=1}^{n} \{a \leftarrow (i : a) \mid a \in \mathcal{A}_i\}$; intuitively, C_0 filters out grounded equilibria. We then run DMCS on M'' at C_0.

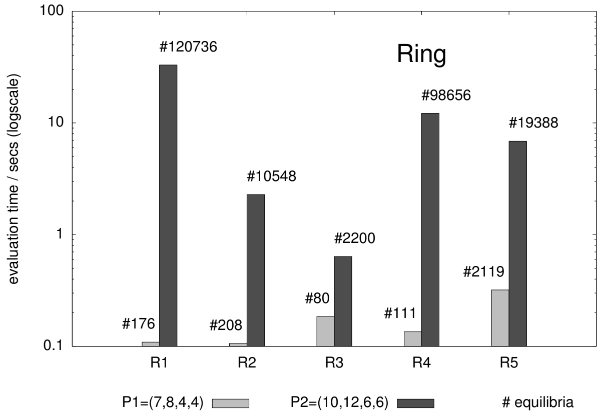
Implementation and Experimental Results

We present initial results for a SAT-solver based prototype implementation of DMCS under Linux, written in C++.[3]

[3]Implementation, experiments, and documentation is available at `www.kr.tuwien.ac.at/research/systems/dmcs`.



(a) Runtime for instances with diamond topology



(b) Runtime for instances with ring topology

Figure 2: Evaluation times for diamond and ring topologies

We used the development version of *clasp* (2010-01-31) as a SAT solver, which accepts DIMACS CNF input (Gebser et al. 2007). Specifically, all $\pi(C_k)$ are encoded as CNF clauses and *clasp* builds all models at a context C_k. Conceptually, any SAT solver that enumerates all models would serve the same purpose. Every context has its own server process running on the machine.

Distributed model computation for nonmonotonic systems is new, and there are no implementations that would serve as a basis for comparison with our approach. We do not plan to use large systems with thousands of contexts, and pay attention to systems that do not have too many contexts (up to several dozens). For this, we randomly created small MCS's and report initial findings on scalability issues.

For initial experimentation, we created random MCS instances of fixed topologies generalizing the diamond and ring in Figure 1(a) and 1(b) (stacking m diamonds in a tower of $3m + 1$ nodes). A parameter setting $P = (n, s, b, r)$ specifies (i) the number n of contexts, (ii) the local alphabet size $|\Sigma_i| = s$ (each C_i has a random ASP program on s atoms with 2^k answer sets, $0 \leq k \leq s/2$), (iii) the maximum interface size b (number of atoms exported), (iv) and the maximum number r of bridge rules per context, each having at most 2 body literals.

Table 1 shows some experimental results for $P_1 = (7, 8, 4, 4)$ and $P_2 = (10, 12, 6, 6)$. Each row D_j (resp., R_j) displays pure computation time (no output) for sequential diamonds (resp., a ring), where the # columns show the numbers of projected partial equilibria computed at C_1 (initiated by sending the request $V^*(1)$ to C_1). The response time varies a lot, especially for P_2, as well as the result size, due to the particular instances. All works quite fast for small interfaces; increasing s should, modulo local solving and projection time, not lead to an overall increase. Stacking multiple diamonds in a tower ($m = 3$ for P_2) models hard instances with many joins. This is reflected in the ratio of running time to result size, compared to the ring. There is a lot of room for improvement; e.g. with the local shared interface optimization for ring, thus fewer results, we gain a significant speedup for P_2 (<1.4 secs in R_1, <0.8 secs in all other cases). Figure 2(a) and 2(b) show a graphical representation of the runtime behavior for the tests above.

topology	P_1 =$(7, 8, 4, 4)$	#	P_2 =$(10, 12, 6, 6)$	#
D_1	0.248	1014	19.520	1944
D_2	0.129	56	3.361	5240
D_3	0.159	32	18.948	15088
D_4	0.121	348	46.249	14596
D_5	0.085	12	11.289	5220
R_1	0.109	176	33.020	120736
R_2	0.106	208	2.285	10548
R_3	0.185	80	0.637	2200
R_4	0.135	111	12.176	98656
R_5	0.320	2119	6.870	19388

Table 1: Runtime (secs) on Pentium M 1.73GHz, 1GB RAM

Related Work

Roelofsen, Serafini, and Cimatti (2004) described evaluation of monotone MCS with classical theories using SAT solvers for the contexts in parallel. They used a (co-inductive) fixpoint strategy to check MCS satisfiability, where a centralized process iteratively combines results of the SAT solvers. Apart from being not truly distributed, an extension to nonmonotonic MCS is non-obvious; also, no caching was used.

Serafini and Tamilin (2005) and Serafini, Borgida, and Tamilin (2005) defined distributed tableaux algorithms for reasoning in distributed ontologies, which are akin to MCS. Distributed ontologies enforce a stronger coupling of the contexts as they provide an alignment of the vocabulary. Thus, the abstract view of MCS is not possible. The algorithms can be used to decide consistency of distributed description logic knowledge bases, provided that the distributed TBox is acyclic. The DRAGO system implements this approach.

Hirayama and Yokoo (2005) dealt with distributed SAT (DisSAT), where the local theories (agents) are propositional clause sets, and communication is based on shared variables between theories. They gave an algorithm for finding a single satisfying assignment as follows. Starting with a random variable assignment, each agent repeatedly finds possible variable flips to reduce conflicts, and sends them to the neighbors; based on their proposals, the agent changes her assignment. This continues until a solution is found or the number of communication rounds reaches a limit. DisSAT can be readily used to find some equilibrium of MCS M that are mappable into distributed SAT (e.g., via our loop transformation $\pi(M)$), while computing multiple (or all) equilibria is not supported.

Adjiman et al. (2006) presented a framework of peer-to-peer inference systems. Local theories of propositional clause sets share atoms, and a special algorithm can be used for consequence finding. As we pursue the dual problem of model building, application for our needs is not straightforward.

Conclusion

We have presented a basic distributed algorithm, DMCS, for computing all (partial) equilibria of an MCS, and reported initial experiments. Future work includes the realization of algorithmic optimizations as discussed, as well as improvements and extensions of the implementation. Also, we plan to build an alternative implementation for ASP logics that uses native ASP solvers, and compare both implementations. In follow-up work, preliminary results on improving scalability by decomposing multi-context systems turn out to be quite effective and lead to significant improvements in several cases (Bairakdar et al. 2010).

Our aim is to compute all models of a nonmonotonic multi-context system. A simplification that should improve scalability is to compute one (block) of model(s) at-a-time.

Regarding possible applications in the Web field, distributed MCS could serve as a host language for distributed SPARQL reasoning (Schenk and Staab 2008). For that, a non-ground semantics must be developed that relies on grounding as customary in logic programming.

Appendix

Proof Sketch for Theorem 1

Lemma 7 *For any context C_k and partial belief state S of an MCS $M = (C_1, \dots, C_n)$, $app(br_k, S) = app(br_k, S|_V)$ for all $V \supseteq V^*(k)$.*

Proof Follows from

$$\begin{aligned}
r \in app(br_k, S) \;&\Leftrightarrow\; \forall (r_i : p_i) \in B^+(r)\colon p_i \in S_{r_i} \wedge \\
&\qquad \forall (r_k : p_k) \in B^-(r)\colon p_k \notin S_{r_k} \\
&\Leftrightarrow\; \forall (r_i : p_i) \in B^+(r)\colon S_{r_i} = S_{r_i}|_{V_{r_i}} \wedge \\
&\qquad \forall (r_k : p_k) \in B^-(r)\colon S_{r_k} = S_{r_k}|_{V_{r_k}} \\
&\qquad \text{as } V \supseteq V^*(k) \wedge S|_{V^*(k)} \subseteq S|_V \\
&\Leftrightarrow\; \forall (r_i : p_i) \in B^+(r)\colon p_i \in S_{r_i}|_V \wedge \\
&\qquad \forall (r_k : p_k) \in B^-(r)\colon p_k \in S_{r_k}|_V \\
&\Leftrightarrow\; r \in app(br_k, S|_V).
\end{aligned}$$

We can now prove Theorem 1.

($\Rightarrow$) We start by showing soundness of DMCS. Let $S' \in C_k.\mathsf{DMCS}(V, \emptyset)$ such that $V \supseteq V^*(k)$. We show now that there is a partial equilibrium S of M w.r.t. C_k such that $S' = S|_V$. We proceed by structural induction on the topology of an MCS, and start with acyclic MCS M.

Base case: C_k is a leaf with $In(k) = \emptyset$ and $br_k = \emptyset$ and $k \notin hist$. This means that (d) is not executed, hence, in (e), lsolve runs exactly once on $(\epsilon, \dots, \epsilon)$, and we get as result the set of all belief states $\mathcal{S} = \mathsf{lsolve}((\epsilon, \dots, \epsilon)) = \{(\epsilon, \dots, \epsilon, T_k, \epsilon, \dots, \epsilon) \mid T_k \in \mathbf{ACC}_k(kb_k)\}$. We further get that $S' \in \mathcal{S}|_V$. Towards a contradiction, assume that there is no partial equilibrium $S = (S_1, \dots, S_n)$ of M w.r.t. C_k such that $S' = S|_V$. From $In(k) = \emptyset$, we get that $IC(k) = \{k\}$, thus the partial belief state $(\epsilon, \dots, \epsilon, T_k, \epsilon, \dots, \epsilon) \in \mathcal{S}$ is a partial equilibrium of M w.r.t. C_k. Contradiction.

Induction step: assume that the import neighborhood of context C_k is $In(k) = \{i_1, \dots, i_m\}$ and

$$\begin{aligned}
\mathcal{S}^{i_1} &= C_{i_1}.\mathsf{DMCS}(V, hist \cup \{k\}), \\
&\vdots \\
\mathcal{S}^{i_m} &= C_{i_m}.\mathsf{DMCS}(V, hist \cup \{k\}),
\end{aligned}$$

such that for every $S'^{i_j} \in \mathcal{S}^{i_j}$, there exists a partial equilibrium S^{i_j} of M w.r.t. C_{i_j} such that $S^{i_j}|_V = S'^{i_j}$. Since $In(k) \neq \emptyset$, step (d) is executed; let

$$\mathcal{T} = \mathcal{S}^{i_1} \bowtie \cdots \bowtie \mathcal{S}^{i_m}$$

be the result of calling all DMCS at $C_{i_1}, \dots, C_{i_m}$. Furthermore, let $\mathcal{S} = \bigcup\{\mathsf{lsolve}(S) \mid S \in \mathcal{T}\}$ be the result of executing step (e). Eventually, $S' \in \mathcal{S}|_V$. Since every DMCS at $C_{i_1}, \dots, C_{i_m}$ returns its partial equilibria w.r.t. C_{i_j} projected to V, we have that every $T \in \mathcal{T}$ is a partial equilibrium w.r.t. C_{i_j} projected to V. M is acyclic and we have visited all contexts from $In(k)$, thus by Lemma 7 we get that for $T \in \mathcal{T}$, $app(br_k, T)$ gives us all applicable bridge rules r regardless of $T_j = \epsilon$ in T, as $j \notin In(k)$. Hence, for all $T \in \mathcal{T}$, $\mathsf{lsolve}(T)$ returns only partial belief states, where each component is projected to V except the kth component. As every $T \in \mathcal{T}$ preserves applicability of the rules by Lemma 7, we get that for every $S' \in \mathcal{S}|_V$, there exists a partial equilibrium S of M w.r.t. C_k such that $S' = S|_V$.

The proof for cyclic M works similarly. In a run we eventually end up in a context C_i such that $i \in hist$ again and guess in step (c). Thus, all possible projected partial belief states $(\epsilon, \dots, \epsilon, S_i, \epsilon, \dots, \epsilon)$ on V will be returned and the calling context can proceed with the result. The result is then propagated throughout the system and will be returned to the first invoked instance of DMCS at C_i, which is the first context of the cycle. Here, another call to lsolve then assures that only partial equilibria survive and will be returned to the calling context, or if $C_i = C_k$, to the user.

($\Leftarrow$) We give now a proof sketch for completeness of DMCS. Let S be a partial equilibrium of M w.r.t. C_k such that $S' = S|_V$. We show now that $S' \in C_k.\mathsf{DMCS}(V, \emptyset)$.

Proof idea is as follows: we proceed as in the soundness part by structural induction on the topology of M, and in the base case for a leaf context C_k, we get that $\mathsf{lsolve}((\epsilon, \dots, \epsilon))$ give us all belief sets at C_k. Since $S_i \in \mathbf{ACC}_i(kb_i \cup \{head(r) \mid r \in app(br_i, S)\})$ for all $i \in IC(k)$, we just have to show that $S'_i = S_i|_V$. Since $V \supseteq V^*(k)$, we get by Lemma 7 that the applicability of bridge rules in br_i is preserved. Thus, at each C_i, the return value $\mathcal{S}|_V$ contains S' and so S'_i.

Proof Sketch for Proposition 4

The proof sketch is as follows: for a context C_k of an MCS $M = (C_1, \ldots, C_n)$, the set $E(k)$ contains all dependencies from contexts C_i for $i \in IC(k)$. Since we visit all $(i,j) \in E(k)$ exactly twice during DFS-traversal of M (once when calling $C_j.\mathsf{DMCS}(V, \mathit{hist})$ at C_i, and once when retrieving $\mathcal{S}|_V$ from C_j in C_i), the claim follows.

Proof Sketch of Theorem 5

($\Rightarrow$) Let $S = (S_1, \ldots, S_n)$ be an equilibrium of the MCS M. We have that S_i is an answer set of

$$R_i = kb_i \cup \{\mathit{head}(r) \mid r \in \mathit{app}(br_i, S)\}.$$

We show now that $\mathbf{S} = \bigcup_{1 \leq i \leq n} S_i$ is a model of $\pi(M)$ by showing that, for all $1 \leq i \leq n$,

(i) $\mathbf{S} \models \kappa(kb_i)$,

(ii) $\mathbf{S} \models \kappa(\ell(br_i))$, and

(iii) $\mathbf{S} \models \lambda(C_i)$

Part (i): From S_i being an answer set of R_i, we get $S_i \models r$ for all $r \in kb_i$. It follows that $S_i \models \kappa(r)$, hence $S_i \models \kappa(kb_i)$, and finally, we get that $\mathbf{S} \models \kappa(kb_i)$.

Part (ii): Take an arbitrary bridge rule $r \in br_i$ of form (1):

- If $r \notin \mathit{app}(br_i, S)$, then either there exists $(c_i : p_i)$ in r such that $p_i \notin S_{c_i}$ for $1 \leq i \leq j$, or there exists $(c_k : p_k)$ in r such that $p_k \in S_{c_k}$ for $j+1 \leq k \leq m$. Since the sets Σ_i are pairwise disjoint, we get that $\mathbf{S}$ does not satisfy the antecedent of $\kappa(\ell(r))$.
- If $r \in \mathit{app}(br_i, S)$, then for all $(c_i : p_i)$ for $1 \leq i \leq j$, $p_i \in S_{c_i}$, and for all $(c_k : p_k)$ for $j+1 \leq k \leq m$, $p_k \notin S_{c_k}$. As the sets Σ_i are pairwise disjoint, we have that $\mathbf{S}$ satisfies the antecedent of $\kappa(\ell(r))$. Furthermore, since r is applicable in S, $\mathit{head}(r)$ was added to R_i to determine S_i, thus $\mathbf{S} \models \mathit{head}(r)$ and so we have that $\mathbf{S}$ satisfies the consequent of $\kappa(\ell(r))$.

In both cases we can derive that $\mathbf{S} \models \kappa(\ell(r))$ for all $r \in br_i$, and eventually $\mathbf{S} \models \kappa(\ell(br_i))$.

Part (iii): Now take an arbitrary loop $\mathcal{L} \subseteq \mathcal{A}_i$ of C_i. We have to show that

$$\mathbf{S} \models \left(\bigvee \mathcal{L}\right) \supset \varepsilon(\mathcal{L}, ER(\mathcal{L}, kb_i) \cup SR(\mathcal{L}, \ell(br_i))) \quad (3)$$

holds. If $\mathcal{L} \cap S_i = \emptyset$, then (3) holds vacuously. Otherwise, $\mathbf{S} \models \bigvee \mathcal{L}$, so we have to show that $\mathbf{S}$ satisfies the consequent of (3), i.e., for some $\mathbf{r} \in ER(\mathcal{L}, kb_i) \cup SR(\mathcal{L}, \ell(br_i))$, we get

$$\mathbf{S} \models \bigwedge B^+(\mathbf{r}) \wedge \bigwedge \neg.B^-(\mathbf{r}) \wedge \bigwedge \neg.(H(\mathbf{r}) \setminus \mathcal{L}) \ . \quad (4)$$

Recall that S_i is an answer set of R_i. We obtain that $T = S_i \setminus \mathcal{L}$ is not a model of $R_i^{S_i}$. There exists an $\bar{r} \in R_i^{S_i}$ such that $T \models B(\bar{r})$ and $T \not\models H(\bar{r})$, thus $B^+(\bar{r}) \subseteq T$ and $B^-(\bar{r}) \cap T = \emptyset$.

We have that there is a rule $r \in R_i$ such that $\bar{r}$ is the reduced rule r. Since $T \subseteq S_i$, we get $S_i \models B(r)$, as $S_i \not\models B^-(r)$ follows from $\bar{r} \in R_i^{S_i}$. This means that $S_i \models H(r)$, hence $S_i \cap H(r) \neq \emptyset$. Since $T \not\models H(\bar{r})$, we also get that $T \cap H(r) = \emptyset$. Thus, $\mathcal{L} \cap H(r) \neq \emptyset$.

We obtain two cases for $r \in R_i$:

- $r \in kb_i$: from $B^+(\bar{r}) \subseteq T$ we get $B^+(r) \cap \mathcal{L} = \emptyset$, and from $\mathcal{L} \cap H(r) \neq \emptyset$ we can then conclude $r \in ER(\mathcal{L}, kb_i)$. We have to show that $S_i \models \varepsilon(\mathcal{L}, r)$, thus
  - $S_i \models \bigwedge B^+(r)$,
  - $S_i \models \bigwedge \neg.B^-(r)$, and
  - $S_i \models \neg.(H(r) \setminus \mathcal{L})$.

  The first two items hold by $S_i \models B(r)$. The last item holds since $T \cap H(r) = \emptyset$, hence $(S_i \cap (\mathcal{A}_i \setminus \mathcal{L})) \cap H(r) = S_i \cap (H(r) \setminus \mathcal{L}) = \emptyset$. Since $r \in ER(\mathcal{L}, kb_i)$, we set $r = \mathbf{r}$ and obtain that (4) is true, hence also (3) holds.
- $r \in \{\mathit{head}(r') \mid r' \in \mathit{app}(br_i, S)\}$: there exists a rule $r' \in \mathit{app}(br_i, S)$ of form (1) such that $r = \mathit{head}(r')$ and for all $(c_i : p_i)$ in r', $1 \leq i \leq j$, we have $p_i \in S_{c_i}$, and for all $(c_k : p_k)$ in r', $j+1 \leq k \leq m$, we have $p_k \notin S_{c_k}$. From $\mathcal{L} \cap H(r) \neq \emptyset$ we have that $\mathcal{L} \cap H(\mathit{head}(r')) \neq \emptyset$ and thus $\ell(r') \in SR(\mathcal{L}, \ell(br_i))$. We have to show for each S_i that $S_i \models \varepsilon(\mathcal{L}, \ell(r'))$, thus
  - $S_i \models \bigwedge B^+(\ell(r'))$,
  - $S_i \models \bigwedge \neg.B^-(\ell(r'))$, and
  - $S_i \models \neg.(H(\ell(r')) \setminus \mathcal{L})$.

  Since the sets Σ_i are pairwise disjoint, the first two hold as $S_i \models B(r)$ and $\mathbf{S}$ is then a model for the body of $\ell(r')$. The last one holds since $T \cap H(\mathit{head}(r')) = \emptyset$, hence $(S_i \cap (\mathcal{A}_i \setminus \mathcal{L})) \cap H(\mathit{head}(r')) = S_i \cap (H(\mathit{head}(r')) \setminus \mathcal{L}) = \emptyset$. Since $\ell(r') \in SR(\mathcal{L}, \ell(br_i))$, we set $\ell(r') = \mathbf{r}$ and obtain that (4) is true, hence also (3) holds.

We have shown that (i)–(iii) holds, and as a result, we get that $\mathbf{S}$ is a model of $\pi(M)$.

($\Leftarrow$) Let $\mathbf{S}$ be a model of $\pi(M)$. We can create a belief state $S = (S_1, \ldots, S_n)$, where each $S_i = \mathbf{S}|_{L_i}$, and show that S is an equilibrium of M (note that the sets S_i are pairwise disjoint as the sets Σ_i are pairwise disjoint). We have to show that each $S_i \in \mathbf{ACC}_i(kb_i \cup H_i)$, where $H_i = \{\mathit{head}(r) \mid r \in \mathit{app}(br_i, S)\}$, i.e., each S_i is an answer set of $kb_i \cup H_i$.

We show the following:

(i) S_i is a model of $(kb_i \cup H_i)^{S_i} = kb_i^{S_i} \cup H_i$ and

(ii) S_i is minimal.

Part (i): By $\mathbf{S} \models \kappa(kb_i)$ we immediately get that $S_i \models \kappa(kb_i)$ (since the sets Σ_i are pairwise disjoint), hence $S_i \models kb_i$ and also $S_i \models kb_i^{S_i}$.

Moreover, we have that $\mathbf{S} \models \kappa(\ell(br_i))$, that is, for all $r_\ell \in \ell(br_i)$ we have $\mathbf{S} \models \kappa(r_\ell)$. Let $r \in br_i$ of form (1) such that $r_\ell = \ell(r)$. We have two cases:

- $\mathbf{S} \not\models \bigwedge B^+(r_\ell)$ or $\mathbf{S} \not\models \bigwedge \neg.B^-(r_\ell)$: there exists a $p_i \in B^+(r_\ell)$ such that $p_i \notin \mathbf{S}$ or a $p_k \in B^-(r_\ell)$ such that $p_k \in \mathbf{S}$. Since p_i, p_k are uniquely determined (follows from our disjoint language assumption), we have that there exists $(c_i : p_i)$ in r, $1 \leq i \leq j$, or $(c_k : p_k)$ in r, $j+1 \leq k \leq m$. Thus, from our construction of S, we obtain $p_i \notin S_{c_i}$ or $p_k \in S_{c_k}$, and so we have that $r \notin app(br_i, S)$. We conclude that $head(r) \notin H_i$.
- $\mathbf{S} \models \bigwedge B^+(r_\ell)$ and $\mathbf{S} \models \bigwedge \neg.B^-(r_\ell)$ and $\mathbf{S} \models \bigvee H(r_\ell)$: for all $p_i \in B^+(r_\ell)$, $p_k \in B^-(r_\ell)$, we have $p_i \in \mathbf{S}$ and $p_k \notin \mathbf{S}$. Moreover, there exists a $p_h \in H(r_\ell)$ such that $p_h \in \mathbf{S}$. As the sets Σ_i are pairwise disjoint, all p_i, p_k, p_h are uniquely determined, i.e., for all of p_i, p_k, p_h we have $(c_i : p_i)$ from r $(1 \leq i \leq j)$, $(c_k : p_k)$ in r $(j+1 \leq k \leq m)$, and a $p_h \in H(s)$, where $s = head(r)$. By our construction of S, we obtain that all $p_i \in S_{c_i}$, and all $p_k \notin S_{c_k}$, thus $r \in app(br_i, S)$, and so is $head(r) \in H_i$. Since there exists a $p_h \in \mathbf{S}$, we obtain that $p_h \in S_i$, and so $S_i \models head(r)$.

  To sum up, $S_i \models H_i$ and $S_i \models kb_i^{S_i}$, therefore (i) holds.

Part (ii): Assume that there is a model $T_i \subset S_i$ of $kb_i^{S_i} \cup H_i$. One can show that this leads to a contradiction and the claim follows.

Proof Sketch of Theorem 6

The proof is similar to the one for Theorem 5. The essential difference is that loops are defined over the whole MCS.

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