Semantics and Complexity of Abduction from Default Theories*

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Abstract

Abductive reasoning (roughly speaking, find an explanation for observations out of hypotheses), has been recognized as an important principle of common-sense reasoning. Since logical knowledge representation is commonly based on nonclassical formalisms like default logic, autoepistemic logic, or circumscription, it is necessary to perform abductive reasoning from theories (i.e. knowledge bases) of nonclassical logics. In this paper, we investigate how abduction can be performed from theories in default logic. In particular, we present a basic model of abduction from default theories. Different modes of abduction are plausible, based on credulous and skeptical default reasoning; they appear useful for different applications such as diagnosis and planning. Moreover, we thoroughly analyze the complexity of the main abductive reasoning tasks, namely finding an explanation, deciding relevance of a hypothesis, and deciding necessity of a hypothesis. These problems are intractable even in the propositional case, and we locate them into the appropriate slots of the polynomial hierarchy. However, we also present known classes of default theories for which abduction is tractable. Moreover, we also consider first-order default theories, based on domain closure and the unique names assumption. In this setting, the abduction tasks are decidable, but have exponentially higher complexity than in the propositional case.

1 Introduction

Abductive reasoning has been recognized as an important principle of common-sense reasoning having fruitful applications in a number of areas such diverse as model-based diagnosis [46, 45, 16], speech recognition [30], maintenance of database views [34], and vision [12]. Various formalizations of abductive reasoning have been proposed, among which set-covering based approaches [43, 7] and logic-based approaches [46, 45, 15, 16] are well-known. These two types basically differ

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in the way domain knowledge is represented. Roughly, in the set-covering approach, the domain knowledge is represented by a function e which maps subsets X of hypotheses, which are atomic entities representing all possible disorders, to subsets $e(X)$ of manifestations, i.e. observed symptoms. The set $e(X)$ can be seen as the explanation power of X; if X explains all manifestations M, i.e. $e(X) = M$, then X is an abductive explanation. On the other hand, in the logic-based approach the domain knowledge is represented by a logical theory T in some language. A subset X of hypotheses is an abductive explanation, if T augmented by X derives the manifestations M. For a more detailed comparison of these and other approaches to abduction, see [22].

Until now, mainly logical abduction from theories of classical logic has been studied. However, logical knowledge representation is commonly based on nonclassical formalisms like default logic, autoepistemic logic, or circumscription. Thus, in such situations it is necessary to perform abductive reasoning from theories (i.e. knowledge bases) of nonclassical logics; in a sense, this is orthogonal to what is known as hybrid reasoning, i.e., reasoning on a knowledge base built using different formalisms [2].

Since default logic [52] is one of the most used logical knowledge representation languages that emerged in the field of nonmonotonic reasoning (cf. [53, 18]), it is important to investigate how abduction can be performed from theories $\langle W, D\rangle$ in default logic. In this paper, we address this problem. We start by first considering some motivating examples; they show that abductive reasoning in default logic is needed, and lead us towards a formal model of abduction from default theories.

Example 1 Consider the following set of default rules, which represent some knowledge about Bill's skiing habits:

$$D = \left\{ \frac{:\neg skiing(Bill)}{\neg skiing(Bill)},\quad \frac{weekend : \neg snowing}{skiing(Bill)},\quad \frac{:\neg snowing}{\neg snowing} \right\}.$$

The defaults intuitively state the following: Bill is usually not out for skiing; on the weekend, Bill is usually out for skiing, unless it is snowing; and usually, it is not snowing. For the certain knowledge $W = \{weekend\}$ (encoding that it is Saturday or Sunday), the default theory $T = \langle W, D\rangle$ has one extension which contains $\neg snowing$ and $skiing(Bill)$.

Suppose now that we observe that Bill is not out for skiing (which is not consistent with the extension). Abduction means to find an explanation for this observation, that is, to identify a set of facts, chosen from a set of hypotheses, whose presence in the theory at hand would derive the observation $\neg skiing(Bill)$, i.e., cause that $\neg skiing(Bill)$ is in the extension. We find such an explanation by adopting the hypothesis *snowing*. Indeed, if we add *snowing* to W, we obtain for the default theory $T' = \langle\{weekend, snowing\}, D\rangle$ a single extension, which contains $\neg skiing(Bill)$. We say that *snowing* is *abduced* from the observation $\neg skiing(Bill)$, or that it is an *abductive explanation* of $\neg skiing(Bill)$. ∎

Observe that the description of the above situation requires the specification of some default properties that cannot be represented properly in classical logic.

Example 2 Assume that information about a computer network is represented using default logic. The default theory $T = \langle W, D\rangle$ described next comprises in a simplified setting knowledge

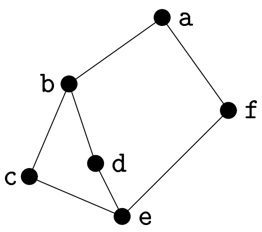

Figure 1: A computer network

about relationships between the status of a site (working or not), connections between sites, and reachability of one site from another (which can be established by a path following direct connections), together with information about connections:

$$W = \left\{ \begin{array}{l} \forall x.\, works(x) \supset path(x,x), \\ \forall x,y.\, \neg works(x) \supset \neg reaches(y,x), \\ conn(a,b),\ conn(a,f),\ conn(b,c), \\ conn(b,d),\ conn(c,e),\ conn(d,e),\ conn(e,f) \\ \forall x.\, conn(x,x), \quad \forall x,y.\, conn(x,y) \supset conn(y,x) \end{array} \right\}$$

The first two formulas in W state that if a site works, then there is a trivial path from this site to itself, and that if a site does not work, then it can not be reached from any node. The facts $conn(s_1,s_2)$ state direct connections between sites. For convenience, reflexivity and symmetry of the connectivity relationship is taken care of by the respective axioms.[1] A graphical representation of these connections is depicted in Figure 1.

The default rules are as follows.

$$D = \left\{ \begin{array}{c} \dfrac{:works(x)}{works(x)},\quad \dfrac{path(x,y):}{reaches(x,y)},\quad \dfrac{:\neg path(x,y)}{\neg reaches(x,y)}, \\ \dfrac{path(x,y) \wedge conn(y,z) \wedge works(z):}{path(x,z)} \end{array} \right\}$$

The first rule states that a site works by default; the next two rules relate paths to reachability, by adopting that x reaches y if a path from x to y provably exists, and does not reach it otherwise. The last rules states that a path can be extended by following a direct connection, if the site at the other end is working; here, it is assumed for simplicity that the connections are reliable, i.e., no communication failures occur.

In this representation, we implicitly adopt the common *domain closure axiom* $\forall x.\, x = c_1 \vee \cdots \vee x = c_n$, where $c_1, \ldots, c_n$ are all the objects (resp. individuals) of the underlying domain, which are those mentioned in T (in our case, all sites), and the axiom of the *unique names assumption* $\bigwedge_{i<j} c_i \neq c_j$, which expresses that all objects mentioned in T are different.

[1]Negated facts $\neg conn(s_1,s_2)$, stating that there is no direct connection between sites s_1 and s_2, are omitted. If desired, they can be derived using the closed world assumption, by introducing defaults $\frac{:\neg conn(x,y)}{\neg conn(x,y)}$ in D.

In this setting, the default theory T has a single extension E, in which all sites work and each site reaches any other site; notice that $reaches(x, y)$ is derivable iff there is a path $x = s_0, s_1, \ldots, s_n = y$ in the network such that $works(s_i)$ is true for all $i = 0, \ldots, n$ and $conn(s_{j-1}, s_j)$ is true for all $j = 1, \ldots, n$.

We remark that the knowledge represented in T cannot be easily represented in classical logic. Indeed, reachability between sites refers to the transitive closure of a graph; it is well-known that transitive closure can not be expressed in classical first-order logic (cf. [1]). Explicit storage in terms of e.g. ground atoms $conn^+(s_1, s_2)$ has the obvious disadvantages of inflexibility and maintenance cost, if the network is extended or its topology changes. However, default logic allows for an elegant representation of transitive closure and reachability.

Suppose now we observe that site a works but site e is not reachable from it, i.e., $works(a)$ and $\neg reaches(a, e)$ are true (which is not the case in the extension of T). If we look for an explanation of these manifestations in terms of sites that are down (while we disallow for assertion of facts $path(x, y)$), we find that –as expected– $\neg works(e)$ is an explanation; indeed, from the formula $\forall . x, y\, \neg works(x) \supset \neg reaches(y, x)$ it follows then immediately $\neg reaches(a, e)$, and, by the first default rule in D, $works(a)$ is true.

Other possible explanations exist besides $\neg works(e)$. In fact, any subset $S \subseteq \{b, \ldots, f\}$ of sites whose failure causes a network partition so that a and e lie in different partitions gives rise to an explanation, by adopting $\neg works(s)$ for every $s \in S$. Thus, e.g. $\neg works(b)$, $\neg works(f)$ and $\neg works(c)$, $\neg works(d)$, $\neg works(f)$ are possible explanations as well; on the other hand, $\neg works(c), \neg works(d)$ is not an explanation. ∎

The examples suggest a formalization of abduction from default theories in a framework similar to the one for abduction from classical theories: Given a theory T, a set of observations (or manifestations) M is to be explained, by adopting a set of formulas (an explanation) E, which are from a set of hypotheses (or abducibles) H, so that $T \cup E$ derives M (where, of course, $T \cup E$ is consistent).

In Example 1, we had $M = \{\neg skiing(Bill)\}$ and $H = \{ snowing \}$, and in Example 2 $M = \{ works(a), \neg reaches(a, e) \}$ and $H = \{\neg works(a), \cdots, \neg works(f) \}$.

In the above examples, the default theories have a single default extension. However, in general, a default theory may have multiple (or even no) extensions; just assume in Example 2 the information is available that either site b or site d is down, which is naturally represented by adding the formula $\neg works(b) \vee \neg works(d)$ to W; Then, the resulting default theory has two extensions, one containing $\neg works(b)$ and the other containing $\neg works(d)$.

As a consequence of multiple extensions, (deductive) inference from default theories comes in different modes; usually, *credulous inference*, under which a formula ϕ is inferred from a default theory T (denoted $T \vdash_c \phi$) iff ϕ belongs to at least one extension of T, and *skeptical inference*, under which ϕ follows from T ($T \vdash_s \phi$) iff ϕ belongs to all extensions of T. Accordingly, the variants of *credulous* abduction, where inference of the observations is based on $\vdash_c$, and *skeptical* abduction, which is based on $\vdash_s$, from default theories arise.

In practice, the user will choose credulous or skeptical abduction on the basis of the particular application domain. Both modes of abduction appear to be useful and relevant. To emphasize this, we argue that credulous abduction is well-suited for *diagnosis*, while skeptical abduction is adequate for *planning* (see Section 3). Notice that the applicability and use of abduction in these domains is well-known, starting from Poole's work on abductive logic-based diagnosis [44, 45, 46] and the work of Eshghi and others on abductive planning and plan recognition, cf. [24, 42].

On the computational side, we have to ask for the complexity of abduction from default logic, in order to find suitable algorithms for abduction. Towards this, we analyze the complexity of the main abductive reasoning tasks, formulated in the context of default abduction. Informally, these tasks amount to deciding whether an explanation exists (called *Consistency*), whether a particular hypothesis belongs to some explanation (*Relevance*) and whether a particular hypothesis belongs to every explanation (*Necessity*).

Consistency is self-explanatory; Relevance is important to separate hypotheses that contribute to an explanation from those which do not, and thus supports focusing the reasoning process. Necessity is important for computing the core of an explanation, which consists of the intersection of all explanations.

For classical theories, the complexity of logic-based abduction has been widely studied in the propositional context [57, 6, 22]. Furthermore, the complexity of reasoning from propositional default theories has been thoroughly analyzed in [58, 35, 27, 59]. The results showed that both abduction and default reasoning are harder than classical logic, and render problems that are complete for classes of the second level of the polynomial hierarchy (in particular, for Σ_2^P and Π_2^P), while classical logic is complete for classes of the first level (in particular, NP and co-NP).

Clearly, abduction from default theories is intractable as well, since default reasoning occurs as a subtask. However, it is not immediately clear how the complexities of abduction and default reasoning combine. We answer this question by precisely determining the complexity of each default abduction task, and locate it in the appropriate slot of the polynomial hierarchy proving completeness for the suitable class. The main complexity results can be summarized as follows:

- Default abduction is at least a hard as (deductive) default reasoning. The "easiest" setting is credulous default abduction, which is complete for Σ_2^P and Π_2^P, respectively; on the other hand, the "hardest" task –Relevance in skeptical default abduction based on minimal explanations (i.e., explanations that do not contain any other explanation properly)– is complete for Σ_4^P, which is loosely speaking the "square" of Σ_2^P. Other settings yield abduction problems that are complete for well-known classes in between Σ_2^P resp. Π_2^P and Σ_4^P;

- Skeptical default abduction is precisely one level harder than credulous default abduction in the polynomial hierarchy. This contrasts with usual results for nonmonotonic formalisms, where skeptical reasoning has complexity complementary to credulous reasoning (typically, Π_2^P and Σ_2^P), and is at the same level in the polynomial hierarchy;

- the Relevance problem has complexity complementary to the Necessity problem; and

- Minimality of explanations is a source of complexity for Relevance, and makes the problem harder. This contrasts with results for abduction from classical theories [22].

Since abduction from default theories is intractable, it is important to find problem restrictions under which polynomial time algorithms are possible. For default reasoning, the border between tractable and intractable fragments has been sharply marked by the excellent work of Kautz and Selman [35] and Stillman [58]. As default abduction builds on default reasoning, tractability of default abduction has to be based on a tractable fragment for default reasoning. Following this approach, we point out two classes of default theories from which abduction tasks are tractable. These are Literal-Horn default theories, where W consists of literals and D contains only so called Horn defaults, and Krom-pf-normal default theories, where W consists of Krom clauses (clauses of size ≤ 2) and D contains only prerequisite-free normal defaults that allow to conclude a conjunction of literals; see Section 5 for details.

We also discuss the complexity of abduction from first-order default theories. Of course, this is undecidable in general as first-order logic is undecidable. However, adopting the common domain closure axiom and the unique names assumption, we obtain that abduction from default theories is decidable. Here, the complexity parallels the complexity of the propositional case, increased by an exponential; this means that the abduction problems we consider are provably intractable.

The rest of this paper is organized as follows. The following Section 2 recalls the necessary concepts of default logic and complexity theory, and introduces some basic notation. Section 3 presents the framework for abduction from default theories and formally states the main abductive reasoning tasks in it. In the course of that, the use of credulous vs skeptical abduction is discussed. Section 4 describes and discusses the results of our analysis of the complexity of default abduction in the propositional context; the subsequent Section 5 addresses the issue of tackling default abduction in practice, and focuses on tractable cases. The complexity of abduction from first-order default theories is addressed in Section 6. Finally, Section 7 concludes the paper and outlines issues for further research.

In order to improve readability, the proofs of the results, except a few short, transparent proofs, have been moved to the appendix.

2 Preliminaries and notation

We assume that the reader knows about the basic concepts of default logic [52] (cf. also [38] for an extensive study of the subject).

Let $\mathcal{L}$ be a first-order predicate language. A default theory is a pair $T = \langle W, D \rangle$ of a set W of formulas from $\mathcal{L}$ without free variables and a set D of default rules d of form

$$\frac{\alpha(\mathbf{x}) : \beta_1(\mathbf{x}), \ldots, \beta_m(\mathbf{x})}{\gamma(\mathbf{x})}, \qquad m \geq 1$$

where $\alpha(\mathbf{x})$ (the *prerequisite*), $\beta_1(\mathbf{x}), \ldots, \beta_m(\mathbf{x})$ (the *justifications*), and $\gamma(\mathbf{x})$ (the *conclusion*) are from $\mathcal{L}$, and the free variables of all these formulas are among the variables in $\mathbf{x} = x_1, \ldots, x_n$. We omit α and any β_i if it is a tautology. If no free variables occur, the default is closed. A default theory $\langle W, D \rangle$ is closed if every default in D is closed.

The semantics of a closed default theory $T = \langle W, D \rangle$ is defined in terms of extensions. A set $E \subseteq \mathcal{L}$ is an *extension* of T iff $E = \bigcup_{i=0}^{\infty} E_i$, where

$E_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{\gamma \mid \tfrac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D,\, E_i \vdash \alpha,\, \neg\beta_1 \notin E, \ldots \neg\beta_m \notin E\right\},^2$$

where $\vdash$ is classical inference and $Th(\cdot)$ denotes classical deductive closure. The extensions of a general default theory $T = \langle W, D\rangle$ are by definition the extensions of the closure of T, $cl(T)$, which is the default theory $cl(T) = \langle W, \{\, d\sigma \mid d(\mathbf{x}) \in D,\ \sigma \text{ is a ground substitution for } \mathbf{x}\}\rangle$.

A default theory may have several (or even no) extensions in general. It is well-known that different extensions $E1$ and $E2$ are incomparable, i.e., $E1 \subseteq E2$ implies $E1 = E2$, and that $\mathcal{L}$ is an extension of T iff W is not consistent.

Each extension of a closed default theory $T = \langle W, D\rangle$ can be characterized in terms of its generating defaults. Let for any set S of closed formulas denote by $GD(S,T)$ the set of all defaults $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D$ such that $S \vdash \alpha$ and $\neg\beta_i \notin S$, for all $i = 1\ldots, m$, and denote by $concl(D')$ the set of all conclusions of defaults in the set D'. Then,

Proposition 2.1 [52] *Let E be an extension of the closed default theory $T = \langle W, D\rangle$. Then, $E = Th(W \cup concl(GD(E,T)))$.*

On the other hand, if a deductively closed formula set S satisfies $S = Th(W \cup concl(GD(S,T)))$, then S is not necessarily an extension; however, it is true if T is prerequisite-free, i.e., the prerequisite of each default in D is a tautology.

A default rule is called *normal* if it is of form $\frac{\alpha:\beta}{\beta}$. The class of *normal default theories*, where each default in D is normal, is considered to be one of the most important fragments of default logic. Normal default theories always have extensions; moreover, they enjoy the property of semi-monotonicity: If E is an extension of a normal default theory $\langle W, D\rangle$ and $D' \supseteq D$ contains only normal defaults, then $\langle W, D'\rangle$ has an extension $E' \supseteq E$ [52].

The standard variants of inference from a default theory T are *credulous inference* (denoted $\vdash_c$), under which ϕ is inferred iff ϕ belongs to at least one extension of T, and *skeptical inference* ($\vdash_s$), under which ϕ follows iff ϕ belongs to all extensions of T.

For NP-completeness and complexity theory, cf. [31]. The classes Σ_k^P and Π_k^P of the polynomial hierarchy are defined as follows:

$$\Sigma_0^P = \Pi_0^P = \mathrm{P} \quad \text{and for all } k > 1, \quad \Sigma_k^P = \mathrm{NP}^{\Sigma_{k-1}^P},\ \ \Pi_k^P = \text{co-}\Sigma_k^P.$$

In particular, $\mathrm{NP} = \Sigma_1^P$, co-NP $= \Pi_1^P$, and $\Sigma_2^P = \mathrm{NP}^{\mathrm{NP}}$. Intuitively, $\mathrm{NP}^{\Sigma_{k-1}^P}$ models computability by a nondeterministic polynomial-time algorithm which may use an oracle (loosely speaking, a subprogram without cost) for solving a problem in Σ_{k-1}^P. In the context of nonmonotonic reasoning, an oracle for NP (which allows to decide classical propositional satisfiability and hence also inference $\vdash$) is common.

The class $\mathrm{D}_\mathrm{k}^\mathrm{P}$ is defined as the class of problems that consist of the conjunction of two (independent) problems from Σ_k^P and Π_k^P. Notice that for all $k \geq 1$, clearly $\Sigma_k^P \subseteq \mathrm{D}_\mathrm{k}^\mathrm{P} \subseteq \Sigma_{k+1}^P$; both inclusions are widely conjectured to be strict, since equality of the left or the right one would imply the collapse of the polynomial hierarchy. Recall that a problem A is complete for a class C iff A belongs to C and every problem in C is reducible to it by a polynomial-time transformation; intuitively, a problem complete for C is the hardest problem in C.

[2] We use here "$E_i \vdash \alpha$ instead of "$\alpha \in E_i$" as usual, which gives rise to the same concept.

It appeared that many computational problems in nonmonotonic reasoning are complete for classes at the lower end of the polynomial hierarchy [9, 41]. In particular, it is well-known that deciding whether a propositional default theory has an extension is Σ_2^P-complete, and that credulous reasoning and skeptical reasoning from default theories are complete for Σ_2^P and Π_2^P, respectively [27, 58]. The complexity remains unchanged if inconsistent extensions are excluded and, for the latter problems, if default theories are in addition normal [27, 59]. Cases of lower complexity and tractable fragments were identified in [35, 58]. The complexity of logic-based abduction from classical theories has been analyzed in [57, 6, 22] (cf. [7] for a comprehensive analysis of the set covering approach). Basically, abduction has the same complexity as default reasoning, and bears Σ_2^P-complete and Π_2^P-complete reasoning problems.

3 Formalizing default abduction

In this section, we describe a basic formal model for abduction from propositional default theories and state the main decisional reasoning tasks for abductive reasoning.

Our formalization of an abduction scenario is as follows.

Definition 3.1 *A default abduction problem (DAP) is a quadruple $\langle H, M, W, D\rangle$ where H is a set of ground (i.e., variable-free) literals (*called hypotheses, or abducibles*), M is a set of ground literals (*called observations, or manifestations*), and $\langle W, D\rangle$ is a default theory.*
We say a DAP $\mathcal{P} = \langle H, M, W, D\rangle$ is normal iff $\langle W, D\rangle$ is normal. ∎

This definition reflects the intention that abduction explains particular observations, stated by possibly negated facts, in terms of a collection of elementary hypotheses, which are possibly negated facts. Such a setting is common in logic-based abduction, cf. [57]. In fact, it is not a shortcoming over allowing arbitrary closed formulas as hypotheses or manifestations, since for each non-literal hypothesis (resp. manifestation) ϕ a new propositional atom a_ϕ might be introduced, and after adding the formula $a_\phi \leftrightarrow \phi$ to W, ϕ can equivalently be replaced by a_ϕ.

If all positive (resp. negative) ground literals $p(t)$ from a predicate p are hypotheses, it is convenient to say that p (resp. $\neg p$) is abducible and include p (resp. $\neg p$) in H rather than all literals $p(t)$ (resp. $\neg p(t)$). Accordingly, in Example 2 $\neg works$ is abducible, and the set H is simply $H = \{\neg works\}$ instead of $H = \{\neg works(a), \ldots, \neg works(f)\}$.

We next formally define explanations, based on credulous resp. skeptical default inference.

Definition 3.2 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a DAP, and let $E \subseteq H$. Then, E is a credulous explanation for $\mathcal{P}$ iff*

(i) $\langle W \cup E, D\rangle \vdash_c M$, and

(ii) $\langle W \cup E, D\rangle$ has a consistent extension.

Similarly, E is a skeptical explanation for $\mathcal{P}$ iff

(i) $\langle W \cup E, D\rangle \vdash_s M$, and

(ii) $\langle W \cup E, D \rangle$ *has a consistent extension.*

The existence of a consistent extension for $\langle W \cup E, D \rangle$ (in this case, all extensions are consistent) assures that the explanation E is consistent with the knowledge represented in $\langle W, D \rangle$. This requirement is analogous to the usual consistency criterion in abduction from classical theories.

It is common in abductive reasoning to prune the set of all explanations and to focus, guided by some principle of explanation preference, on a set of preferred explanations. The most important such principle is, following Occam's principle of parsimony, to prefer nonredundant explanations, i.e., explanations which do not contain any other explanation properly (cf. [43, 57, 37]). We refer to such explanations as *minimal explanations.* (See [43, 7, 22] for other preferences on explanations).

Example 1 (ctd.) Suppose the certain knowledge W is extended by $\forall x. broken\_leg(x) \supset \neg skiing(x)$ to $W1$. Then, for $M = \{ \neg skiing(Bill) \}$ $H = \{ snowing, broken\_leg \}$, the explanations for $\langle H, M, W1, D \rangle$, under credulous as well as skeptical inference, are: $E1 = \{snowing\}$, $E2 = \{broken\_leg(Bill)\}$, and $E3 = \{snowing, broken\_leg(Bill) \}$. The minimal explanations are $E1$ and $E2$; they are clearly preferred to $E3$.

Example 2 (ctd.) Reconsider to explain $M = \{ works(a), \neg reaches(a, e) \}$ from $H = \{ \neg works \}$. The explanations for $\langle H, M, W, D \rangle$ correspond 1-1 to the subsets of sites $S \subseteq \{b, \ldots, f\}$ whose simultaneous failure causes a and e to be in two different partitions of the network. The minimal explanations thus correspond to the minimal such sets S, which are: $\{e\}$, $\{b, f\}$, and $\{c, d, f\}$. Intuitively, they amount to the explanations a rational agent is willing to accept, namely that e is down, that b and f are down, or that c, d and f are down.

In the sequel, we will write $\mathit{Exp}(\mathcal{P})$ for the set of explanations for the *DAP* $\mathcal{P}$, abstracting from the chosen type of explanations (credulous, skeptical, minimal credulous, or minimal skeptical).

3.1 Credulous vs skeptical abduction

As already mentioned in Section 1, both modes of abduction –credulous and skeptical abduction– appear to be useful in practice. For example, credulous abduction is suitable for diagnosis, while skeptical planning is appropriate for planning, which we discuss next.

3.1.1 Credulous abduction: diagnosis

Consider a system represented by a default theory $T = \langle W, D \rangle$. If the system receives some input, reflected by adding a set A of facts to W, then each extension of $T' = \langle W \cup A, D \rangle$ is a possible evolution of the system, i.e., each extension represents a possible reaction of the system to A.

Abductive diagnosis consists, loosely speaking, in deriving from an observed system state (given by M), a suitable input A which caused this evolution (cf. [46]). Now, since *each* extension of $\langle W \cup A, D \rangle$ is a possible evolution of the system with input A, we can assert that A is a possible input that caused F if $\langle W \cup A, D\rangle \vdash_c F$. Thus, diagnostic problems can be naturally represented by abductive problems with credulous inference.

In the network example above, finding an explanation for the observations $works(a)$ and $\neg reaches(a,e)$ is a diagnostic problem; minimal explanations correspond to the intuitive diagnoses, i.e., alternative collections of possibly broken machines. However, each of the explanations E_i is acceptable under both credulous and skeptical inference, since the default theory $\langle W \cup E_i, D\rangle$ has a single extension.

To show the use of credulous inference for diagnosis, we consider the scenario where the additional information is available that either b or d is down, i.e., W contains also the formula $\neg works(b) \vee \neg works(d)$. Then, the default theory $\langle W, D\rangle$ has two extensions, one in which b is down (while d is up) and another one in which d is down (while b is up). Suppose now we want to find a diagnosis for the observations $(works(a),\ \neg reaches(a,e))$ in which the actual status of b and d is open; that is, we focus on a partial, consistency-based diagnosis in terms of hypotheses from $\{\neg works(c),\ \neg works(e),\ \neg works(f)\}$, which might be extendible to a complete diagnosis, in which the status of each site is pinned down. Intuitively, the fact $\neg works(f)$ is then an acceptable diagnosis; in fact, it allows for a complete diagnosis (in which $\neg works(b)$ is true and thus all paths from a to e are broken). As easily checked, $E = \{\neg works(f)\}$ is a credulous explanation for the *DAP* $\mathcal{P} = \langle H, M, W, D1\rangle$ where $M = \{\, works(a), \neg reaches(a,e)\,\}$ and $H = \{\,\neg works(c),\ \neg works(e),\ \neg works(f)\,\}$. Note, however, that E is not a skeptical explanation, since $\langle W \cup E, D\rangle$ has an extension in which $reaches(a,e)$ is true.

Other diagnostic tasks in which credulous default abduction is appropriate arise in a slightly modified scenario; for example, if the knowledge about connections between sites is not complete. Assume it is known that f is connected to either e or b, but not to which of them, and a closed world assumption default $\frac{:\neg conn(x,y)}{\neg conn(x,y)}$ is present in the default theory. Then, if we concentrate on (partial) diagnoses in terms of site failures, $\neg works(b)$ would be acceptable for the above manifestations M. In fact, $\{\neg works(b)\}$ is a credulous explanation $\mathcal{P} = \langle H, M, W, D\rangle$ where $H = \{\neg works\}$.

3.1.2 Skeptical abduction: planning

Suppose now that we want that the system evolves into a certain state (described by a set F of facts), and we have to determine the "right" input (i.e., actions) that enforces this state of the system (planning). In this case it is not sufficient to choose an input A such that F is true in some possible evolution of the system; rather, we look for an A such that F is true in *all* possible evolutions, as we want be sure that the system reacts in that particular way. In other words, we look for A such that $\langle W \cup A, D\rangle \vdash_s F$. Hence, planning activities can be represented by abductive problems with skeptical inference.

Let us consider a simple planning problem in the network example. Suppose that a packet should be sent from site a to e. In particular, a route for sending the packet from a to b should be generated, taking into account possible knowledge that some sites might not work.

To model this, we augment the default theory as follows. First, a formula is added to W that states a packet can be sent to a single node only, yielding

$$W_1 = W \cup \{\, \forall x,y,z.\, send(x,y) \wedge send(x,z) \supset y = z \,\}\,;$$

we augment D to

$$D_1 = D \cup \left\{ \begin{array}{ll} \dfrac{:\neg send(x,y)}{\neg send(x,y)}, & \dfrac{send(x,y) : \neg conn(x,y) \vee \neg reaches(x,y)}{\perp} \\ \dfrac{send(x,y) : t(x,y)}{t(x,y)}, & \dfrac{t(x,y) \wedge send(y,z) : t(x,z)}{t(x,z)} \end{array} \right\};$$

here, $\perp$ is a symbol for contradiction and $t(x,y)$ states that the packet is transported from x to y. The augmented default theory is $T_1 = \langle W_1, D_1 \rangle$.

The planning problem can be solved by finding a (skeptical) explanation for $M = \{\, t(a,e) \,\}$ using $H = \{\, send \,\}$ as hypotheses, i.e., ground facts $send(s_1, s_2)$. For the original W (which describes the network in Figure 1, and no site failures are known), we then obtain three minimal explanations for $\langle H, M, W, D_1 \rangle$, namely $E_1 = \{send(a,b), send(b,c), send(c,e)\}$, $E_2 = \{send(a,b), send(b,d),\ send(d,e)\}$, and $E_3 = \{send(a,f), send(f,e)\}$; all three are in fact credulous as well as skeptical explanations. Minimality of explanations means each routing is minimal, i.e., contains no superfluous send actions; in particular, it avoids looping (e.g. $E_4 = \{send(a,b)$, $send(b,e)$, $send(c,e)$, $send(e,d)$, $send(b,d)\}$ is a nonminimal explanation that amounts to a routing in a loop). Clearly, looping could also be explicitly excluded by an axiom in W_1 stating that each node can receive a packet from only one node.

Notice that adding the information $\neg works(b) \vee \neg works(d)$ to W_1 eliminates two out of the three minimal explanations, namely E_1 and E_2; only E_3 remains as a skeptical explanation. This matches perfectly our intuitive expectations: There is an extension in which b is down, and therefore all routes for sending the packet that pass through b are not fully reliable. In particular, the routes corresponding to E_1 and E_2 are not fully reliable and have to be dropped.

Finally, if it known that besides $\neg works(b) \vee \neg works(d)$ also $\neg works(f)$ is true, then longer a skeptical explanation exists. This perfectly matches our intuitive expectation: there is an extension in which b and f are down simultaneously, and since all paths from a to e go through either b or f, it is impossible to send in this extension the packet from a to e.

3.2 Abductive reasoning tasks

The following properties of a hypothesis in a *DAP* $\mathcal{P}$ are important with respect to computing explanations.

Definition 3.3 *Let $\mathcal{P} = \langle H, M, W, D \rangle$ be a DAP and $h \in H$. Then, h is relevant for $\mathcal{P}$ iff $h \in E$ for some $E \in Exp(\mathcal{P})$, and h is necessary for $\mathcal{P}$ iff $h \in E$ for every $E \in Exp(\mathcal{P})$.*

The opposite of necessity is also termed dispensability (cf. [32]). In Example 1, *snowing* is both relevant and necessary, as $E = \{snowing\}$ is the only explanation.

The main decisional problems in abductive reasoning amount to the following. Given a *DAP* $\mathcal{P} = \langle H, M, W, D \rangle$,

(Consistency): does there exist a explanation for $\mathcal{P}$?

(Relevance): is a given ground hypothesis $h \in H$ relevant for $\mathcal{P}$, i.e., does h contribute to some explanation of $\mathcal{P}$?

(Necessity): is a given ground hypothesis $h \in H$ necessary for $\mathcal{P}$, i.e., is h contained in all explanations of $\mathcal{P}$?

In the following section, we address the complexity of these problems in the propositional context.

4 Complexity results: propositional case

In this section, we analyze the complexity of the main abductive reasoning tasks in the propositional context.

A propositional default abduction problem (*PDAP*) is thus a *DAP* $\mathcal{P} = \langle H, M, W, D\rangle$ such that H and M are sets of propositional formulas, and $\langle W, D\rangle$ is a propositional default theory. For solving the abductive reasoning tasks, we suppose that the input *PDAP* $\mathcal{P}$ is finite, i.e., H, M, W, and D are finite.

The main results on the complexity of abduction from general propositional default theories are summarized in Table 1. Each entry C represents completeness for the class C. In our analysis, we pay particular attention to normal *PDAP*s, since this class corresponds to the most important fragment of default logic. The results in Table 1, with the exception of recognizing a minimal credulous explanation, hold even if the underlying default theory $\langle W, D\rangle$ is normal. Thus like deduction, abduction from normal default theories is as hard as abduction from arbitrary default theories.

PDAP $\mathcal{P} = \langle H, M, W, D\rangle$	arbitrary explanations		minimal explanations	
Problem:	credulous	skeptical	credulous	skeptical
Consistency	$\mathbf{\Sigma}_2^P$	$\mathbf{\Sigma}_3^P$	$\mathbf{\Sigma}_2^P$	$\mathbf{\Sigma}_3^P$
Recognition: $E \in \mathit{Exp}(\mathcal{P})$?	$\mathbf{\Sigma}_2^P$	D_2^P	D_2^P	$\mathbf{\Pi}_3^P$
$E \in \mathit{Exp}(\mathcal{P})$ is minimal ?	$\mathbf{\Pi}_2^P$	$\mathbf{\Pi}_3^P$	—	—
Relevance	$\mathbf{\Sigma}_2^P$	$\mathbf{\Sigma}_3^P$	$\mathbf{\Sigma}_3^P$	$\mathbf{\Sigma}_4^P$
Necessity	$\mathbf{\Pi}_2^P$	$\mathbf{\Pi}_3^P$	$\mathbf{\Pi}_2^P$	$\mathbf{\Pi}_3^P$

Table 1: Complexity of abduction from propositional default theories (completeness results)

The results can be commented and explained as follows. The “easiest” abductive reasoning tasks have the same complexity as (deductive) inference from default theories, namely Σ_2^P and Π_2^P.

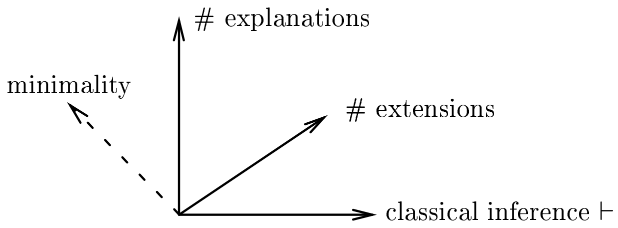

Figure 2: Orthogonal sources of complexity for **Relevance**, in skeptical abduction using minimal explanations

respectively, while the "hardest" task has complexity Σ_4^P (**Relevance** under minimal credulous explanations), which is, loosely speaking the "square" of Σ_2^P. The other tasks have intermediate complexity.

As argued in [9, 20], the level of the polynomial hierarchy at which a problem resides shows that the problem suffers from (at least) that many sources of inherent complexity, and they are intermingled so that, in a sense, they act "orthogonally." For the hardest problem in Table 1 (Σ_4^P-completeness), we thus must have four such orthogonal sources of complexity. In fact, they can be identified as follows (cf. Figure 2): (1.) classical deductive inference ($\vdash$), (2.) the number of extensions of $\langle W \cup E, D\rangle$, (3.) the number of candidates E for a skeptical explanation, and (4.) the number of possible smaller explanations (i.e., minimality); here these numbers can be exponential in the problem size.

If we compare credulous abduction to skeptical abduction, we find that for the main reasoning tasks, skeptical abduction is always precisely one level harder than credulous abduction. This contrasts with the results for many nonmonotonic reasoning formalisms (default logic, autoepistemic logic, circumscription, ...) obtained so far (see [9, 41]), where skeptical reasoning has complexity complementary to credulous reasoning (typically, complete for Π_2^P while credulous reasoning is complete for Σ_2^P). The reason for this interesting phenomenon is that a credulous explanation and an extension proving the explanation property can be nondeterministically generated simultaneously, which saves one level in the hierarchy over skeptical explanations (see below for a more formal account).

Minimality of extensions is a source of complexity for **Relevance** and causes an increase in complexity by one level in the polynomial hierarchy. However, it does not add to the complexity of **Necessity** due to the following simple fact.

Proposition 4.1 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a finite PDAP and let $h \in H$. Then, h is necessary for $\mathcal{P}$ under minimal credulous (resp. skeptical) explanations iff h is necessary for $\mathcal{P}$ under credulous (resp. skeptical) explanations.*

Therefore, we shall not deal in our analysis explicitly with **Necessity** in the case of minimal explanations.

The results for **Relevance** contrast with corresponding results for abduction from classical theories: there, minimality of explanation is not a source of complexity [22]. This discrepancy can be explained by the fact that the inference relation $\vdash$ underlying classical abduction is monotonic, while $\vdash_c$ and $\vdash_s$ are nonmonotonic; monotonicity effects that the explanation space is well-structured, so that minimality of an explanation can be decided by looking at few candidates for smaller explanations. This benign property does not apply to $\vdash_c$ and $\vdash_s$ inference, where still an exponential candidate space remains to be checked.

By the results in Table 1, we are able to draw some conclusions about the relationship of default abduction to other forms of nonmonotonic reasoning. First, by virtue of complexity theory, the reasoning tasks that are complete for Σ_2^P or Π_2^P can be polynomially transformed into an equivalent reasoning task in any of the classical nonmonotonic formalisms such as default logic, autoepistemic logic, circumscription, etc. Thus, theorem provers for such formalisms, e.g., the DeReS system for default logic (cf. [13]), can be fruitfully used for performing default abduction as well. Such polynomial transformations can be extracted from the proofs of the complexity results, by simply composing transformations. In many cases, however, a direct and much simpler transformation can be found. In fact, we show below how default abduction can be transformed to classical default reasoning by means of a plain transformation.

For the classes Σ_3^P and Π_3^P, not many nonmonotonic logics of this complexity have been known until recently. It was known that moderately grounded AEL has this complexity [21]; it appeared that a family of similar so called ground logics has the same complexity [19]. Moreover, the recent extension of default logic by trans-epistemic defaults [50] is complete for Σ_3^P resp. Π_3^P [51]. Thus, the default abduction tasks in Table 1 that have this complexity can be efficiently translated into this logics and vices versa; implementations lack today, however.

Finally, for the hardest reasoning task (Σ_4^P-complete), there is no well-known nonmonotonic logic of similar complexity; it landmarks a region of very hard nonmonotonic reasoning problems, and shows that intuitive reasoning tasks can grow quite complex.

Besides the quantitative account, the above completeness results have also a qualitative aspect. They tell that a simply ad hoc algorithm for solving the reasoning tasks, which exhaustively searchs an exponential candidate space for each source of complexity, can not be substantially improved with concern of solving the problem on all instances. For example, consider the straightforward algorithm in Table 2. There, it is assumed that a procedure **skept-inf**(W, D, ϕ) for skeptical inference of ϕ from $\langle W, D\rangle$ is available (cf. the DeReS system [13]), and $\perp$ is any contradiction. This algorithm is exponential, even if **skept-inf** is seen as an zero-cost oracle; the complexity result for this problem (Σ_4^P-completeness, which loosely speaking means Σ_2^P-completeness modulo skeptical inference), however, indicates that there is basically no substantially better algorithm that avoids the costly for-loops. (Of course, the loops can be optimized, but overall still an exponential effort is needed. An algorithm that uses exponential space might be able to get away with a single for-loop, e.g. by systematic and exhaustive state-space search for extensions.)

Since all the results in Table 1 are negative with respect to efficient computation, we have to ask how to deal with the reasoning problems in practice. This will be addressed in Section 5.

4.1 Arbitrary explanations

A useful observation is that in general, the problems **Relevance** and **Necessity** can be easily reduced to **Consistency** and its complement, respectively, for both credulous and skeptical

Algorithm Min-Skeptical-Relevance(H, M, W, D, h)
input: A propositional *DAP* $\mathcal{P} = \langle H, M, W, D\rangle$, $h \in H$.
output: "Yes" if h is relevant for $\mathcal{P}$ under minimal skeptical explanations, "No" otherwise.

```
  procedure explanation(E): boolean    (* check if E is skeptical explanation *)
      begin φ := conjunction of all literals in M;
      return skept-inf(W ∪ E, D, φ) and not skept-inf(W ∪ E, D, ⊥);
  end explanation;
begin
  for each E ⊆ H s.t. h ∈ E do begin
      if explanation(E) then
      begin minexp := true;          (*E is candidate for a minimal explanation *)
          for each E' ⊂ E do if explanation(E') then minexp := false;
      end;
      if minexp = true then begin output("Yes"); halt end;
  end;
  output("No");
end Min-Skeptical-Relevance.
```

Table 2: Algorithm for **Relevance**

explanations. Indeed, the following is easily verified.

Proposition 4.2 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP based on credulous or skeptical explanations, and let $h \in H$. Then, E is an explanation for the PDAP $\langle H \setminus \{h\} M, W \cup \{h\}, D\rangle$ iff $E \cup \{h\}$ is an explanation for $\mathcal{P}$.*

Proof. "$\Rightarrow$": all extensions of $\langle W \cup \{h\} \cup E, D\rangle$ are consistent and contain M, and extensions exist. Hence, $E \cup \{h\}$ is an explanation for $\mathcal{P}$.
"$\Leftarrow$": $E \cup \{h\}$ is an explanation for $\mathcal{P}$ means that some consistent extension of $\langle W \cup E \cup \{h\}, D\rangle$ exists and that all (some) extensions contain M. Hence, E is an explanation for $\langle H \setminus \{h\}, M, W \cup \{h\}, D\rangle$. ∎

Corollary 4.3 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP and $h \in H$. Then, (i) h is relevant for $\mathcal{P}$ iff $\langle H \setminus \{h\}, M, W \cup \{h\}, D\rangle$ has an explanation and (ii) h is necessary for $\mathcal{P}$ iff $\langle H \setminus \{h\}, M, W, D\rangle$ has no explanation.*

Thus, if **Consistency** lies in complexity class C, then **Relevance** is also in C and **Necessity** is in co-C.

Our first result shows that abduction from default theories based on credulous explanations can be efficiently reduced to deductive reasoning from propositional default theories. This is somewhat unexpected and surprising, since in case of classical theories, abduction can not be efficiently reduced to deduction.

Given a *PDAP* $\mathcal{P} = \langle H, M, W, D \rangle$, we construct a default theory $T_{\mathcal{P}} = \langle W_{\mathcal{P}}, D_{\mathcal{P}} \rangle$ such that the credulous explanations of $\mathcal{P}$ correspond to the (consistent) extensions of $T_{\mathcal{P}}$. Indeed, define

$$\begin{aligned} W_{\mathcal{P}} &= W \cup \{a_h \supset h \mid h \in H\}, \\ D_{\mathcal{P}} &= D \cup \{\tfrac{:\neg m}{\perp} \mid m \in M\} \cup \left\{\tfrac{:a_h}{a_h}, \tfrac{:\neg a_h}{\neg a_h} \mid h \in H\right\} \end{aligned}$$

where for each $h \in H$, a_h is a new propositional atom. Informally, the atom a_h means that hypothesis h is adopted. Each credulous explanation E of $\mathcal{P}$ is verified by a consistent extension A of $\langle W \cup E, D \rangle$ that contains M. This extension A gives rise to a consistent default extension A' of $T_{\mathcal{P}}$ which augments A be new formulas. The default $\frac{:\neg m}{\perp}$ prevents an augmentation of an extension A which does not contain M, while the defaults $\frac{:a_h}{a_h}$, $\frac{:\neg a_h}{\neg a_h}$ serve to choose the explanation E, where $\frac{:a_h}{a_h}$ (resp. $\frac{:\neg a_h}{\neg a_h}$) is applied to include h in E (resp. exclude h from E). Note that using the defaults $\frac{:h}{h}$, $\frac{:\neg h}{\neg h}$ instead of $\frac{:a_h}{a_h}$, $\frac{:\neg a_h}{\neg a_h}$ does not work; in particular, applying $\frac{:\neg h}{\neg h}$ would mean to include the negation of h in E, which is different from excluding h from E.

More formally, the construction has the following property:

Theorem 4.4 *Let $\mathcal{P} = \langle H, M, W, D \rangle$ be a PDAP. Then, (i) if E is a credulous explanation for $\mathcal{P}$, then there exists a consistent extension A' of $T_{\mathcal{P}}$ such that $E = \{h \in H \mid a_h \in A'\}$; (ii) if A' is a consistent extension of $T_{\mathcal{P}}$, then $E = \{h \in H \mid a_h \in A'\}$ is a credulous explanation for $\mathcal{P}$.*

Proof. (i) Let E be a credulous explanation for $\mathcal{P}$. Hence, $\langle W \cup E, D \rangle$ has a consistent extension A which contains M. Define $A' = Th(A \cup \{a_h \mid h \in E\} \cup \{\neg a_h \mid h \in H \backslash E\})$. Clearly, A' is consistent. Moreover, A' is an extension of $\langle W_{\mathcal{P}}, D_{\mathcal{P}} \rangle$. Indeed, each default $d \in D$ that can be applied in A_i from the sequence $A_0, A_1, \ldots$ for A can also be applied in A'_i from $A'_0, A'_1, \ldots$ On the other hand, if $d \in D$ is applied in some A'_i, then it is also applied in some A_j. Since $M \subseteq A'$, no default $\frac{:\neg m}{\perp}$ can be applied. Thus, $A' = \bigcup_{i=0}^{\infty} A'_i$, and hence A' is an extension of $\langle W_{\mathcal{P}}, D_{\mathcal{P}} \rangle$. Obviously, $E = \{h \mid a_h \in A'\}$. This proves (i).

(ii) Let A' be a consistent extension of $\langle W_{\mathcal{P}}, D_{\mathcal{P}} \rangle$, and let $E = \{h \mid a_h \in A'\}$. Let A be the restriction of A' to the language of $\mathcal{P}$. Then, A is an extension of $\langle W \cup E, D \rangle$. Indeed, first notice that $E \subseteq A$. Each default $d \in D$ which is applicable in A'_i from $A'_0, A'_1, \ldots$ is also applicable in A_i from $A_0, A_1, \ldots$ On the other hand, each $d \in D$ which is applicable in A_i can be clearly applied in A'_{i+1} as well. We conclude that $A = \bigcup_{i=0}^{\infty} A_i$, and hence A is an extension of $\langle W \cup E, D \rangle$. We show that $M \subseteq A$, by which (ii) is proven. But this is easy: $M \subseteq A'$, for otherwise a default $\frac{:\neg m}{\perp}$ would be applicable; hence, by definition of A, $M \subseteq A$. ∎

Using (i) and (ii), the main decisional abductive reasoning tasks can be efficiently transformed to similar deductive reasoning tasks in default logic.

Corollary 4.5 *Let $\mathcal{P}$ be a PDAP based on credulous explanations. Then, (i)* **Consistency**, *(ii)* **Relevance**, *and (iii)* **Necessity** *are equivalent to (i^*) existence of a consistent extension of $T_{\mathcal{P}}$, (ii^*) membership of a_h in some consistent extension of $T_{\mathcal{P}}$, and (iii^*) membership of a_h in all extensions of $T_{\mathcal{P}}$, respectively.*

By the results on the complexity of propositional default logic [27, 59], it follows that (i) and (ii) are in Σ_2^P and that (iii) is in Π_2^P. We also obtain matching hardness by reductions from deductive default reasoning. Let $T = \langle W, D\rangle$ be a normal default theory such that W is a consistent set of literals (in fact, W can be even empty), and ϕ a formula. Let h, q be new propositional atoms. Then, the following facts are clear: the *PDAP*

$$\langle \emptyset, \{q\}, W \cup \{\phi \supset q\}, D\rangle \tag{1}$$

has a credulous explanation iff $T \vdash_c \phi$; h is relevant for the *PDAP*

$$\langle \{h\}, \{q\}, W \cup \{\phi \supset q\}, D\rangle \tag{2}$$

iff $T \vdash_c \phi$; and h is necessary for the *PDAP*

$$\langle \{h\}, \{q\}, W \cup \{\phi \vee h \supset q\}, D\rangle \tag{3}$$

iff $T \not\vdash_c \phi$. Since the reasoning problems for T in (1), (2) are Σ_2^P-hard and the one in (3) is Π_2^P-hard [27], we obtain the following results.

Theorem 4.6 *Let $\mathcal{P}$ be a PDAP based on credulous explanations. The problem (i)* **Consistency** *is Σ_2^P-complete, (ii)* **Relevance** *is Σ_2^P-complete and (iii)* **Necessity** *is Π_2^P-complete, with hardness even for normal $\mathcal{P}$.*

Remark. The complexity result for **Consistency** shows that a polynomial-time reduction of credulous explanations to classical extensions of a default theory $T_{\mathcal{P}}$ as above, but where the default theory is normal, is not possible in general. Indeed, if $W_{\mathcal{P}}$ is consistent, then $T_{\mathcal{P}}$ has a consistent extension, and thus the *PDAP* would also have an explanation. This would imply an NP algorithm deciding existence of a credulous explanation, and NP $= \Sigma_2^P$ would follow, which is generally believed to be false.

It is interesting to note that verifying a credulous explanation is as hard as finding one. The former problem can be easily reduced to the latter; moreover, $\emptyset$ is the only possible credulous explanation for the *PDAP* (1). Thus,

Theorem 4.7 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if $E \subseteq H$ is a credulous explanation for $\mathcal{P}$ is Σ_2^P-complete, with hardness holding even for normal $\mathcal{P}$.*

Now consider abduction based on skeptical reasoning. It would be useful to have a reduction of abductive reasoning to deductive reasoning which can be computed efficiently. However, by using skeptical reasoning the abductive reasoning tasks grow more complex, by one level of the polynomial hierarchy. This strongly suggests that such an efficient reduction is not possible.

We first consider the problem of recognizing skeptical explanations. Clearly, this reduces to deciding if a certain default theory has a consistent extension (which is in Σ_2^P) and if each extension includes all manifestations (Π_2^P). Thus, the problem is a logical conjunction of a problem in Σ_2^P and a problem in Π_2^P, and hence in the class $\mathrm{D}_2^{\mathrm{P}}$. Moreover, it is also hard for this class.

Theorem 4.8 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if $E \subseteq H$ is a skeptical explanation for $\mathcal{P}$ is* $\mathrm{D}_2^{\mathrm{P}}$*-complete.*[3]

Thus, as in the case of credulous explanations, recognizing a skeptical explanation is at the second level of the polynomial hierarchy. However, since this problem involves both a Σ_2^P and a Π_2^P-hard subtask (as opposed to only a Σ_2^P-hard one), finding a skeptical explanation resides at the third level.

Theorem 4.9 *Let $\mathcal{P}$ be a PDAP based on skeptical explanations. The problem (i)* **Consistency** *is Σ_3^P-complete, (ii)* **Relevance** *is Σ_3^P-complete and (iii)* **Necessity** *is Π_3^P-complete, with hardness even for normal $\mathcal{P}$.*

How does this result compare to other nonmonotonic logics, in particular, which nonmonotonic logic has similar complexity ? We know that Konolige's moderately grounded autoepistemic logic [36] and several other ground nonmonotonic modal logics have the same complexity [21, 19]; thus, we can use a theorem prover for such logics to perform abductive reasoning from default theories based on skeptical explanations.

4.2 Minimal explanations

As mentioned above, one is usually interested in *minimal* explanations for observations. The results in [22] were that the complexity of abduction from classical theories does not increase if minimal explanations are used instead of arbitrary explanations. However, this is not true for abduction from default logic. Here, checking minimality of an explanation is a source of complexity, which causes an increase in complexity by one level of the polynomial hierarchy.

Consider first credulous explanations. Checking minimality of an explanation E has complementary complexity of checking the explanation property.

Theorem 4.10 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if a credulous explanation E for $\mathcal{P}$ is minimal is Π_2^P-complete, with hardness even for normal $\mathcal{P}$.*[4]

To see this, notice that E is *not* minimal iff for some $h \in E$, the *PDAP* $\langle E\setminus \{h\}, M, W, D\rangle$ has a credulous explanation; hence, it follows that the problem is in Π_2^P. On the other hand, reconsider the *PDAP* (3). Clearly, $\{h\}$ is a credulous explanation; moreover, it is minimal iff h is necessary for $\mathcal{P}$. Thus, Π_2^P-hardness follows.

Note that recognizing minimal credulous explanations, which consists in checking the explanation property and testing minimality, is in $\mathrm{D}_2^{\mathrm{P}}$; it is also complete for this class (Theorem A.1). Thus, this problem can be transformed into recognition of skeptical explanations for a certain *PDAP* and vice versa.

[3] For normal *PDAP*s, the problem is "only" Π_2^P-complete, since then deciding whether $\langle W \cup E, D\rangle$ has a consistent extension is in NP.

[4] Strictly speaking, it is "promised" that E is an explanation; in fact, since recognizing explanations is $\mathrm{D}_2^{\mathrm{P}}$-complete, already recognizing correct instances would be harder than Π_2^P, unless $\Pi_2^P = \mathrm{D}_2^{\mathrm{P}}$. However, the result holds also on deciding given any E, whether no $E' \subset E$ is a credulous explanation.

Due to the complexity of minimality checking, **Relevance** migrates to the next level of the polynomial hierarchy.

Theorem 4.11 *Let $\mathcal{P}$ be a PDAP based on minimal credulous explanations. Then,* **Relevance** *is Σ_3^P-complete, and hardness holds even for normal $\mathcal{P}$.*

Now let us consider minimal skeptical explanations. Testing minimality of a skeptical explanation is much more involved than of a credulous explanation. While the latter has roughly the same complexity as testing the explanation property, the former is harder by one level of the polynomial hierarchy. Intuitively, this can be explained as follows. Since verifying a credulous explanation E is in Σ_2^P, it has a polynomial-size "proof" which can be checked with an NP oracle in polynomial time. Thus, if we ask for a smaller explanation $E' \subset E$, we can simultaneously guess E' and its proof, and check the proof in polynomial time with the NP oracle. However, verifying a skeptical explanation E is Π_2^P-hard, and hence E does not have such a "proof". Here, verification needs the full power of a Π_2^P oracle.

Theorem 4.12 *Let $\mathcal{P} = \langle H, M, W, D \rangle$ be a PDAP. Deciding if a skeptical explanation E for $\mathcal{P}$ is minimal is Π_3^P-complete, with hardness holding even for normal $\mathcal{P}$.*[5]

Note that recognizing minimal skeptical explanations is in Π_3^P, since the complexity of deciding minimality (Π_3^P) dominates the complexity of the explanation property ("only" D_2^P), and is also complete for this class (See Theorem A.2).

The complexity of deciding relevance of a hypothesis increases by the same amount as testing minimality if skeptical explanations are used instead of credulous explanations. In fact, the problem resides at the fourth level of the polynomial hierarchy.

Theorem 4.13 *Let $\mathcal{P}$ be a PDAP based on minimal skeptical explanations. Then, problem* **Relevance** *is Σ_4^P-complete, with hardness even for normal $\mathcal{P}$.*

There is no well-known nonmonotonic logic that has similar complexity, and thus one can not take advantage of theorem provers for such logics to perform skeptical abduction from default theories.

5 Tractable cases

From the practical side, the results of the previous section are discouraging, since abduction from default theories has even higher complexity than deduction, in particular for skeptical explanations. As a consequence, an implementation that works efficiently on all inputs is not possible. But this does not mean tht it is impossible to come up with solutions that work in practice. Several methods to cope with NP-hard problems are known [41, 9], among which

[5]Note that here, as opposed to checking minimality of credulous explanations, recognizing correct problem instances (D_2^P) is not harder than deciding the minimality (cf. footnote 4).

the following two are most popular: identifying (natural) problem restrictions that guarantee tractability and approximation methods.

Concerning approximation, two directions are viable. One is to find a notion of approximate inference that reduces complexity, in the spirit of [55, 10]. The other is to use heuristic algorithms that work well in practice; in case of the classical satisfiability problem, the greedy algorithm (GSAT) [56] has proven to be a valuable method for solving moderately-sized instances efficiently. However, GSAT and similar algorithms are designed for problems that are in NP; it is not known how to apply GSAT in order to solve problems that are hard for Σ_2^P or Π_2^P efficiently. In general, little is known about approximation of problems in the polynomial hierarchy, as well as PSPACE-complete problems [14].

For fragments of default logic from which the abductive reasoning tasks are in NP, GSAT is applicable. An example is credulous abduction from default theories where all propositional formulas are from a tractable fragment of the propositional language, e.g. *Horn* formulas or *Krom* formulas (conjunctions of clauses with at most two literals). In such a case, classical inference $\vdash$ vanishes as source of complexity. In particular, the Σ_2^P-complete abductive reasoning tasks fall back to NP. Therefore, after a polynomial-time transformation of the reasoning task to the classical satisfiability problem, we can apply GSAT to solve the reasoning tasks efficiently.

Identifying tractable cases in terms of suitable restrictions on the input is the "classical" way of tackling a hard problem. This approach requires that all sources of complexity are dried up; default abduction suffers from up to four such sources. In particular, the underlying default reasoning tasks (which embody the complexity of classical inference and the number of extensions) must be tractable. The excellent work of Kautz and Selman [35] and Stillman [58] gives a very detailed picture of polynomial vs. intractable cases of default reasoning. The following two classes of default theories $\langle W, D\rangle$ are among the many fragments of default logic they considered:

Literal-Horn [35]**:** W is a set of literals and each default in D is Horn, i.e., of form $\frac{a_1\wedge\cdots\wedge a_k:\ell}{\ell}$, where the a_i's are atoms and ℓ is a literal.

Krom-pf-normal [58]**:** W is a set of Krom formulas, and each default in D is of form $\frac{:\ell_1\wedge\cdots\wedge\ell_k}{\ell_1\wedge\cdots\wedge\ell_k}$, where all ℓ_i's are literals.

For those classes, tractability of credulous inference $\langle W, D\rangle \vdash_c \ell'$ for a single literal ℓ' was shown. In fact, these classes are the maximal classes in the hierarchy of default classes for which this inference task is tractable, cf. [58]. Note that Literal-Horn theories include factual knowledge bases, extended by simple default rules; however, they allow to express e.g. the close world assumption (CWA), or rules so that their application does not lead to inconsistency.

Based on generalizations of the quoted results, we obtain some tractable cases of credulous default abduction. Similar tractability results for skeptical default abduction are unlikely, since the underlying skeptical inference $\langle W, D\rangle \vdash_s \ell$ is co-NP-complete in both cases (for Literal-Horn, cf. [35, Theorem 7.3]; for Krom-pf-normal, see Proposition B.3 in Appendix B).

The following lemma generalizes [35, Theorem 6.5], which is formulated for a single literal ℓ rather than a conjunction $\ell_1 \wedge \cdots \wedge \ell_m$ of literals (see Appendix B for the proof).

Lemma 5.1 *Let $\langle W, D\rangle$ be a Literal-Horn default theory, and let $\ell_1, \ldots, \ell_m$ be literals. Then, deciding $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ is possible in $O(m \cdot n)$ time, where n is the length of the input.*

This result is established by reducing $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$, apart from the trivial cases where W or $\ell_1 \wedge \cdots \wedge \ell_m$ is not consistent, to a classical inference problem $H \vdash \ell_1 \wedge \cdots \wedge \ell_m$ where H is a Horn theory that is linear time constructible (see the proof in Appendix B); applying standard linear time algorithms for Horn satisfiability (e.g. unit resolution [40]), $H \vdash \ell_i$ can be decided in linear time, and thus $H \vdash \ell_1 \wedge \cdots \wedge \ell_m$ is decidable in $O(m \cdot n)$ time.[6] Thus, we obtain the lemma.

For Krom-pf-normal, an analogous generalization is not evident; indeed, the problem $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ is NP-hard (see Proposition B.1 in Appendix B). However, a generalization is possible if the conjunction is small. In what follows, we call a set L of literals *small* iff $|L| \leq b$ for some fixed constant b.

Lemma 5.2 *Let $\langle W, D\rangle$ be Krom-pf-normal, and let $L = \{\ell_1, \ldots, \ell_k\}$ be a small set of literals. Then, deciding $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_k$ is possible in $O(|D|^b \cdot n)$ time, i.e., in polynomial time.*

For convenience, we introduce some additional notation for the rest of this section. For any literal ℓ, we denote by $\sim\ell$ the literal opposite to ℓ, i.e., $\sim\ell = x$ if $\ell = \neg x$, and $\sim\ell = x$ if $\ell = \neg x$.

5.1 Credulous abduction from Literal-Horn default theories

For this class, the main abductive reasoning tasks are tractable.

Theorem 5.3 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP based on credulous explanations, such that $\langle W, D\rangle$ is Literal-Horn. Then,* **Consistency**, **Relevance**, *and* **Necessity** *can be solved in $O(|M| \cdot n)$ time, where n is the length of $\mathcal{P}$.*

Proof. Construct a Literal-Horn $T1 = \langle W, D1\rangle$, where

$$D1 = \left\{ \tfrac{b_h : h}{h}, \tfrac{: \neg b_h}{\neg b_h}, \tfrac{: b_h}{b_h} \mid h \in H \right\},$$

where each b_h is a new propositional atom. It is not hard to see that $\mathcal{P}$ has an explanation iff W is consistent and $T1 \vdash_c m_1 \wedge \cdots \wedge m_k$, where $M = \{m_1, \ldots, m_k\}$. $T1$ can be constructed in $O(n)$ time, and by Lemma 5.1, $T1 \vdash_c m_1 \wedge \cdots \wedge m_k$ can be decided in $O(|M| \cdot n)$ time. Hence, **Consistency** can be decided in $O(|M| \cdot n)$ time. By Corollary 4.3, **Relevance** and **Necessity** can be easily reduced (in fact, in linear time) to **Consistency** resp. its complement. Thus, they are also solvable in $O(|M| \cdot n)$ time. ∎

Notice that a polynomial algorithm for *finding* an explanation, (even containing a given hypothesis), can be extracted from the proof of Theorem 5.3 and Lemma 5.1, which is outlined in Table 3.

[6] This is even possible in $O((m^- + 1) \cdot n)$ time, where m^- is the number of negative literals ℓ_i (cf. proof of Lemma 5.1); however, to our knowledge, no $O(n)$, i.e., linear time algorithm is known. The bounds for the polynomial-time results in Section 5.1 can be sharpened accordingly, by replacing "$|M|$" with "$|M^-| + 1$", where M^- is the set of negative literals in M.

Algorithm LH-Cred-Explanation(H, M, W, D)**;**
input: A propositional Literal-Horn *DAP* $\mathcal{P} = \langle H, M, W, D \rangle$.
output: A credulous explanation E for $\mathcal{P}$, if one exists.
begin
for each $\ell \in W$ **do if** $\sim\ell \in W$ **then stop;** *(* W inconsistent; no explanation *)*
for each $\ell \in M$ **do if** $\sim\ell \in M$ **then stop;** *(* M inconsistent; no explanation *)*
$\phi :=$ conjunction of all literals in M**;**
construct $T1 = \langle W, D1 \rangle$ from $\mathcal{P}$; *(* $\mathcal{P}$ has an explanation iff $T1 \vdash_c \phi$ *)*
construct Horn theory H s.t. $H \vdash \phi$ iff $T1 \vdash_c \phi$**;** *(* proof of Lemma 5.1 *)*
if $H \not\vdash \phi$ **then stop;** *(* no explanation exists *)*
$M := \{x \mid \text{ atom } x, H \vdash x\}$**;** *(* M is the least model of H *)*
$C := W \cup M \cup \{\neg x \mid (\alpha : \neg x / \neg x) \in D1,\ M \models \alpha,\ M \not\models x\}$**;**
$E := \{h \in H \mid h \in C,\ b_h \in C\}$**;** *(* E is an explanation *)*
output(E)**;**
end LH-Cred-Explanation.

Table 3: Algorithm for finding a credulous explanation (Literal-Horn)

Algorithm LH-Min-Cred-Explanation(H, M, W, D)**;**
input: A propositional Literal-Horn *DAP* $\mathcal{P} = \langle H, M, W, D \rangle$.
output: A minimal credulous explanation E for $\mathcal{P}$, if one exists.
begin $E :=$ LH-Cred-Explanation(H, M, W, D);
if E is undefined **then stop else** $R := E$; *(* try to remove each $h \in R$ from E *)*
while $R \neq \emptyset$ **do begin** select $h \in R$; *(* try to remove h from E *)*
$E1 :=$ **LH-Cred-Explanation**$(E \setminus \{h\}, M, W, D)$;
if $E1$ is undefined **then** $R := R \setminus \{h\}$ *(* h is in all explanations $E' \subseteq E$ *)*
else begin $E := E1$; $R := E$; **end**; *(* take the smaller explanation $E1$ *)*
end;
output(E)**;**
end LH-Min-Cred-Explanation.

Table 4: Algorithm for finding a minimal credulous explanation (Literal-Horn)

Each step of the algorithm can be done in $O(n)$ time, except the test $H \vdash \phi$, which is possible in $O(|M| \cdot n)$ time; hence, the total running time is $O(|M| \cdot n)$. The algorithm can also be used to solve **Relevance** and **Necessity**, utilizing Corollary 4.3, in $O(|M| \cdot n)$ time.

Moreover, even a *minimal* credulous explanation can be found in polynomial time. Indeed, it is easy to see that an explanation E for a *PDAP* $\mathcal{P} = \langle H, M, W, D \rangle$ is minimal if and only if $\langle E \setminus \{h\}, M, W, D \rangle$ has no explanation for each $h \in E$. Thus, given an explanation E for $\mathcal{P}$ as above, one can find a minimal one by trying to subsequently eliminate hypotheses from E, as done in the algorithm in Table 4. Thus, we obtain the following result.

Theorem 5.4 *Let $\mathcal{P} = \langle H, M, W, D \rangle$ be a PDAP where $\langle W, D \rangle$ is Literal-Horn. Then, a minimal credulous explanation for $\mathcal{P}$ can be found in $O(|H| \cdot |M| \cdot n)$ time, where n is the length of $\mathcal{P}$.*

However, **Relevance** based on minimal credulous explanations for *PDAP*s with Literal-Horn default theories can be shown to be NP-complete (Proposition B.4). Recall that necessity of a hypothesis h in minimal explanations coincides with necessity of h in arbitrary explanations (Proposition 4.1), and thus also **Necessity** is polynomial.

5.2 Credulous abduction from Krom-pf-normal default theories

Tractability of credulous inference of a literal from a Krom-pf-normal theory allows us to obtain another tractable fragment of abduction from default theories.

The following lemma on inference of a literal from a Krom theory augmented by literals is important for credulous inference of a set of literals from a Krom-pf-normal default theory.

Lemma 5.5 *Let S be a (propositional) Krom theory and ℓ a literal, and let $L = \{\ell_1, \ldots, \ell_k\}$ be a set of literals such that (i) $S \cup L$ is consistent, (ii) $S \cup L \vdash \ell$ and $S \not\vdash \ell$. Then, $S \cup \{\ell_i\} \vdash \ell$ for some $\ell_i \in L$.*

From the lemma, we obtain the following criterion for credulous inference of a literal. For convenience, let for each default $d = \frac{:\ell_1 \wedge \cdots \wedge \ell_k}{\ell_1 \wedge \cdots \wedge \ell_k}$ denote $lit(d) = \{\ell_1, \ldots, \ell_k\}$, and let for each set D of such defaults denote $lit(D) = \bigcup_{d \in D} lit(d)$.

Proposition 5.6 *Let $T = \langle W, D \rangle$ be a Krom-pf-normal default theory and let $L = \{\ell_1, \ldots, \ell_m\}$ be a set of literals. Let $L1 = \{\ell_i \in L \mid W \not\vdash \ell_i\}$. Then, $T \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ iff (a) $L1$ is empty, or (b) there exists a subset $D1 \subseteq D$ such that*

- *(b.1) $|D1| \leq |L1|$,*
- *(b.2) $W \cup lit(D1)$ is consistent, and*
- *(b.3) $W \cup lit(D1) \vdash \ell$, for each $\ell \in L1$.*

Based on this proposition, the algorithm in Table 5 decides whether a conjunction of literals is a credulous consequence of a Krom-pf-normal default theory. The analysis of its running time yields the following result.

Theorem 5.7 *Let $T = \langle W, D \rangle$ be a Krom-pf-normal default theory, and let $L = \{\ell_1, \ldots, \ell_m\}$ be a set of literals. Then, $T \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ can be decided using* **Krom-PFN-Cred-Inf** *in $O(|D|^m \cdot m \cdot n)$ time, where n is the length of the input.*

Thus, in particular, if $L = \{\ell_1, \ldots, \ell_k\}$ is small, we obtain a polynomial bound $O(|D|^k \cdot n)$; this proves Lemma 5.2 at the beginning of this section. Recall that tractability of credulous literal inference can most likely not be generalized to an arbitrary conjunction of literals, since this problem is NP-complete (Proposition B.1).

By a reduction to credulous default reasoning, one can show that credulous abduction from a Krom-pf-normal default theory is polynomial if the set of manifestations is small.

```
Algorithm Krom-PFN-Cred-Inf(W, D, φ);
input:   A propositional Krom-pf-normal default theory ⟨W, D⟩,
         a conjunction of literals φ = ℓ1 ∧ ··· ∧ ℓm.
output:  "Yes" if ⟨W, D⟩ ⊢c φ, "No" otherwise.
begin L1 := { ℓi | W ⊬ ℓi, 1 ≤ i ≤ m };
  if L1 = ∅ then output("Yes")          (* W ⊢ φ; thus φ is in any extension *)
  else for each D1 ⊆ D s.t. |D1| ≤ |L1| do
      begin S := W ∪ lit(D1);       (* lit(D1) = all literals in D1; S is Krom *)
          if S ⊬ ⊥ and S ⊢ ℓi for all ℓi ∈ L1          (* m + 1 × linear time *)
          then begin output("Yes"); stop; end;
      end;
  output("No");
end Krom-PFN-Cred-Inf.
```

Table 5: Algorithm for credulous inference of a conjunction of literals (Krom-pf-normal)

Theorem 5.8 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP based on credulous explanations such that $M = \{\ell_1, \ldots, \ell_k\}$ is small and $\langle W, D\rangle$ is Krom-pf-normal. Then,* **Consistency**, **Relevance**, *and* **Necessity** *can be solved in $O(|D|^b \cdot n)$ time, where n is the length of $\mathcal{P}$.*

Unfortunately, Theorem 5.8 can not be generalized to an arbitrary set M of literals. In fact, due to the NP-hardness of $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ for Krom-pf-normal $\langle W, D\rangle$, the problem is clearly NP-hard.

Like in the case of Literal-Horn default theories, a polynomial time algorithm for finding a credulous explanation can be extracted from the proof. As shown below, there are efficient and simple algorithms for finding all minimal credulous explanations of a such a PDAP. Thus, we do not further elaborate on this point.

Interestingly, the number of hypotheses in a minimal credulous explanation is bounded by the number of manifestations. Intuitively, this is explained by the fact that a manifestation can always be explained by a single hypothesis.

Proposition 5.9 *Let E be any minimal credulous explanation for a PDAP $\mathcal{P} = \langle H, M, W, D\rangle$ where $\langle W, D\rangle$ is Krom-pf-normal and M is finite. Then, $|E| \leq |M|$.*

In particular, for a single manifestation ($M = \{\ell\}$), the minimal explanations consist of single hypotheses, if hypotheses are needed at all for an explanation. A consequence of this characterization is that all minimal credulous explanations can be computed by testing all subsets $E \subseteq H$ with $|E| \leq |M|$ time, which is done by the algorithm in Table 6. The correctness of the algorithm follows from Proposition 5.9 and the correctness of **Krom-PFN-Cred-Inf**. If the set of manifestations M is small, then the algorithm runs in polynomial time, which is captured by the next theorem.

Theorem 5.10 *Given a PDAP $\mathcal{P} = \langle H, M, W, D\rangle$ where $\langle W, D\rangle$ is Krom-pf-normal and M is small, all minimal credulous explanations for $\mathcal{P}$ can be computed in time $O(|H|^b \cdot |D|^b \cdot n)$, where n is the length of $\mathcal{P}$.*

Algorithm All-KPFN-Min-Cred-Exp(H, M, W, D);
input: A propositional Krom-pf-normal *DAP* $\mathcal{P} = \langle H, M, W, D \rangle$.
output: All minimal credulous explanations E for $\mathcal{P}$.
begin $\phi :=$ conjunction of all literals in M; $Min\_Expl := \emptyset$;
  if $W \vdash \perp$ or $\phi \vdash \perp$ **then stop;** *(* 2 × linear time; no explanation exists *)*
  for $i := 0$ **to** $|M|$ **do**
    for each $E \subseteq H$ s.t. $|E| = i$ **do**
      if $W \cup E \not\vdash \perp$ **and** $E' \notin Min\_Expl$, **for every** $E' \subset E$ **then**
        if Krom-PFN-Cred-Inf$(W \cup E, D, \phi) =$ "Yes" **then**
        begin output(E);
          $Min\_Expl := Min\_Expl \cup \{ E \}$;
        end;
end All-KPFN-Min-Cred-Exp.

Table 6: Algorithm for finding all minimal credulous explanations (Krom-pf-normal)

(Standard search data structures are used to represent Min_Expl.) As a consequence, also **Relevance** and **Necessity** for minimal explanations are polynomial if M is small.

6 Complexity of first-order case

In the previous sections, we considered the computational cost of abduction from propositional *DAP*s. However, for knowledge representation in practice, a language allowing for predicates is more relevant and realistic, as in the examples we considered above.

Since default logic subsumes first-order logic, default reasoning based on unrestricted first-order logic is undecidable in general; in fact, as shown already by Reiter, the credulous consequences are even not recursively enumerable [52]. Hence, also abduction from default logic is highly undecidable in general.

However, default logic is decidable in the common setting of the domain closure axiom ($\forall x. x = c_1 \vee \cdots \vee x = c_n$) and the unique names assumption ($\bigwedge_{i<j} c_i \neq c_j$), where the language is function-free. In the following, we implicitly assume this setting for first-order default logic; notice that this was intuitive for the examples we considered.

In this setting, an existential quantifier $\exists x \phi(x)$ is equivalent to $\phi(c_1) \vee \cdots \vee \phi(c_n)$, and a universal quantifier $\forall x \psi(x)$ to $\psi(c_1) \wedge \cdots \wedge \psi(c_n)$. Therefore, every first-order formula can be rewritten to a quantifier-free formula, which, however, is in general exponentially larger. Each default rule d with free variables $\mathbf{x} = x_1, \ldots, x_m$ gives rise to an exponential number (n^m) of ground defaults. As a consequence, a first-order default theory $T = \langle W, D \rangle$ reduces to an equivalent propositional default theory $T^* = \langle W^*, D^* \rangle$ that can be exponentially larger.

Intuitively, this means that the complexity of first-order default reasoning is (at most) exponentially higher than the complexity of propositional default reasoning (Σ_2^P). The appropriate class for such complexity is the exponential analogue of $\Sigma_2^P = \mathrm{NP}^{\mathrm{NP}}$, which is the class $\Sigma_2^E = \mathrm{NEXP}^{\mathrm{NP}}$, i.e. nondeterministic *exponential* time ($O(2^{p(n)})$) rather than polynomial time ($O(p(n))$) with an oracle for NP ("exponential" comes in by the reduction of T to T^*, after which the algorithms for propositional *DAP*s can be applied.)

In fact, it is shown in [29] that credulous first-order default reasoning is complete for Σ_2^E; as a simple consequence, skeptical reasoning is complete for $\Pi_2^E = \text{co-}\Sigma_2^E$. The proof in [29] establishes that these results hold even if W is a universal theory and no quantifiers occur in D. Moreover, like in the propositional case, they hold if in addition the default theory is normal and W is a consistent set of literals (this is immediate from [29] and results in [8]).

The polynomial-time transformations of default reasoning to the main credulous default abduction tasks in 1–3 established the hardness parts of Theorem 4.6. Clearly, these transformations also work for first-order theories. Therefore, from the complexity results for first-order default reasoning, it follows that **Consistency** and **Relevance** are Σ_2^E-hard and **Necessity** is Π_2^E-hard for credulous default abduction in the first-order case. On the other hand, it is clear from the above discussion that any first-order *DAP* $\mathcal{P}$ can be reduced in exponential time to an equivalent propositional *DAP* $\mathcal{P}^*$; as a consequence, the main credulous first-order default abduction problems belong to the classes Σ_2^E and Π_2^E, respectively. Thus, we arrive at the following result.

Theorem 6.1 *Let $\mathcal{P}$ be a first-order DAP under the domain closure axiom and unique names assumption, based on credulous explanations. Then (i)* **Consistency** *is Σ_2^E-complete, (ii)* **Relevance** *is Σ_2^E-complete, and (iii)* **Necessity** *is Π_2^E-complete.*

Analogous results follow for recognizing a credulous explanation and checking minimality of a credulous explanation. Thus, the complexity of credulous abduction is exponentially higher in the first-order case than in the propositional context. Note that this *definitely proves* intractability, even if $P = \text{NP}$ would be true. Indeed, EXPTIME, i.e., the class of all problems that can be decided in exponential time, is included in Σ_2^E and Π_2^E, and so all EXPTIME-complete problems. Those problems have provably exponential running time, which implies that every problem that is complete for Σ_2^E or Π_2^E has provably an exponential lower bound on its running time.

For skeptical first-order abduction and minimal explanations, an analogous exponential increase in complexity (from Σ_k^P to Σ_k^E, Π_k^P to Π_k^E, etc) strongly suggests itself. From the discussion above, it is clear that for each problem, the exponential analogue is an upper complexity bound. Matching hardness results can be obtained by applying the results on evaluating second-order predicate formulas over finite structures in [29] (see end of Appendix A for more details).

We close this section with some remarks on tractable first-order default abduction. Like the intractability results, also the polynomial-time results in Section 5 scale up from the propositional case. A –quite moderate– extension of propositional Literal-Horn and Krom-pf-normal default theories $\langle W, D\rangle$ to the first-order case is to allow also universally quantified Krom clauses (resp. literals) besides propositional (ground) clauses (resp. literals) in W, and to allow also nonground literals $\ell(\mathbf{x})$ in defaults besides ground literals. Unfortunately, reducing a first-order *DAP* $\mathcal{P}$ to the equivalent propositional *DAP* $\mathcal{P}^*$, on which the polynomial time algorithms in Section 5 can be applied, does not give a polynomial algorithm, since the reduction is exponential. However, no substantially better algorithms are available, since even under this restrictive setting, abduction is intractable.

To substantiate this claim, we first consider the class of first-order Literal-Horn default theories. It is easy to see that logic programs without negations (i.e., datalog programs) are a fragment of this class. Indeed, each program rule $A_0 \leftarrow A_1, \ldots, A_n$, where the A_i are atoms, can be equivalently represented by the Horn default $\frac{A_1 \wedge \cdots \wedge A_n \,:\, A_0}{A_0}$. As a consequence, credulous as well as skeptical deductive inference from a default theory of this form is EXPTIME-hard, since inference of a fact from a datalog program is known to be EXPTIME-hard (this is implicit in the work of [11, 61] and part of the folklore now). Based on this, it is easily established that the main credulous abduction tasks from a first-order Literal-Horn default theory are EXPTIME-complete problems. Again, this definitely proves intractability, since no polynomial-time algorithms for such problems are possible.

Similarly, also abduction from Krom-pf-normal default theories is intractable in the first order case. In fact, one can show that credulous inference of a single literal from such a default theory is NP-hard, even for instances of simple form (see Proposition B.5 in the Appendix B). Therefore, also credulous abduction is NP-hard (resp. co-NP-hard), even if only a single manifestation has to be explained.

Tractability as in the propositional case can be gained, however, for both classes by imposing a constant upper bound on the arity of predicates; this guarantees that a first-order *DAP* $\mathcal{P}$ reduces to $\mathcal{P}^*$ in polynomial time. Thus, by applying the polynomial algorithms from Section 5 on $\mathcal{P}^*$ afterwards, we have a polynomial overall algorithm.

7 Conclusion and further research

We have investigated abduction from default theories. Starting from motivating examples, we looked how abduction can be reasonably performed; this led us to a basic model of abduction from default theories. The two inference modalities of classical default reasoning, credulous and skeptical inference, give rise to two credulous and skeptical abduction; both are useful in practice. Credulous abduction appears to be useful e.g. for diagnosis, while skeptical abduction is appealing for planning. This corresponds to previous applications of abduction in diagnosis (cf. [44, 46]) and planning and plan recognition (cf. [24, 42]).

Moreover, we have analyzed the computational complexity of the abduction model, both in the propositional as well as in the predicate case. The main results on the complexity of abduction from propositional default theories are summarized Table 1. They are discouraging from the practical point, since they show that the main reasoning tasks are highly intractable, and hence polynomial time algorithms are unrealistic. In fact, the complexity ranges from the second level of the polynomial hierarchy (Σ_2^P, Π_2^P), which is the level of deductive default reasoning and abduction from classical theories, up to fourth level (Σ_4^P), the "square" of the complexity of default reasoning. Interestingly, credulous abduction is less complex than skeptical abduction, by one level in the polynomial hierarchy; thus, abductive default diagnosis is less complex than abductive default planning. A somewhat positive finding is that the least complex variant, credulous default abduction, can be shortly transformed into classical default reasoning, and thus theorem provers like the DeReS system [13] can be efficiently used to perform abduction. However, this property is lost as soon as one uses minimal explanations, which are more complex.

In the first-order case, under some common restrictions (in full first-order logic, all abduction problems are trivially undecidable) the complexity scales up by an exponential factor, and is provably intractable.

Exploring the tractability frontier, we have also shown that credulous abduction from the previously known classes of propositional Literal-Horn and Krom-pf-normal default theories [35, 58] is tractable. Unfortunately, these results do not carry over to the first-order case.

Several issues remain for future work. One is to find more applications for abduction from default theories, and to see how credulous and skeptical abduction apply therefore. It would also be interesting to see whether other modes of abduction, besides credulous and skeptical abduction, are relevant.

Further issues more on the foundational side are other concepts of abductive explanation, as well variants of classical default logic for the underlying theory. In the present paper, we focused on the most simple concepts of abduction explanation that have been considered in the literature. A number of different notions of acceptable explanations exist, cf. [43, 7, 22, 48], which are based on a preference between explanations; however, all single out minimal explanations. Most well-known are explanations that have minimal size (cf. [43]) and explanations that have minimal cost (or weight resp. probability [48, 47]). Results for abduction from classical theories [22] suggest that abduction from default theories using explanations of smallest size or smallest cost yield problems complete for classes Δ_k^P and $\Delta_k^P[O(\log n)]$ of the (refined) polynomial hierarchy [31]. In particular, these concepts are not strong enough to lower the complexity of default abduction.

On the other hand, several variants and extensions of Reiter's default logic have been introduced, motivated by some weaknesses of the original approach, e.g., [5, 4, 49, 26, 39]. It would be interesting to investigate how abduction behaves on these formalisms. In particular, stationary default logic [4, 49] would be interesting, since there the complexity of skeptical reasoning is lower than in classical default logic [28]; as a consequence, also skeptical abduction has lower complexity.

Also on the algorithmic side, interesting issues remain for further work. One important is, of course, to find other fragments of default logic on which abduction is tractable. In particular, fragments that are not within a propositional language would be interesting. Apart from that, approximation methods that allow more efficient abductive default reasoning are needed, either on a semantical basis, cf. [55, 10, 54] or on a computational approach [56]. Anyway, the results about tractability make clear that good approximation algorithms are needed. Note that quite a bit is known about approximation of probabilistic inference problems [17, 54] as well #P-complete problems [60] (see also [31]). However, not much is known about approximation of problems complete for classes of the polynomial hierarchy above NP and for PSPACE-complete problems cf. [14].

Finally, it would be worthwhile to thoroughly investigate abduction from theories in knowledge representation formalisms different from default logic. In the context of logic programming, the study and use of abduction is meanwhile established, cf. [33]. Complexity results for abductive logic programming [23] that are similar to those for abduction from default theories support the view of the intimate relationship between these formalisms. Similar results for other closely related formalisms such as autoepistemic logic are plausible.

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A Appendix: Proofs of general case complexity

The classes of the polynomial hierarchy have problems complete under polynomial-time transformations involving quantified Boolean formulas (QBFs), which we will refer to in many proofs. A QBF is an expression of the form

$$Q_1 X_1 Q_2 X_2 \cdots\ Q_k X_k\, E, \quad k \geq 1, \tag{4}$$

where E is a Boolean expression whose (propositional) variables are from pairwise disjoint nonempty sets of variables $X_1, \ldots, X_k$, and the Q_i's are alternating quantifiers from $\{\exists, \forall\}$, for all $i = 1, \ldots, k$. If $Q_1 = \exists$ the we say the QBF is k-existential, otherwise it is k-universal. Validity of QBFs is defined in the obvious way by recursion to variable-free Boolean expressions. We denote by $\mathrm{QBF}_{k,\exists}$ (resp., $\mathrm{QBF}_{k,\forall}$) the set of all valid k-existential (resp., k-universal) QBFs.

Deciding, given a k-existential QBF Φ (resp. a k-universal QBF Ψ), whether $\Phi \in \mathrm{QBF}_{k,\exists}$ (resp. $\Psi \in \mathrm{QBF}_{k,\forall}$), is a classical Σ_k^P-complete (resp. Π_k^P-complete) problem; deciding the conjunction $(\Phi \in \mathrm{QBF}_{k,\exists}) \wedge (\Psi \in \mathrm{QBF}_{k,\forall})$ is complete for $\mathrm{D}_\mathrm{k}^\mathrm{P}$.

We introduce some additional notation. For a set A of propositional atoms, we denote by $\neg A$ the set $\{\neg a \mid a \in A\}$ and by A' the set of atoms $\{a' \mid a \in A\}$. Moreover, for a Boolean formula F and a set $X1 \subseteq X$ of atoms, we denote by F_{X1}^X the formula obtained from F by replacing each occurrence of an atom $x \in X1$ in F by *true* and each occurrence of an atom $x \in X \backslash X1$ by *false*; as X will be understood, we write simply F_{X1}.

Theorem 4.8 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if $E \subseteq H$ is a skeptical explanation for $\mathcal{P}$ is D_2^P-complete.*

Proof. As already argued, the problem is in D_2^P. To show hardness, we reduce deciding whether, given two propositional default theories $T1 = \langle W1, D1\rangle$ and $T2 = \langle W2, D2\rangle$, it holds that $T1$ has a consistent extension and $T2 \vdash_s x$ to the problem; since the problems are Σ_2^P-hard and Π_2^P-hard [27], respectively, this will clearly prove D_2^P-hardness. Without loss of generality, $T1$ and $T2$ have different propositional atoms, x does not occur in $T1$, and $W2 = \emptyset$ (thus $T2$ has only consistent extensions), cf. [27]. Define $\mathcal{P} = \langle \emptyset, \{x\}, W1 \cup W2, D1 \cup D2\rangle$. It is easily seen that $\emptyset$ is a skeptical explanation of $\mathcal{P}$ iff $T1$ has a consistent extension and $T2 \vdash_s x$; this proves the result. ∎

Theorem 4.9 *Let $\mathcal{P}$ be a PDAP based on skeptical explanations. The problem (i)* **Consistency** *is Σ_3^P-complete, (ii)* **Relevance** *is Σ_3^P-complete and (iii)* **Necessity** *is Π_3^P-complete, with hardness even for normal $\mathcal{P}$.*

Proof. *Membership.* A guess E for a skeptical explanation for $\mathcal{P}$ can be verified by two calls to Σ_2^P oracle, by checking whether $\langle W \cup E, D\rangle$ has a consistent extension and and whether $\langle W \cup E, D\rangle \vdash_s \phi$, where ϕ is the conjunction of all formulas in M. Hence, it is easy to see that (i), (ii), and the complement of (iii) are in Σ_3^P.

Hardness. (i) We describe here a transformation of deciding if a QBF $\Phi = \exists X \forall Y \exists Z F$ is valid. Define

$$W = \{ f \leftrightarrow F \}, \qquad D = \left\{ \frac{:\neg a}{\neg a}, \ \frac{:a}{a} \mid a \in X \cup Y \right\} \cup \left\{ \frac{:F}{F} \right\}$$

where f is a new atom, and $\mathcal{P} = \langle X \cup \neg X, \{f\}, W, D\rangle$. Note that $\mathcal{P}$ is a normal *PDAP*.

We claim that $\mathcal{P}$ has a skeptical explanation iff Φ is valid.

"$\Rightarrow$": Assume E is a skeptical explanation for $\mathcal{P}$. Let $X1 = E \cap X$, and let $Y1 \subseteq Y$ be arbitrary. Then, clearly

$$A = Th(W \cup E \cup X1 \cup \neg(X \backslash X1) \cup Y1 \cup \neg(Y \backslash Y1))$$

is an extension of the default theory $\langle W \cup E, D \backslash \left\{ \frac{:F}{F} \right\}\rangle$. By semi-monotonicity of normal default theories ([52]; see Section 2), $\langle W \cup E, D\rangle$ must have an extension B such that $A \subseteq B$. Clearly, B must be consistent. Moreover, since E is a skeptical explanation for $\mathcal{P}$, $f \in B$, and hence $F \in B$. Consequently, $F_{X1 \cup Y1}$ is satisfiable. Since $Y1$ is arbitrary, it follows that $\forall Y \exists Z F_{X1}$ is valid. Hence, Φ is valid.

"$\Leftarrow$": Let $X1 \subseteq X$ such that $\forall Y \exists Z F_{X1}$ is valid. Define $E = X1 \cup \neg(X \backslash X1)$. Clearly, $W \cup E$ is consistent. Hence, every extension A of $\langle E, D\rangle$ is consistent; an extension exists, since $\langle E, D\rangle$ is normal. Moreover, each such A and contains $Y1 \cup \neg(Y \backslash Y1)$ for some $Y1 \subseteq Y$. From the characterization of A in terms of its generating defaults and the hypothesis that $\forall Y \exists Z F_{X1}$ is valid, it follows that F is consistent with A. Consequently, $\frac{:F}{F} \in GD(A, \langle E, D\rangle)$, and hence $F \in A$; it follows $f \in A$. Since the extension A was arbitrary, it follows that E is a skeptical explanation for $\mathcal{P}$. This proves the claim, which also concludes the proof of (i).

To show (ii), add h to the hypotheses in an instance of $\mathcal{P}$ in (i), and ask if h is relevant; for (iii), add h to the hypotheses, $f \vee h \supset q$ to the third component, change the manifestations to $\{q\}$ and ask if h is necessary. Indeed, h is necessary iff $\mathcal{P}$ has no skeptical explanation. ∎

Theorem A.1 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if $E \subseteq H$ is a minimal credulous explanation E for $\mathcal{P}$ is $\mathrm{D}_2^{\mathrm{P}}$-complete, with hardness even for normal $\mathcal{P}$.*

Proof. It remains to show hardness. For that, consider *PDAP*s $\mathcal{P}1 = \langle \emptyset, \{q_1\}, W1, D1\rangle$ as in (1) and $\mathcal{P}2 = \langle \{h\}, \{q_2\}, W2, D2\rangle$ as in (3) preceding Theorem 4.6, and assume that they are on distinct atoms. Then, $\{h\}$ is a minimal credulous explanation of $\mathcal{P}3 = \langle \{h\}, \{q_1, q_2\}, W1 \cup W2, D1 \cup D2\rangle$ iff $\mathcal{P}1$ has a credulous explanation and h is necessary for $\mathcal{P}2$. Since the latter problems are Σ_2^P-hard and Π_2^P-hard, respectively, and $\mathcal{P}3$ is clearly normal, $\mathrm{D}_2^{\mathrm{P}}$-hardness of the problem follows. ∎

Theorem 4.11 *Let $\mathcal{P}$ be a PDAP based on minimal credulous explanations. Then,* **Relevance** *is Σ_3^P-complete, and hardness holds even for normal $\mathcal{P}$.*

Proof. *Membership.* A guess E for a minimal credulous explanation for $\mathcal{P}$ such that $h \in E$ can be verified by two calls to an Π_2^P oracle (cf. Theorems 4.6 and 4.10). Hence, the problem is in Σ_3^P.

Hardness. We outline a reduction from deciding validity of a QBF $\Phi = \exists X \forall Y \exists Z F$. Let s and q be new atoms, and define

$$\begin{array}{rcl} W & = & \emptyset \\ D & = & \left\{ \frac{:\neg s}{\neg s}, \frac{s \wedge Y : q}{q}, \frac{\neg s \wedge \neg F : q}{q} \right\} \cup \left\{ \frac{x : x'}{x'}, \frac{\neg x : x'}{x'} \mid x \in X \right\} \cup \left\{ \frac{:\neg y}{\neg y} \mid y \in Y \right\}. \end{array}$$

Let $\mathcal{P} = \langle X \cup \neg X \cup Y \cup \{s\}, X' \cup \{q\}, W, D \rangle$; note that $\mathcal{P}$ is normal.

Claim: s is relevant for a minimal credulous explanation iff Φ is valid.

The following facts, which are straightforward, are useful for a proof of this claim:

Fact 1: If E is a credulous explanation for $\mathcal{P}$, then (i) $x \in E$ or $\neg x \in E$, for each $x \in X$; (ii) if $s \in E$, then $Y \subseteq E$; and (iii) if $s \notin E$, then $\neg s$ belongs to each extension of $\langle W \cup E, D \rangle$.

Fact 2: for each $X1 \subseteq X$, the set $E_{X1} = X1 \cup \neg(X \backslash X1) \cup Y \cup \{s\}$ is a credulous explanation for $\mathcal{P}$. (Indeed, the set $Th(W \cup E_{X1} \cup X' \cup \{q\})$ is an extension of $\langle W \cup E_{X1}, D \rangle$.)

We now prove the claim.

"$\Rightarrow$": Relevance of s implies by Facts 1 and 2 that for some $X1 \subseteq X$, E_{X1} is a minimal credulous explanation. Thus, for each $Y1 \subseteq Y$, the set $E_{X1,Y1} = X1 \cup \neg(X \backslash X1) \cup Y1$ is not a credulous explanation. Since $W \cup E_{X1,Y_1}$ is consistent, it follows by semi-monotonicity of normal default theories that $\langle W \cup E_{X1,Y1}, D \rangle$ has a consistent extension A that contains

$$B = \{\neg s\} \cup X1 \cup \neg(X \backslash X1) \cup Y1 \cup \neg(Y \backslash Y1);$$

however, $q \notin A$ (otherwise, $E_{X1,Y1}$ would be an explanation.) Thus, from the characterization of A by its generating defaults, it follows that $B \not\vdash \neg F$, i.e., $B \cup F$ is consistent. Consequently, the formula $F_{X1 \cup Y1}$ is satisfiable. It follows that $\exists X \forall Y \exists Z F$ is valid.

"$\Leftarrow$": Let $X1 \subseteq X$ such that $\forall Y \exists Z F_{X1}$ is valid. Then, E_{X1} is a minimal credulous explanation for $\mathcal{P}$. To show this, assume to the contrary that some smaller explanation $E \subset E_{X1}$ exists. Then, from Fact 1 we have $s \notin E$. The default theory $\langle W \cup E, D \rangle$ must have an extension that contains q; by the properties of default logic, it follows that $\neg F$ is provable from $X1 \cup \neg(X \backslash X1) \cup Y1 \cup \neg(Y \backslash Y1)$, where $Y1 = Y \cap E$. It follows that $\forall Y \exists Z F_{X1}$ is not valid. Contradiction.

This proves the claim. As $\mathcal{P}$ can be constructed in polynomial time, Σ_3^P-hardness of the problem follows. ∎

Theorem 4.12 *Let $\mathcal{P} = \langle H, M, W, D \rangle$ be a PDAP. Deciding if a skeptical explanation E for $\mathcal{P}$ is minimal is Π_3^P-complete, with hardness holding even for normal $\mathcal{P}$.*

Proof. *Membership.* A guess for a smaller skeptical explanation $E' \subset E$ can be verified with two calls to a Σ_2^P oracle, and hence deciding the existence of such an E' is in Σ_3^P. Consequently, the problem is in Π_3^P.

Hardness. We describe here a reduction from deciding whether a QBF $\Phi = \forall X \exists Y \forall Z F$ is valid. Let s and q be new atoms, and define

$$D = \left\{ \tfrac{:\neg s}{\neg s}, \tfrac{s \wedge X : q}{q}, \tfrac{\neg s : \neg F \wedge q}{\neg F \wedge q} \right\} \cup \left\{ \tfrac{:\neg x}{\neg x} \mid x \in X \right\} \cup \left\{ \tfrac{:y}{y}, \tfrac{:\neg y}{\neg y} \mid y \in Y \right\}.$$

Let $\mathcal{P} = \langle X \cup \{s\}, \{q\}, \emptyset, D \rangle$. Check that $E = X \cup \{s\}$ is a skeptical explanation for $\mathcal{P}$.

Claim: E is a minimal skeptical explanation for $\mathcal{P}$ iff Φ is valid.

In the proof of this claim, we use the following facts, which are easily established. *Fact 3*: E is the only skeptical explanation for $\mathcal{P}$ that contains s.

Fact 4: For each $X1 \subseteq X$, (i) the default theory $\langle X1, D \rangle$ has for each $Y1 \subseteq Y$ an extension containing $X1 \cup \neg(X \backslash X1) \cup Y1 \cup \neg(Y \backslash Y1)$ (follows from semi-monotonicity); (ii) each extension of $\langle X1, D \rangle$ contains $\neg s$.

Proof of the claim:

"$\Rightarrow$": Let $X1 \subseteq X$. Then, by hypothesis, $X1$ is not a skeptical explanation. Hence, there exists some extension A of $\langle X1, D \rangle$ such that $q \notin A$, which implies by (ii) of Fact 4 that $\neg F \wedge q \notin A$. From the characterization of A in terms of its generating defaults, it follows that $X1 \cup \neg(X \backslash X1) \cup Y1 \cup \neg(Y \backslash Y1) \vdash F$, where $Y1 = Y \cap A$. Thus, $\exists Y \forall Z F_{X1}$ is valid. Since $X1$ is arbitrary, it follows that Φ is valid.

"$\Leftarrow$": Assume that E is not minimal. Then, by Fact 3, some $X1 \subseteq X$ exists which is a skeptical explanation. By (i) and (ii) of Fact 4, we conclude that for each $Y1 \subseteq Y$, it holds that $X1 \cup \neg(X \backslash X1) \cup Y1 \cup \neg(Y \backslash Y1)$ is consistent with $\neg F$. Hence, $\forall Y \exists Z \neg F_{X1}$ is valid. This implies that Φ is not valid. Thus, by contraposition, validity of Φ implies that E is minimal.

This proves the claim. The result follows. ∎

Theorem A.2 *Given a PDAP $\mathcal{P} = \langle H, M, W, D \rangle$ and a subset $E \subseteq H$, deciding whether E is a minimal skeptical explanation for $\mathcal{P}$ is Π^P_3-complete, with hardness holding even for normal $\mathcal{P}$.*

Proof. *Membership.* If E is not a minimal skeptical explanation, then either E is not a skeptical explanation, which can be checked with two calls to a Σ^P_2 oracle, or there exists a guess $E' \subset E$ such that E' is a smaller skeptical explanation, which can be checked by another two calls to a Σ^P_2 oracle. Hence, it follows that deciding whether E is *not* a minimal skeptical explanation is in Σ^P_3; hence, deciding whether E is a minimal skeptical explanation is in Π^P_3.

Hardness. Follows immediately from the reduction in the hardness part of the proof of Theorem 4.12: take $E = X \cup \{s\}$. ∎

Theorem 4.13 *Let $\mathcal{P}$ be a PDAP based on minimal skeptical explanations. Then, problem* **Relevance** *is Σ^P_4-complete, with hardness even for normal $\mathcal{P}$.*

Proof. *Membership.* A guess for a minimal skeptical explanation E for $\mathcal{P}$ such that $h \in E$ can be verified with one call to a Σ^P_3 oracle (cf. Theorem A.2).

Hardness. We outline a reduction from deciding validity of a QBF $\Psi = \exists R \forall X \exists Y \forall Z F$, which is an extension to the reduction in the proof of Theorem 4.12. Let as there be s and q new atoms, and define

$$D1 \;=\; D \cup \left\{ \tfrac{r':r\wedge r''}{r\wedge r''},\ \tfrac{\neg r':\neg r\wedge r''}{\neg r\wedge r''} \mid r \in R \right\},$$

where D is the same set of defaults as in the proof of Theorem 4.12. Define $\mathcal{P} = \langle H, R'' \cup \{q\}, \emptyset, D1\rangle$, where $H = R' \cup \neg R' \cup X \cup \{s\}$. Note that $\mathcal{P}$ is normal; if W would be empty, then $\mathcal{P}$ would be identical to the *PDAP* in the proof of Theorem 4.12). We observe the following facts on $\mathcal{P}$:

Fact 5: Let E be any skeptical explanation for $\mathcal{P}$. Then, either $r' \in E$ or $\neg r' \in E$, but not both simultaneously, for each $r \in R$.

Proof: Note that the manifestation r'' can be only explained by either r' or $\neg r'$, which can not be derived by applying defaults. Clearly, it is not possible that r and $\neg r$ are both in E.

Claim: Let $E1 \subseteq H$ and $R1 \subseteq R$ such that $E1 \cap (R' \cup \neg R') = R1' \cup \neg(R \backslash R1)'$. Then, $E1$ is a skeptical explanation for $\mathcal{P}$ iff $E = E1 \backslash (R' \cup \neg R')$ is a skeptical explanation for the *PDAP* $\mathcal{P}_\Phi = \langle H_\Phi, M_\Phi, W_\Phi, D_\Phi\rangle$ constructed for $\Phi = \forall X \exists Y \forall Z F_{R1}$ in the proof of Theorem 4.12.

Proof of the claim: Let $D2$ be the set of defaults obtained from $D1$ by replacing $\neg F \wedge q$ with $\neg F_{R1} \wedge q$.

"$\Rightarrow$": Each extension of $\langle E1, D1\rangle$ must contain $R1 \cup \neg(R\backslash R1)$. (Otherwise, R'' would not be explained.) From this, it is not difficult to see that $\langle E1, D2\rangle$ has the same extensions as $\langle E1, D1\rangle$. Moreover, from the occurrences of r, r' and r'' in $\langle E1, D2\rangle$, it is clear that the extensions of $\langle E1, D2\rangle$ restricted to the language over the atoms $X \cup Y \cup Z \cup \{s, q\}$ are precisely the extensions of $\langle E, D_\Phi\rangle$. Hence, it follows that E is a skeptical explanation for $\mathcal{P}_\Phi$.

"$\Leftarrow$": Assume that E is aa skeptical explanation of $\mathcal{P}_\Phi$. Let B be an arbitrary extension of $\langle E1, D1\rangle$. Clearly, B is consistent. To show that $E1$ is a skeptical explanation for $\mathcal{P}$, it suffices to prove that $R'' \cup \{q\} \subseteq B$. We consider two cases.

(i) $s \in E1$. Then, $s \in E$; since E is a skeptical explanation for $\mathcal{P}_\Phi$, we conclude that $X \subseteq E$, and hence $X \subseteq E1$. Consequently, $q \in B$ by the default $\frac{s\wedge X:q}{q}$; moreover, $R'' \subseteq B$ since for each $r \in R$, one of the defaults $\frac{r':r\wedge r''}{r\wedge r''}$ and $\frac{\neg r':\neg r\wedge r''}{\neg r\wedge r''}$ is applicable. (Note that the default $\frac{\neg s\,:\,\neg F\wedge\neg q}{\neg F\wedge\neg q}$ is not applicable).

(ii) $s \notin E1$. Then, $\neg s \in B$. Let $Y1 = B \cap Y$. Since E is a skeptical explanation for $\mathcal{P}_\Phi$, we conclude that the set

$$E \cup \{\neg F_\Phi \wedge q\} \cup \neg(X\backslash E) \cup Y1 \cup \neg(Y\backslash Y1) \cup \{\neg s\}$$

is consistent. This can be easily established by semi-monotonicity of normal default logic: start to build an extension of $\langle E, D_\Phi\rangle$ by applying the appropriate default from $\frac{:y}{y}$, $\frac{:\neg y}{\neg y}$ for each $y \in Y$, after that all defaults $\frac{:\neg x}{\neg x}$, for $x \in X\backslash E$, and then $\frac{:\,\neg s}{\neg s}$; the default $\frac{\neg s:\neg F_\Phi\wedge q}{\neg F_\Phi\wedge q}$ must be applicable thereafter, as this is the only possibility to join q to the extension.) Consequently, also the set

$$C = E1 \cup \{r \wedge r'' \mid r \in R1\} \cup \{\neg r \wedge r'' \mid r \in R\backslash R1\} \cup \{\neg F \wedge q\} \cup \neg(X\backslash E) \cup Y1 \cup \neg(Y\backslash Y1) \cup \{\neg s\}$$

must be consistent. Thus, by the characterization of an extension in terms of its generating defaults, we conclude that $B = Th(C)$. Hence, it follows that $R'' \cup \{q\} \subseteq B$.

It follows that $E1$ is a skeptical explanation for $\mathcal{P}$. This proves the claim.

From Fact 5 and the claim, it follows that s is relevant for some minimal skeptical explanation for $\mathcal{P}$ iff for some $R1 \subseteq R$, the formula $\forall X \exists Y \forall Z F_{R1}$ is valid, i.e., Ψ is valid. As $\mathcal{P}$ is efficiently constructible from Ψ, this proves Σ_4^P-hardness. The result follows. ∎

We close this part of the appendix with a remark on the complexity of first-order default abduction problems under domain closure and unique names assumption (cf. Section 6). Most of the hardness results from above are proved by polynomial-time transformations of evaluating particular quantified Boolean formulas that are hard for Σ_k^P resp. Π_k^P. Note that such formulas can be seen as second-order propositional formulas, since the quantifiers are on propositional atoms. It is shown in [29] that evaluating second-order formulas of predicate logic over a fixed finite relational structure (i.e., a first-order structure with finite domain for a language without functions and constants) has precisely exponentially higher complexity, and yields problems hard for the exponential analogues Σ_k^E and Π_k^E of Σ_k^P and Π_k^P, respectively.

Utilizing these results, the hardness proofs from above can be easily generalized to corresponding hardness proofs for the first-order case. In fact, a suitable replacement of QBFs by corresponding second-order predicate formulas is feasible. We show this on the example of the transformation in the proof of Theorem 4.9. Instead of the QBF $\Phi = \exists X \forall Y \exists Z\, F$, we have a second-order predicate formula

$$\Psi = \exists p_{1,1} \cdots \exists p_{1,n_1} \forall p_{2,1} \cdots \forall p_{2,n_2} \exists p_{3,1} \cdots \exists p_{3,n_3} \phi$$

where the $p_{i,j}$ are predicate variables and ϕ is a closed first-order formula (in which no function and constant symbols occur).

Evaluating Ψ on a fixed finite relational structure $\mathcal{A}$ is Σ_3^E-hard [29]. $\mathcal{A}$ can be represented in default logic by the fixed first-order theory

$$W_{\mathcal{A}} = \{r(\mathbf{t}) \mid \mathcal{A} \models r(\mathbf{t}), r(\mathbf{t}) \text{ is a ground atom}\} \cup \{\neg r(\mathbf{t}) \mid \mathcal{A} \not\models r(\mathbf{t}), r(\mathbf{t}) \text{ is a ground atom}\}$$

(i.e., $W_{\mathcal{A}}$ yields the closed world assumption applied to the relations in $\mathcal{A}$). Define now defaults

$$D = \left\{ \frac{:\neg p_{i,j}(\mathbf{x})}{\neg p_{i,j}(\mathbf{x})}, \; \frac{: p_{i,j}(\mathbf{x})}{p_{i,j}(\mathbf{x})} \mid i = 1,2, \; j = 1, \ldots, n_i \right\} \cup \left\{ \frac{:\ \phi}{\phi} \right\}$$

(here the $\mathbf{x} = x_1, \ldots, x_n$ is assumed to match the arity of $p_{i,j}$) and a first-order *DAP*

$$\mathcal{P} = \langle \{ p_{1,j}, \neg p_{1,j} \mid j = 1, \ldots, n_1 \}, \{f\}, W_{\mathcal{A}} \cup \{ f \leftrightarrow \phi \}, D \rangle,$$

where f is a new propositional atom. It holds that $\mathcal{P}$ has a skeptical explanation iff $\mathcal{A} \models \Psi$. This proves Σ_3^E-hardness of **Consistency** for skeptical abduction in the first-order case.

Similar generalizations are straightforward in the other proofs.

B Appendix: Proofs for tractable cases

Lemma 5.1 *Let $\langle W, D \rangle$ be a Literal-Horn default theory, and let $\ell_1, \ldots, \ell_m$ be literals. Then, deciding $\langle W, D \rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ is possible in $O(m \cdot n)$ time, where n is the length of the input.*

Proof. If W is not consistent (which holds iff it contains a pair of opposite literals) then $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$; if $\ell_1 \wedge \cdots \wedge \ell_m$ is not satisfiable, then $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ iff W is not consistent. Both consistency of W and satisfiability of $\ell_1 \wedge \cdots \wedge \ell_m$ can be checked in linear time; in what follows, we thus assume that W is consistent and that $\ell_1 \wedge \cdots \wedge \ell_m$ is satisfiable, i.e., $\ell_1, \ldots, \ell_m$ does not contain a pair of opposite literals. Define

$$H = W \cup \{\alpha \supset \ell \mid (\alpha : \ell/\ell) \in D \text{ and } \sim\ell \notin W \text{ and } [\ell \in \{\ell_1, \ldots, \ell_m\} \text{ or } (\ell \text{ is positive and } \ell \notin \{\sim\ell_1, \ldots, \sim\ell_m\})]\}.$$

Note the important fact that H does not contain simultaneously clauses with opposite literal in the head, i.e., clauses $\beta \supset \ell$ and $\gamma \supset \sim\ell$ (where β and/or γ may be empty).

Claim: $\langle W, D\rangle$ has an extension containing $\ell_1, \ldots, \ell_m$ iff $H \vdash \ell_1 \wedge \cdots \wedge \ell_m$.

Note that for $n = 1$, i.e., a single literal ℓ_1, H is the Horn theory of [35, Lemma 6.4], which states the claimed property for a single literal.

H can be constructed in linear time from $\langle W, D\rangle$; notice that H is Horn. It is well-known that for a Horn theory T and a literal ℓ, the test $T \vdash \ell$ can be decided in time linear in the length of T, e.g. by positive unit resolution, cf. [40]. Consequently, deciding $H \vdash \ell_1 \wedge \cdots \wedge \ell_m$ is possible in time linear in the length of H times m, and hence in time $O(m \cdot n)$. (In fact, the set of all positive literal ℓ such that $T \vdash \ell$ (i.e., the least model) can computed in time linear in the length of T [40], and thus the bound could be sharpened to $O((m^- + 1)n)$ where m^- is the number of negative literals ℓ_i.) Therefore, the lemma is an immediate consequence of this claim.

Prior to the proof of the claim, define subsets $H_0, H_1, \ldots, H_i, \ldots$ of H as follows:

$$\begin{aligned}
H_0 &= W \\
H_1 &= H_0 \cup \{\, \alpha \supset \ell \in H \backslash H_0 \mid H_0 \vdash \alpha \,\} \\
&\vdots \\
H_{i+1} &= H_i \cup \{\, \alpha \supset \ell \in H \backslash H_i \mid H_i \vdash \alpha \,\} \\
&\vdots
\end{aligned}$$

Note that the sequence $\langle H_i\rangle$, $i \geq 0$, converges to $\bigcup_{i=0}^{\infty} H_i$; this set relates to H as follows. Since H is a Horn theory, it has a least model M; the following characterization of M is not difficult to establish:

$$(0) \quad M = \left\{ x \mid \alpha \supset x \in \textstyle\bigcup_{i\geq 0}^{\infty} H_i \right\},$$

i.e., the least model of H is given by the atoms x in the heads of the clauses with positive head (resp. positive facts) in $H_0, H_1 \ldots$ Indeed, let $N = \left\{ x \mid \alpha \supset x \in \bigcup_{i\geq 0} H_i \right\}$. Then, one can check that N is a model of H; moreover, by induction on i, one can show that M must contain all positive heads of the clauses in $H_0 \cup \cdots \cup H_i$, and hence N is contained in M.

We now prove the claim.

"$\Leftarrow$": Assume that $H \vdash \ell_1 \wedge \cdots \wedge \ell_m$. Define the set E by

$$E = Th(\, W \cup M \cup \{\neg x \mid (\alpha : \neg x/\neg x) \in D,\ M \models \alpha,\ M \not\models x \,\}).$$

Notice that E is consistent. We claim that E is an extension of $\langle W, D\rangle$ such that $\ell_i \in E$, for every $i = 1, \dots n$, which proves the *If*-direction. To prove this claim, we first verify that $\ell_i \in E$, for every $i = 1, \dots, n$. If ℓ_i is a positive literal, then $\ell_i \in M$ and hence, by definition of E, clearly $\ell_i \in E$; otherwise, if ℓ_i is a negative literal $\neg x$, then, since $H \vdash \ell_i$, there must exist a clause $\alpha \supset \neg x$ in H such that $M \models \alpha$ (for otherwise, a model of H in which x is true exists). Thus, $(\alpha : \neg x \,/\, \neg x) \in D$, $M \models \alpha$ and $M \not\models x$ all hold; by definition, $\neg x \in E$.

It remains to show that E is an extension, i.e., $E = \bigcup_{i=0}^{\infty} E_i$, where the sets E_i are those in the definition of extension. We obtain

$$\begin{aligned}
E_0 &= W = W \cup \{x \mid x \in H_0\} \\
E_1 &= Th(E_0) \cup \{\ell \mid (\alpha : \ell \,/\, \ell) \in D,\ E_0 \vdash \alpha,\ \neg\ell \notin E\} \\
&= Th(E_0) \cup \left\{ \neg x \mid \frac{\alpha : \neg x}{\neg x} \in D,\ E_0 \vdash \alpha,\ x \notin E \right\} \cup \{ x \mid \alpha \supset x \in H_1 \} \\
&\ \ \vdots \\
E_{i+1} &= Th(E_i) \cup \left\{ \neg x \mid \frac{\alpha : \neg x}{\neg x} \in D,\ E_i \vdash \alpha,\ x \notin E \right\} \cup \{ x \mid \alpha \supset x \in H_{i+1} \} \\
&\ \ \vdots
\end{aligned}$$

Indeed, by induction on i, it is straightforward that the set of positive literals in E_i, P_i, is $\{ x \mid \alpha \supset x \in H_i \}$.

For $i = 0$, this is obvious. Consider thus $i' = i + 1$, and first $\alpha \supset x \in H_{i+1}$. If $\alpha \supset x \in H_i$, then by the induction hypothesis $x \in E_i$, and hence $x \in E_{i+1}$; otherwise, a default $(\alpha : x/x) \in D$ exists, by the induction hypothesis $E_i \vdash \alpha$, and since $x \in M$, $\neg x \notin E$; consequently, $x \in E_{i+1}$. It follows that: (+) $\{x \mid \alpha \supset x \in H_{i+1}\} \subseteq P_{i+1}$.

On the other hand, consider a positive atom x from E_{i+1}. If $x \in Th(E_i)$, then from the induction hypothesis, it follows that $x \in \{y \mid \alpha \supset y \in H_i\}$. Otherwise, there is a default $(\alpha : x/x) \in D$, s.t. $E_i \vdash \alpha$ and $\neg x \notin E$. By the induction hypothesis, it follows that $H_i \vdash \alpha$, and we have $\neg x \notin W$ (since $W \subseteq E$). Since $x \in E$ (as E_{i+1}) and, as shown above, $\ell_i \in E$ for every $i = 1, \dots, n$, it follows that either $x \in \{\ell_1, \dots, \ell_m\}$ or $x \notin \{\sim\ell_1, \dots, \sim\ell_m\}$. Consequently, $\alpha \supset x \in H_{i+1}$. It follows that (++) $P_{i+1} \subseteq \{x \mid \alpha \supset x \in H_{i+1}\}$. Conditions (+) and (++) imply $P_{i+1} = \{x \mid \alpha \supset x \in H_{i+1}\}$, i.e., the statement of the induction. This concludes the induction.

It follows that

$$\bigcup_{i=0}^{\infty} E_i = Th\left(W \cup \left\{ x \mid \alpha \supset x \in \bigcup_{i=0}^{\infty} H_i \right\} \cup \left\{ \neg x \mid \frac{\alpha : \neg x}{\neg x} \in D,\ \bigcup_{i=0}^{\infty} E_i \vdash \alpha,\ x \notin E \right\} \right).$$

For any conjunction α of positive literals, it follows from (0) that

$$(1) \quad \textstyle\bigcup_{i=0}^{\infty} E_i \vdash \alpha \ \Leftrightarrow\ \{x \mid \alpha \supset x \in \bigcup_{i=0}^{\infty} H_i\} \vdash \alpha \ \Leftrightarrow\ M \models \alpha.$$

Furthermore, from the definition of E, it is easily verified that

$$(2) \quad x \notin E \Leftrightarrow M \not\models x.$$

By (0), (1), and (2), we therefore obtain

$$\begin{aligned}\bigcup_{i=0}^{\infty} E_i &= Th(\, W \cup \underbrace{\{\, x \mid \alpha \supset x \in \textstyle\bigcup_{i=0}^{\infty} H_i\}}_{=M} \cup \{\neg x \mid \tfrac{\alpha:\neg x}{\neg x} \in D,\ M \models \alpha,\ M \not\models x\}\,) \\ &= E.\end{aligned}$$

Thus, E is an extension of $\langle W, D\rangle$. This proves the *If*-direction.

"$\Rightarrow$": Assume that $\langle W, D\rangle$ has an extension E containing $\ell_1, \ldots, \ell_m$. We have to show that $H \vdash \ell_i$, for every $i = 1, \ldots, n$. Without loss of generality, we assume that E satisfies the following condition:

(*) If $\frac{\alpha:x}{x} \in D$ such that $E \vdash \alpha$, and $x \notin \{\sim\ell_1, \ldots, \sim\ell_m\}$, then $\neg x \notin E \backslash W$.

Indeed, assume that $(\alpha : x/x)$ is a default in D that violates (*). Since $\langle W, D\rangle$ is Literal-Horn, it is not hard to see that $\langle W, D\rangle$ has an extension $E' \supseteq Th(E \backslash \{\neg x\} \cup \{x\})$ that contains $\ell_1, \ldots, \ell_m$ (take any extension of $\langle E \backslash \{\neg x\} \cup \{x\}, D\rangle$.) Using e.g. a (well)-ordering on the defaults D, one can single out a particular such extension. This gives rise to a monotonic operator, which takes E to an extension E' with a larger part of positive atoms; the least fixpoint of this operator is an extension that satisfies (*). This justifies our assumption.

Let M_0 be the set of positive literals in E. We claim that M_0 is the least model of H (i.e., $M_0 = M = \{\, x \mid \alpha \supset x \in \bigcup_{i=0}^{\infty} H_i\})$.

We proof this claim showing by induction on $i \geq 0$ that

$$M_0 \cap Th(E_i) = \{x \mid \alpha \supset x \in H_i\}\,,$$

i.e., the positive literals in E_i are the positive heads of clauses in H_i. Since $M = \bigcup_{i=0}^{\infty}\{\, x \mid \alpha \supset x \in H_i\,\}$, the claim is an immediate consequence thereof.

(Basis) For $i = 0$, we have $E_0 = W$ and $H_0 = W$. Thus, the statement is clearly true.

(Induction) For $i' = i + 1$, we have

$$\begin{aligned} M_0 \cap Th(E_{i+1}) &= M_0 \cap Th(\, Th(E_i) \cup \{\ell \mid (\alpha : \ell/\ell) \in D,\ E_i \vdash \alpha,\ \neg\ell \notin E\}\,) \\ &= \{x \mid \alpha \supset x \in H_i\} \cup \underbrace{\left\{x \mid \frac{\alpha : \neg x}{\neg x} \in D,\ E_i \vdash \alpha,\ \neg x \notin E\right\}}_{=_{def} B} \end{aligned}$$

The induction hypothesis is applied to obtain the left expression on the second line.

Consider $x \in B$. Then, $\neg x \notin W$ and either $x \in \{\ell_1, \ldots, \ell_m\}$ or $x \notin \{\sim\ell_1, \ldots, \sim\ell_m\}$ (as $\ell_i \in E$, $x \in E$, and E must be consistent). Since $E_i \vdash \alpha$, it follows from the induction hypothesis that $H_i \vdash \alpha$. Therefore, $\alpha \supset x \in H_{i+1}$, which implies: (**) $\quad x \in \{\, y \mid \alpha \supset y \in H_{i+1}\,\}$.

On the other hand, consider any $\alpha \supset y \in H_{i+1}$. If $y \in \{x \mid \beta \supset x \in H_i\}$, it is clear from the equation above that $y \in M_0 \cap Th(E_{i+1})$. Otherwise, if $\alpha \supset x \in H_{i+1}$, then $H_i \vdash \alpha$, which by the induction hypothesis implies that $E_i \vdash \alpha$. Moreover, either (a) $y \in \{\ell_1, \ldots, \ell_m\}$ or (b) $y \notin \{\sim\ell_1, \ldots, \sim\ell_m\}$. In case (a), $y \in E$ and $\neg y \notin E$, as $E \vdash \ell_1 \wedge \cdots \wedge \ell_m$; thus $y \in B$. In case

(b) we obtain from the assumption (*) on E that $\neg y \notin E \backslash W$; moreover, by the definition of H, $\neg y \notin W$. Therefore, $\neg y \notin E$, which implies $y \in B$. Since $y \in B$ holds in both cases (a) and (b), it follows that (***) $\quad y \in M_0 \cap Th(E_{i+1})$.

Now (**) and (***) imply

$$M_0 \cap Th(E_{i+1}) = \{\, x \mid \alpha \supset x \in H_{i+1} \}\,;$$

this proves the induction step and the claim $M_0 = M$.

Eventually, we verify that $H \vdash \ell_i$ for every $i = 1, \ldots, n$. If ℓ_i is positive, then $\ell_i \in M_0 = M$; therefore, $H \vdash \ell_i$. Otherwise, ℓ_i is a negative literal $\neg x$. If $\ell_i \in W$, then $\ell_i \in H$ and therefore $H \vdash \ell_i$; if $\ell_i \notin W$, then some $\frac{\alpha : \ell_i}{\ell_i} \in D$ exists such that $E \vdash \alpha$. Since $\ell_i \in E$, we have $\sim \ell_i \notin W$; hence, it follows $\alpha \supset \ell_i \in H$. Moreover, since α is a conjunction of positive literals, $E \vdash \alpha$ and $M_0 = M$ implies $M \models \alpha$; consequently, $H \vdash \alpha$. Since $\alpha \supset \ell_i \in H$, it follows $H \vdash \ell_i$.

This concludes the proof of the *Only-If* direction and the proof of the lemma. ∎

Proposition B.1 *Given a Krom-pf-normal default theory $\langle W, D\rangle$ and literals $\ell_1, \ldots, \ell_m$, deciding whether $\langle W, D\rangle \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ is NP-complete.*

Proof. Concerning membership, a consistent extension E such that $\ell_1, \ldots, \ell_m \in E$ can be guessed and verified in polynomial time. Indeed, since $\langle W, D\rangle$ is prerequisite-free and normal, a guess $D' \subseteq D$ for $GD(E,T)$ is proper if $C = W \cup \mathit{concl}(D')$ is consistent, $C \vdash \ell_i$, for all $i = 1, \ldots, m$, and no $\frac{:\beta}{\beta} \in D \backslash D'$ exists such that $C \cup \{\,\beta\,\}$ is consistent. Since W is Krom and all formulas in defaults are Krom, each of these tests can be done in polynomial time. (Indeed, satisfiability of a Krom theory can be decided in linear time [25, 3]). This proves membership in NP.

To show hardness, we reduce the classical satisfiability problem (SAT) to this problem as follows. For a set of propositional clauses $C = \{C_1, \ldots, C_m\}$ where $C_i = \ell_{i_1} \vee \cdots \vee \ell_{i_{n_i}}$, define

$$D = \left\{ \frac{:\ \ell_{i_j} \wedge c_i}{\ell_{i_j} \wedge c_i} \mid i = 1, \ldots, m,\ j = 1, \ldots, n_i \right\} \cup \left\{ \frac{:x}{x},\ \frac{:\neg x}{\neg x} \mid x \text{ occurs in } C \right\},$$

where the c_i, $i = 1, \ldots, m$, are new atoms. It is easy to see that $\langle \emptyset, D\rangle \vdash_c c_1 \wedge \cdots \wedge c_m$ iff C is satisfiable. Since D is efficiently obtained from C, this proves NP-hardness. ∎

From the proof of this proposition, the following result can be easily established.

Corollary B.2 *Deciding if a given PDAP $\mathcal{P} = \langle H, M, W, D\rangle$ has a credulous explanation is NP-hard, even if all defaults are prerequisite-free, $W = H = \emptyset$, and M is a set of atoms.*

Proposition B.3 *Given a Krom-pf-normal default theory $\langle W, D\rangle$ and a literal ℓ, deciding whether $\langle W, D\rangle \vdash_s \ell$ is* co-NP*-complete.*

Proof. Membership in co-NP follows from the fact that a guess for an extension E such that $\ell \notin E$ can be verified in polynomial time.

To show co-NP-hardness, we reduce the unsatisfiability problem (UNSAT) to this problem: Decide whether a given set of propositional clauses $C = \{C_1, \ldots, C_m\}$, $C_i = \ell_{i_1} \vee \cdots \vee \ell_{i_{n_i}}$, is not satisfiable. Define a set of defaults D by

$$D = \left\{ \frac{:\ c \wedge {\sim}\ell_{i_1} \wedge \cdots \wedge {\sim}\ell_{i_{n_i}}}{c \wedge {\sim}\ell_{i_1} \wedge \cdots \wedge {\sim}\ell_{i_{n_i}}} \mid i = 1, \ldots, m \right\} \cup \left\{ \frac{:x}{x}, \ \frac{:\neg x}{\neg x} \mid x \text{ occurs in } C \right\}.$$

where c is a new atom. It holds that $\langle \emptyset, D \rangle \vdash_s c$ iff C is unsatisfiable. Since D can be constructed efficiently, this proves the result. ∎

Proposition B.4 Relevance *based on minimal credulous explanations for PDAPs $\mathcal{P} = \langle H, M, W, D \rangle$ where $\langle W, D \rangle$ is Literal-Horn is* NP*-complete.*

Proof. *Membership.* A guess for a minimal credulous explanation E such that $h \in E$ can be verified in polynomial time, by testing that E is an credulous extension and that $\langle E \setminus \{h'\}, M, W, D \rangle$ has no credulous explanation, for every $h' \in E$; both tests are polynomial, as follows from Lemma 5.1 and Theorem 5.3.

Hardness. The hardness part is a simple reduction of abductive reasoning from classical theories (cf. [22]): Given a definite Horn *PAP* $\langle V, H, M, T \rangle$, decide if a certain hypothesis h is relevant in a minimal explanation. Here T is a set of definite Horn clauses $a_1 \wedge \cdots \wedge a_n \supset a_0$ on a set of variables V, $H \subseteq V$, $M \subseteq V$, and the underlying inference is classical consequence $\vdash$. The constructed *PDAP* is $\mathcal{P}' = \langle H, M, W, D \rangle$, where $W = \{a \in V \mid a \in T\}$ and $D = \left\{ \frac{a_1 \wedge \cdots a_k : b}{b} \mid a_1 \wedge \cdots \wedge a_k \supset b \in T \right\}$. Observe that the credulous explanations for $\mathcal{P}'$ coincide with the skeptical ones, since $\langle W \cup E, D \rangle$ has a unique extension for every $E \subseteq H$. It is easy to see that the credulous explanations for $\mathcal{P}'$ coincide with the explanations for $\mathcal{P}$. Therefore, h is relevant in a minimal credulous explanation for $\mathcal{P}'$ iff h is relevant in a minimal explanation for $\mathcal{P}$. Since deciding the latter is NP-hard [22], **Relevance** for Literal-Horn *PDAP*s is also NP-hard. ∎

Lemma 5.5 *Let S be a (propositional) Krom theory and ℓ a literal, and let $L = \{\ell_1, \ldots, \ell_k\}$ be a set of literals such that (i) $S \cup L$ is consistent, (ii) $S \cup L \vdash \ell$ and $S \not\vdash \ell$. Then, $S \cup \{\ell_i\} \vdash \ell$ for some $\ell_i \in L$.*

Proof. This is obvious if $\ell \in L$. Assume thus that $\ell \notin L$. Consider any resolution proof for ℓ from $S \cup L$. (Such a proof must exist). In the last step, we have that $\{\ell\}$ is the resolvent of clauses C_1 and C_2 that have a resolution proof from $S \cup L$. It is easy to see that a clause C of size 2 has a resolution proof from a consistent Krom theory T iff it has a resolution proof from the set $T' \subseteq T$ of all clauses in T of size 2. Hence, we conclude that without loss of generality C_2 is a literal, i.e., $C_1 = \{\ell, {\sim}\ell_0\}$ and $C_2 = \{\ell_0\}$ (for otherwise, $S \vdash \ell$ would hold). Since $S \cup L$ is Krom, C_1 has a resolution proof from S, and therefore $S \cup \{\ell_0\} \vdash \ell$. By induction on the length r of a shortest resolution proof for ℓ from $S \cup L$, we conclude that $\ell_0 \in L$. Indeed, if $r = 0$, then $\ell_0 \in L$; if $r > 0$, then $\ell_0 \notin L$; since ℓ_0 has a shorter resolution proof than ℓ, it follows from the induction hypothesis that $S \cup \{\ell'\} \vdash \ell_0$ where $\ell' \in L$; hence, $S \cup \{\ell'\} \vdash \ell$, which concludes the induction and proves the lemma. ∎

Proposition 5.6 *Let $T = \langle W, D\rangle$ be a Krom-pf-normal default theory and let $L = \{\ell_1, \dots, \ell_m\}$ be a set of literals. Let $L1 = \{\ell_i \in L \mid W \not\vdash \ell_i\}$. Then, $T \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ iff (a) $L1$ is empty, or (b) there exists a subset $D1 \subseteq D$ such that*

(b.1) $|D1| \leq |L1|$,

(b.2) $W \cup lit(D1)$ *is consistent, and*

(b.3) $W \cup lit(D1) \vdash \ell$, *for each* $\ell \in L1$.

Proof. "$\Rightarrow$": If $T \vdash_c \ell$, then there exists an extension E of T such that $L \subseteq E$. By the characterization of E in terms of its generating defaults (see Section 2), $E = Th(W \cup lit(GD(E,T)))$. Consider two cases.

(i). E is not consistent. Then, by the well-known properties of default extensions [52], W is not consistent. Thus, from the definition, we have $L1 = \emptyset$; hence, (a) holds.

(ii). E is consistent. Then $W \cup lit(GD(E,T))$ is consistent. From Lemma 5.5, we infer that for each $\ell \in L1$ there exists some $\mu \in lit(GD(E,T))$ such that $W \cup \{\mu\} \vdash \ell$. Consequently, there exists a subset $D1 \subseteq GD(E,T)$, and hence $D1 \subseteq D$, such that $|D1| \leq |L1|$, $W \cup lit(D1)$ is consistent and $W \cup lit(D1) \vdash \ell$, for each $\ell \in L1$. Hence, (b) holds. This proves the *Only-If* direction.

"$\Leftarrow$": Assume that (a) holds, i.e., $L1 = \emptyset$. This means $W \vdash \ell$ for each $\ell \in L$. Since $W \subseteq E$ for every extension E of T and E is deductively closed, it follows that $L \subseteq E$ for every extension of T. As T is normal, an extension exists, and hence $T \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$.

Assume now that (b) does hold, i.e. for some $D1 \subseteq D$ we have $|D1| \leq |L1|$, $W \cup lit(D1)$ is consistent, and $W \cup lit(D1) \vdash \ell$ for every $\ell \in L1$. Since T is prerequisite-free and normal, it follows from semi-monotonicity that T has a consistent extension E such that $W \cup lit(D1) \subseteq E$. Since E is deductively closed, $L \subseteq E$; it follows $T \vdash_c \ell$. This proves the result. ∎

Theorem 5.7 *Let $T = \langle W, D\rangle$ be a Krom-pf-normal default theory, and let $L = \{\ell_1, \dots, \ell_m\}$ be a set of literals. Then, $T \vdash_c \ell_1 \wedge \cdots \wedge \ell_m$ can be decided using* **Krom-PFN-Cred-Inf** *in $O(|D|^m \cdot m \cdot n)$ time, where n is the length of the input.*

Proof. Correctness of the algorithm follows from Proposition 5.6. Concerning the complexity, satisfiability of a Krom theory can be decided in linear time [25, 3]. Consequently, also inference of a literal from a Krom theory can be decided in linear time. Therefore, $L1$ can be computed in $O(m \cdot n)$ time. The for-loop checks at most $|D|^{|L1|} \leq |D|^m$ sets $D1$; for each $D1$, computing S is possible in $O(n)$ time, and the tests $S \not\vdash \perp$, $S \vdash \ell_i$ take $O(m \cdot n)$ time. Hence, the overall running time is $O(m \cdot n + |D|^m(n + m \cdot n))$, which is $O(|D|^m \cdot m \cdot n)$. ∎

Theorem 5.8 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP based on credulous explanations such that $M = \{\ell_1, \dots, \ell_k\}$ is small and $\langle W, D\rangle$ is Krom-pf-normal. Then,* **Consistency**, **Relevance**, *and* **Necessity** *can be solved in $O(|D|^b \cdot n)$ time, where n is the length of $\mathcal{P}$.*

Proof. Construct a Krom-pf-normal default theory $T2 = \langle W2, D2\rangle$, given by

$$W2 = W \cup \{c_h \supset h \mid h \in H\}, \quad D2 = D \cup \left\{ \frac{: c_h}{c_h}, \frac{: \neg c_h}{\neg c_h} \mid h \in H \right\}$$

where each c_h is a new propositional atom. It is not hard to see that $\mathcal{P}$ has an explanation iff $W2$ is consistent and $T2 \vdash_c \ell_1 \wedge \cdots \wedge \ell_k$. Clearly, $W2$, $D2$, and $T2$ can be constructed in $O(n)$ time. Consistency of $W2$ can be checked in linear time [25, 3] and by Lemma 5.2, $T2 \vdash_c \ell_1 \wedge \cdots \wedge \ell_k$ can be decided in $O(|D2|^b \cdot n')$ time, where n' is the length of $T2$ (which is $O(n)$). Hence, **Consistency** can be decided in $O(|D|^b \cdot n)$ time. By Corollary 4.3, **Relevance** and **Necessity** can be easily reduced to **Consistency** resp. its complement in $O(n)$ time; hence, these problems are also solvable in $O(|D|^b \cdot n)$ time. ∎

Proposition 5.9 *Let E be any minimal credulous explanation for a PDAP $\mathcal{P} = \langle H, M, W, D\rangle$ where $\langle W, D\rangle$ is Krom-pf-normal and M is finite. Then, $|E| \leq |M|$.*

Proof. Let E be any credulous explanation. Then, the default theory $T1 = \langle W \cup E, D\rangle$ has a consistent extension A such that $M \subseteq E$. By the characterization of extensions, we have $A = Th(W \cup E \cup L)$, where $L = lit(GD(A, T1))$. Hence, $W \cup E \cup L$ is consistent and $W \cup E \cup L \vdash \ell$, for each $\ell \in M$. From Lemma 5.5, it follows that for each $\ell \in M$ such that $W \not\vdash \ell$, there exists a $\mu_\ell \in E \cup L$ such that $W \cup \{\mu_\ell\} \vdash \ell$. Fix any such μ_ℓ for each ℓ, and let $E0 \subseteq E$ be the set $E_0 = \{\mu_\ell \mid \ell \in M\} \cap E$; notice that $|E0| \leq |M|$. Then, $E0$ is a credulous explanation for $\mathcal{P}$. Indeed, the default theory $T2 = \langle W \cup E0, D\rangle$ has a consistent extension which contains M: Since the set $W \cup E0 \cup lit(GD(A, T1))$ is consistent and all defaults in D are prerequisite-free and normal, $T2$ has a consistent extension $E3$ containing $W \cup E_0 \cup lit(GD(A, T1))$. Moreover, since $W \cup E0 \cup lit(GD(A, T1)) \vdash \ell$ for each $\ell \in M$, we have $M \subseteq E3$. Thus, $E0$ is a credulous explanation of $\mathcal{P}$.

Since E was arbitrary, it follows that for each explanation E for $\mathcal{P}$, there exists an explanation $E0 \subseteq E$ such that $|E0| \leq |M|$. In particular, if E is a minimal explanation, then $E = E0$, and hence $|E| \leq |M|$. This proves the result. ∎

Theorem 5.10 *Given a PDAP $\mathcal{P} = \langle H, M, W, D\rangle$ where $\langle W, D\rangle$ is Krom-pf-normal and M is small, all minimal credulous explanations for $\mathcal{P}$ can be computed in time $O(|H|^b \cdot |D|^b \cdot n)$, where n is the length of $\mathcal{P}$.*

Proof. Consider algorithm **All-KPFN-Min-Cred-Exp**. In total, at most $|H|^b$ subsets $E \subseteq H$ are considered. For each set E, the test $W \cup E \not\vdash \perp$ takes $O(n)$ time ($W \cup E$ is Krom); the tests $E' \notin Min\_Expl$ can be done in $O(2^b \cdot \log|Min\_Expl|)$ time, if Min_Expl is stored in a standard search data structure (e.g., an AVL-tree) on which $E0 \notin Min\_Expl$ is checked for every subset $E0 \subset E$. The test **Krom-PFN-Cred-Inf**$(W \cup E, D, \phi)$ takes $O(|D|^b \cdot n)$. Since $|Min\_Expl|$ is bounded by $|H|^b$, this amounts to $O(|H|^b \cdot (2^b \log|H|^b + |D|^b \cdot n))$ time. Building and maintaining the data structure for Min_Expl takes in total $O(|H|^b \cdot \log|H|^b)$ time. Since $\log|H|^b = b\log|H|$ is $O(n)$, it follows that total running time of the algorithm is $O(|H|^b \cdot |D|^b \cdot n)$. ∎

Proposition B.5 *Given a first-order Krom-pf-normal default theory $T = \langle W, D \rangle$ and a ground literal ℓ, deciding whether $T \vdash_c \ell$ is* NP*-hard, even if all formulas in W are (universally quantified) atoms and D contains a single default.*

Proof. Transform the classical satisfiability problem (SAT) into this problem as follows. Let $C = \{C_1, \ldots, C_m\}$ be a propositional clause set on atoms $a_1, \ldots, a_n$, where without loss of generality no atom occurs positively and negatively in the same clause, and neither all clauses are positive or negative. For each clause C_i in C, put the formula $\phi_i = \forall x_1 \cdots \forall x_n p(t_1, \ldots, t_n)$ in W, where

$$t_j = \begin{cases} 0, & \text{if } a_j \in C_j; \\ 1, & \text{if } \neg a_j \in C_j; \\ x_j, & \text{otherwise.} \end{cases}$$

In D, put the single default $d = \frac{:\, \neg p(x_1,\ldots,x_n) \wedge s}{\neg p(x_1,\ldots,x_n) \wedge s}$, where s is a propositional atom. Intuitively, each ground atom $p(t_1, \ldots, t_n)$, where $t_i \in \{0, 1\}$, corresponds to a truth value assignment to the propositional atoms $a_1, \ldots, a_n$ such that a_i is true if $t_i = 1$ and $a_i = 0$ if $t_i = 0$. Each formula ϕ_i describes all truth assignments that do *not* satisfy clause C_i; the default d is applicable precisely if there is a truth assignment that is compatible with each clause C_i, i.e., if it satisfies C. Therefore, C is satisfiable if and only if $\langle W, D \rangle \vdash_c s$. This proves NP-hardness. ∎

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