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Semantics and Complexity of Abduction from Default Theories

(Extended abstract*)

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Abstract

Since logical knowledge representation is commonly based on nonclassical formalisms like default logic, autoepistemic logic, or circumscription, it is necessary to perform abductive reasoning from theories of nonclassical logics. In this paper, we investigate how abduction can be performed from theories in default logic. Different modes of abduction are plausible, based on credulous and skeptical default reasoning; they appear useful for different applications such as diagnosis and planning. Moreover, we analyze the complexity of the main abductive reasoning tasks. They are intractable in the general case; we also present known classes of default theories for which abduction is tractable.

1 Introduction

Abductive reasoning has been recognized as an important principle of common-sense reasoning having fruitful applications in a number of areas such diverse as model-based diagnosis [Poole, 1989], speech recognition [Hobbs *et al.*, 1988], maintenance of database views [Kakas and Mancarella, 1990], and vision [Charniak and McDermott, 1985]. Until now, mainly abduction from theories of classical logic has been studied. However, logical knowledge representation is commonly based on nonclassical formalisms like default logic, autoepistemic logic, or circumscription. Thus, in such situations it is necessary to perform abductive reasoning from theories (i.e. knowledge bases) of nonclassical logics.

Since default logic is widely proposed for knowledge representation, it is important to investigate how abduction can be performed from theories $\langle W, D\rangle$ in default logic. We informally pursue this on an example.

Example 1 Consider the following set of default rules, which represent knowledge about Bill's skiing habits:

$$D = \left\{ \frac{:\neg skiing\_Bill}{\neg skiing\_Bill}, \frac{weekend : \neg snowing}{skiing\_Bill}, \frac{:\neg snowing}{\neg snowing} \right\}$$

The defaults intuitively state the following: (i) Bill is usually not out for skiing; (ii) Bill is out for skiing on weekends, if we can assume that it is not snowing; (iii) usually it is not snowing. For the certain knowledge $W = \{ weekend \}$ (encoding that it is Saturday or Sunday), the default theory $T = \langle W, D\rangle$ has one extension which contains $\neg snowing$ and $skiing\_Bill$.

Suppose now we observe that Bill is not out for skiing (which is inconsistent with the extension). Abduction means to find an explanation for this observation, that is, to identify a set of facts, chosen from a set of hypotheses, whose presence in the theory at hand would entail the observation $\neg skiing\_Bill$, i.e., cause that $\neg skiing\_Bill$ is in the extension. We find such an explanation by adopting the hypothesis $snowing$. Indeed, if we add $snowing$ to W, the default theory $T' = \langle\{ weekend, snowing \}, D\rangle$ has a single extension, which contains $\neg skiing\_Bill$. We say that $snowing$ is *abduced* from the observation $\neg skiing\_Bill$, or that it is an *abductive explanation* of $\neg skiing\_Bill$. ∎

Observe that the description of the above situation requires the specification of some default properties that can not be represented properly in classical logic.

In general, as opposed to the example, a default theory may have several or even no extensions. For deductive entailment, this gives rise to *credulous entailment*, under which ϕ is entailed from a default theory T (denoted $T \vdash_c \phi$) iff ϕ belongs to at least one extension of T, and to *skeptical entailment*, under which ϕ follows from T ($T \vdash_s \phi$) iff ϕ belongs to all extensions of T. Accordingly, two variants of abduction from default theories arise: *credulous* abduction, where entailment of an observation is based on $\vdash_c$, and *skeptical* abduction, which is based on $\vdash_s$. In practice, the user will choose credulous or skeptical abduction on the basis of the particular application domain.

We argue that credulous abduction is well suited for *diagnosis*, while skeptical abduction is adequate for *planning*. (Cf. [Poole, 1989] and [Eshghi, 1988; Ng and Mooney, 1991] for abduction in logic-based diagnosis and planning & plan recognition, respectively.) In fact, consider a system represented by a default theory $\langle W, D\rangle$. If it receives some input, reflected by adding a set A of facts to W, then each extension of $\langle W \cup A, D\rangle$ is a possible evolution of the system, i.e., each extension represents a possible reaction of the system to A.

*A more elaborate version including proofs is available on email request to the authors.

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Abductive diagnosis consists, loosely speaking, in deriving from an observed system state (characterized by the truth of a set F of facts), a suitable input A which caused this evolution (cf. [Poole, 1989]). Now, since *each* extension of $\langle W \cup A, D\rangle$ is a possible evolution of the system with input A, we can assert that A is a possible input that caused F if $\langle W \cup A, D\rangle \vdash_c F$. Thus, diagnostic problems can be naturally represented by abductive problems with credulous entailment.

Example 2 Assume there are two sky routes, $rv1$ and $rv2$, between Rome and Vienna, and three sky routes $mv1$, $mv2$, and $mv3$ between Milan and Vienna. Route $mv1$ intersects route $rv1$, and $mv2$ intersects $rv2$. On normal speed and flight conditions, two planes from Milan and Rome to Vienna will collide if the plane from Milan takes off 20 minutes after the plane from Rome and they fly on intersecting routes. This knowledge about possible collisions is represented in simplified form by the following set D of propositional defaults:

$$D = \left\{ \frac{mv1 \wedge rv1 \wedge m\_20min\_later : collision}{collision}, \frac{mv2 \wedge rv2 \wedge m\_20min\_later : collision}{collision}, \frac{: \neg collision}{\neg collision}, \frac{: \neg rv1}{\neg rv1}, \frac{: \neg rv2}{\neg rv2}, \frac{: \neg mv1}{\neg mv1}, \frac{: \neg mv2}{\neg mv2}, \frac{: \neg mv3}{\neg mv3} \right\}$$

Now, you are informed that planes flying from Milan and Rome to Vienna collided. A diagnosis for the collision can be obtained by abducing an explanation for the observation *collision* from the theory $T = \langle \emptyset, D\rangle$. In this case, we want to know possible flight schedules that can have caused the collision. In other words, we are looking for schedules S such that *collision* is in some extension of the theory $T' = \langle S, D\rangle$ ($T' \vdash_c collision$). Credulous abduction correctly identifies such explanations. For instance, it is easy to recognize that both $E1 = \{mv1, rv1, m\_20min\_later\}$ and $E2 = \{mv2, rv2, m\_20min\_later\}$ are credulous explanations for *collision*. ∎

Suppose now we want that the system evolves into a certain state (described by a set F of facts), and we have to determine the "right" input that enforces this state of the system (planning). In this case it is not sufficient to choose an input A such that F is true in some possible evolution of the system; rather, we look for an input A such that F is true in *all* possible evolutions, as we want be sure that the system reacts in that particular way. In other words, we look for A such that $\langle W \cup A, D\rangle \vdash_s F$. Hence, planning activities can be represented by abductive problems with skeptical entailment.

Example 3 We know that a plane from Rome to Vienna left at 7.50 ($r\_7.50$), but we do not know on which route. We have to schedule the flight of a plane from Milan to Vienna, where takeoff is possible at 8.10 ($m\_8.10$) and at 8.20 ($m\_8.20$). The collision-free schedules can be obtained by finding an abductive explanation out of the hypotheses $m\_8.10$, $m\_8.20$, $mv1$, $mv2$, $mv3$ for the observation $\neg collision$ from the theory $T = \langle W, D1\rangle$,

$$\begin{aligned} W &= \{\, r\_7.50, \quad rv1 \vee rv2, \quad m\_8.10 \vee m\_8.20, \\ &\qquad mv1 \vee mv2 \vee mv3, \\ &\qquad r\_7.50 \wedge m\_8.10 \supset m\_20min\_later \,\} \\ D1 &= D \cup \left\{ \frac{: \neg m\_8.10}{\neg m\_8.10}, \frac{: \neg m\_8.20}{\neg m\_8.20} \right\} \end{aligned}$$

As we can not risk a collision, we want that every possible evolution of the system is collision-free. Thus, we have to look for skeptical explanations of $\neg collision$. For instance, both $E3 = \{m\_8.20\}$ and $E4 = \{mv3\}$ are skeptical explanations for $\neg collision$; that is, takeoff at 8.20 or using route $mv3$ prevents a collision, where the route in $E3$ and the time in $E4$ can be chosen freely. ∎

The two examples above support the intuition that credulous abduction is feasible for diagnosis, while skeptical abduction is well-suited for planning. On the other hand, Section 4 shows that skeptical abduction has most likely a higher complexity than credulous abduction; thus, from the above point of view, planning is most likely harder than diagnosis.

For space reasons, we only present some proof sketches. Proofs of all results are given in the full paper.

2 Preliminaries and Notation

We assume that the reader knows the basic concepts of default logic [Reiter, 1980] (cf. also [Marek and Truszczyński, 1993] for an extensive study). We focus on propositional default theories $T = \langle W, D\rangle$ over a propositional language $\mathcal{L}$ (including $\perp$ for falsity), i.e. W is a subset of $\mathcal{L}$ and D a set of defaults $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}$, $m \geq 1$ where α, $\beta_1, \ldots, \beta_m$, γ are from $\mathcal{L}$. The extensions of T, which are deductively closed sets $E \subseteq \mathcal{L}$, are defined by a fixpoint equation; in particular, $\mathcal{L}$ is an extension of T (and, in this case, unique) iff W is not consistent. Recall that T is *normal* iff each default in D is *normal*, i.e., of form $\frac{\alpha:\beta}{\beta}$; a normal T always has an extension.

For NP-completeness and complexity theory, cf. [Johnson, 1990]. The classes Σ_k^P and Π_k^P of the polynomial hierarchy are defined as follows: $\Sigma_0^P = \Pi_0^P = \mathrm{P}$, and

$$\Sigma_k^P = \mathrm{NP}^{\Sigma_{k-1}^P}, \quad \Pi_k^P = \text{co-}\Sigma_k^P, \quad \text{for all } k \geq 11.$$

In particular, NP $= \Sigma_1^P$ and co-NP $= \Pi_1^P$. The class $\mathrm{D_k^P}$, which is defined as the class of problems that consist of the conjunction of two (independent) problems from Σ_k^P and Π_k^P, respectively, is considered to be further restricted in computational power. For all $k \geq 1$, clearly $\Sigma_k^P \subseteq \mathrm{D_k^P} \subseteq \Sigma_{k+1}^P$; both inclusions are believed to be strict. Many nonmonotonic reasoning problems are complete for classes at the lower end of the polynomial hierarchy [Cadoli and Schaerf, 1993; Nebel, 1994]. It is well-known that deciding whether a propositional default theory has an extension is Σ_2^P-complete, and that credulous and skeptical reasoning from default theories are complete for Σ_2^P and Π_2^P, respectively. This remains true if inconsistent extensions are excluded and, for the latter problems, if default theories are in addition normal [Gottlob, 1992;

Stillman, 1992]. Cases of lower complexity and tractable fragments were identified in [Kautz and Selman, 1991; Stillman, 1990].

3 Formalizing default abduction

In this section, we describe a basic formal model for abduction from propositional default theories and state the main decisional reasoning tasks for abductive reasoning.

Our formalization of an abduction scenario is as follows.

Definition 1 *A propositional default abduction problem (PDAP) is a quadruple $\langle H, M, W, D\rangle$ where H is a set of propositional literals (*called hypotheses, or abducibles*), M is a set of propositional literals (*observations, or manifestations*), and $\langle W, D\rangle$ is a propositional default theory. $\mathcal{P}$ is normal iff each default in D is normal.* ∎

Note that hypotheses and manifestations may be literals rather than atoms. Allowing literals as hypotheses is common in abduction, cf. [Selman and Levesque, 1990]. However, this has no effect on the expressive power or complexity of the formalism in general.

Credulous and skeptical explanations are as follows.

Definition 2 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP, and let $E \subseteq H$. Then, E is a credulous explanation for $\mathcal{P}$ iff (i) $\langle W \cup E, D\rangle \vdash_c M$, and (ii) $\langle W \cup E, D\rangle$ has a consistent extension. Similarly, E is a skeptical explanation for $\mathcal{P}$ iff (i) $\langle W \cup E, D\rangle \vdash_s M$ and (ii) $\langle W \cup E, D\rangle$ has a consistent extension.*

The existence of a consistent extension for $\langle W \cup E, D\rangle$ (in this case, all extensions are consistent) assures that the explanation E is consistent with the knowledge represented in $\langle W, D\rangle$. This is analogous to the consistency criterion in abduction from classical theories.

It is common in abductive reasoning to prune the set of all explanations and to focus, guided by some principle of explanation preference, on a set of preferred explanations. The most important such principle is, following Occam's principle of parsimony, to prefer nonredundant explanations, i.e., explanations which do not contain any other explanation properly, cf. [Peng and Reggia, 1990; Selman and Levesque, 1990; Konolige, 1992]. We refer to such explanations as *minimal explanations*. In Example 3 $E3 = \{m\_8.20\}$ and $E4 = \{mv3\}$ are the minimal explanations; they represent the smallest partial schedules that can be arbitrarily completed to collision-free schedules, and thus provide the greatest flexibility.

In the sequel, we will write *Exp*($\mathcal{P}$) for the set of explanations for the *PDAP* $\mathcal{P}$, abstracting from the chosen type of explanations (credulous, skeptical, minimal credulous, or minimal skeptical).

The following properties of a hypothesis in a *PDAP* $\mathcal{P}$ are important with respect to computing explanations.

Definition 3 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP and $h \in H$. Then, h is relevant (resp. necessary) for $\mathcal{P}$ iff $h \in E$ for some (resp. every) $E \in Exp(\mathcal{P})$.*

The opposite of necessity is also termed dispensability (cf. [Josephson *et al.*, 1987]). In Example 2, m_20min_later is necessary, while each hypothesis $rv1, rv2, mv1, mv2$ is relevant, but not necessary. Moreover, in Example 3 $mv3$ is relevant w.r.t. minimal (skeptical) explanations, but not necessary. Note that in the same example $rv1$ is relevant under arbitrary explanations, but not relevant under minimal explanations.

The main decisional problems in abductive reasoning amount to the following. Given a *PDAP* $\mathcal{P} = \langle H, M, W, D\rangle$,

(Consistency): does there exist an explanation for $\mathcal{P}$?

(Relevance): is a given hypothesis $h \in H$ relevant for $\mathcal{P}$, i.e., does h contribute to some explanation of $\mathcal{P}$?

(Necessity): is a given hypothesis $h \in H$ necessary for $\mathcal{P}$, i.e., is h contained in all explanations of $\mathcal{P}$?

Due to the following simple fact, we shall not deal in our analysis explicitly with **Necessity** in the case of minimal explanations.

Proposition 1 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP and let $h \in H$. Then, h is necessary for $\mathcal{P}$ under minimal credulous (resp. skeptical) explanations iff h is necessary for $\mathcal{P}$ under credulous (resp. skeptical) explanations.*

4 Results

The main results on the complexity of abduction from general propositional default theories are summarized in Table 1. In our analysis, we pay particular attention to normal *PDAP*s, since this class corresponds to the most important fragment of default logic. All hardness results in Table 1 have been derived for the case where the underlying default theory $\langle W, D\rangle$ is normal. Thus like deduction, abduction from normal default theories is as hard as abduction from arbitrary default theories.

We introduce some additional notation. For a set A of propositional atoms, we denote by $\neg A$ the set $\{\neg a \mid a \in A\}$ and by A' the set of atoms $\{a' \mid a \in A\}$.

4.1 Arbitrary explanations

Our first result shows that abduction from default theories based on credulous explanations can be efficiently reduced to deductive reasoning from propositional default theories. This is somewhat unexpected and surprising, since in case of classical theories, abduction can not be efficiently reduced to deduction.

Given a *PDAP* $\mathcal{P} = \langle H, M, W, D\rangle$, we construct a default theory $T_{\mathcal{P}} = \langle W_{\mathcal{P}}, D_{\mathcal{P}}\rangle$ such that the credulous explanations of $\mathcal{P}$ correspond to the extensions of $T_{\mathcal{P}}$. Indeed, define

- $W_{\mathcal{P}} = W \cup \{a_h \supset h \mid h \in H\}$,
- $D_{\mathcal{P}} = D \cup \left\{\frac{:\neg m}{\perp} \mid m \in M\right\} \cup \left\{\frac{:a_h}{a_h}, \frac{:\neg a_h}{\neg a_h} \mid h \in H\right\}$

where for each $h \in H$, a_h is a new propositional atom. Then, we have:

Theorem 1 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Then, (i) if E is a credulous explanation for $\mathcal{P}$, then there exists a consistent extension E' of $T_{\mathcal{P}}$ such that $E = \{h \in H \mid a_h \in E'\}$; (ii) if E' is a consistent extension of $T_{\mathcal{P}}$, then $E = \{h \in H \mid a_h \in E'\}$ is a credulous explanation for $\mathcal{P}$.*

PDAP $\mathcal{P} = \langle H, M, W, D\rangle$	arbitrary explanations		minimal explanations	
Problem:	credulous	skeptical	credulous	skeptical
$Exp(\mathcal{P}) \neq \emptyset$	$\mathbf{\Sigma}_2^P$	$\mathbf{\Sigma}_3^P$	$\mathbf{\Sigma}_2^P$	$\mathbf{\Sigma}_3^P$
$E \in Exp(\mathcal{P})$	$\mathbf{\Sigma}_2^P$	D_2^P	D_2^P	$\mathbf{\Pi}_3^P$
$E \in Exp(\mathcal{P})$ is minimal	$\mathbf{\Pi}_2^P$	$\mathbf{\Pi}_3^P$		
$h \in H$ is relevant for $\mathcal{P}$	$\mathbf{\Sigma}_2^P$	$\mathbf{\Sigma}_3^P$	$\mathbf{\Sigma}_3^P$	$\mathbf{\Sigma}_4^P$
$h \in H$ is necessary for $\mathcal{P}$	$\mathbf{\Pi}_2^P$	$\mathbf{\Pi}_3^P$	$\mathbf{\Pi}_2^P$	$\mathbf{\Pi}_3^P$

Table 1: Complexity results for abduction from propositional default theories

Using (i) and (ii), the main decisional abductive reasoning tasks can be efficiently transformed to similar deductive reasoning tasks in default logic.

Corollary 1 *Let $\mathcal{P}$ be a PDAP based on credulous explanations. Then, (i)* **Consistency**, *(ii)* **Relevance**, *and (iii)* **Necessity** *are equivalent to (i') existence of a consistent extension of $T_\mathcal{P}$, (ii') membership of a_h in some consistent extension of $T_\mathcal{P}$, and (iii') membership of a_h in all extensions of $T_\mathcal{P}$, respectively.*

By the results on the complexity of propositional default logic [Gottlob, 1992; Stillman, 1992], it follows that (i) and (ii) are in Σ_2^P and that (iii) is in Π_2^P. We also obtain matching hardness by reductions from deductive default reasoning. Let $T = \langle W, D\rangle$ be a normal default theory such that W is consistent, and ϕ a formula. Let h, q be new propositional atoms. Then, the *PDAP*

$$(*) \qquad \langle \emptyset, \{q\}, W \cup \{\phi \supset q\}, D\rangle$$

has a credulous explanation iff $T \vdash_c \phi$; h is relevant for the *PDAP*

$$(**) \qquad \langle \{h\}, \{q\}, W \cup \{\phi \supset q\}, D\rangle$$

iff $T \vdash_c \phi$; and h is necessary for the *PDAP*

$$(***) \qquad \langle \{h\}, \{q\}, W \cup \{\phi \vee h \supset q\}, D\rangle$$

iff $T \not\vdash_c \phi$. Since the reasoning problems for T in (*), (**) are Σ_2^P-hard and the one in (***) is Π_2^P-hard [Gottlob, 1992], the hardness results follow.

It is interesting to note that verifying a credulous explanation is as hard as finding one. The former problem can be easily reduced to the latter; moreover, $\emptyset$ is the only possible credulous explanation for the *PDAP* (*).

Thus,

Theorem 2 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if $E \subseteq H$ is a credulous explanation for $\mathcal{P}$ is Σ_2^P-complete, with hardness holding even for normal $\mathcal{P}$.*

Now consider abduction based on skeptical reasoning. It would be useful to have a reduction of abductive reasoning to deductive reasoning which can be computed efficiently. However, by using skeptical reasoning the abductive reasoning tasks grow more complex, by one level of the polynomial hierarchy. This strongly suggests that such an efficient reduction is not possible.

We first consider the problem of recognizing skeptical solutions. Clearly, this reduces to deciding if a certain default theory has a consistent extension (which is in Σ_2^P) and if each extension includes all manifestations (Π_2^P). Thus, the problem is a logical conjunction of a problem in Σ_2^P and a problem in Π_2^P, and hence in the class D_2^P. Moreover, it is also hard for this class.

Theorem 3 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if $E \subseteq H$ is a skeptical explanation for $\mathcal{P}$ is D_2^P-complete.*

Thus, as in the case of credulous explanations, recognizing a skeptical explanation is at the second level of the polynomial hierarchy. However, since this problems involves both a Σ_2^P and a Π_2^P-hard subtask (as opposed to only a Σ_2^P-hard one), finding a skeptical explanation resides at the third level.

We sketch here the Σ_3^P-hardness proof for **Consistency** by a transformation of deciding if a *quantified Boolean formula* (QBF) $\Phi = \exists X \forall Y \exists Z F$ is valid (cf. [Johnson, 1990] for a definition of QBFs). Define

$$D = \left\{ \tfrac{:\neg a}{\neg a}, \tfrac{:a}{a} \mid a \in X \cup Y \right\} \cup \left\{ \tfrac{:F}{F} \right\},$$
$$\mathcal{P} = \langle X \cup \{\neg x \mid x \in X\}, \{f\}, \{f \equiv F\}, D\rangle,$$

where f is a new atom. Then, $\mathcal{P}$ has a skeptical explanation iff Φ is valid.

How does this result compare to other nonmonotonic logics, in particular, which nonmonotonic logic has similar complexity ? We know that Konolige's moderately grounded autoepistemic logic [Konolige, 1988] and several other ground nonmonotonic modal logics have the same complexity [Eiter and Gottlob, 1992; Donini *et al.*, 1995]; thus, we can use a theorem prover for such logics to perform abductive reasoning from default theories based on skeptical explanations.

4.2 Minimal explanations

As mentioned above, one is usually interested in *minimal* explanations for observations. The results in [Eiter and Gottlob, 1995] were that the complexity of abduction from classical theories does not increase if minimal explanations are used instead of arbitrary explanations. However, this is not true in for abduction from default logic. Here, checking minimality of an explanation is a source of complexity, which causes an increase in complexity by one level of the polynomial hierarchy.

Consider first credulous explanations. Checking minimality of an explanation E has complementary complexity of checking the explanation property. Notice that E is *not* minimal iff for some $h \in E$, the *PDAP* $\langle E - \{h\}, M, W, D\rangle$ has a credulous solution; hence, it follows that the problem is in Π_2^P. On the other hand, reconsider (***). Clearly, $\{h\}$ is a credulous explanation; moreover, it is minimal iff h is necessary for $\mathcal{P}$. Thus, Π_2^P-hardness follows.

Note that recognizing minimal credulous explanations, which consists in checking the solution property and testing minimality, is in $\mathrm{D}_2^{\mathrm{P}}$, and also complete for this class. Thus, this problem can be transformed into recognition of skeptical explanations for a certain *PDAP* and vice versa. Due to the complexity of minimality checking, problem **Relevance** migrates to the next level of the polynomial hierarchy.

Theorem 4 *Let $\mathcal{P}$ be a PDAP based on minimal credulous explanations. Then, problem* **Relevance** *is Σ_3^P-complete, with hardness holding even for normal $\mathcal{P}$.*

Proof. (Sketch) *Membership.* A guess E for a minimal credulous explanation for $\mathcal{P}$ such that $h \in E$ can be verified by two calls to a Π_2^P oracle. Hence, the problem is in Σ_3^P.

Hardness. We outline a reduction from deciding validity of a QBF $\Phi = \exists X \forall Y \exists Z F$. Let s and q be new atoms, and define

$$D = \left\{ \frac{:\neg s}{\neg s}, \frac{s \wedge Y : q}{q}, \frac{\neg s \wedge \neg F : q}{q} \right\} \cup \left\{ \frac{x : x'}{x'}, \frac{\neg x : x'}{x'} \mid x \in X \right\} \cup \left\{ \frac{:\neg y}{\neg y} \mid y \in Y \right\}.$$

Let $\mathcal{P} = \langle X \cup \neg X \cup Y \cup \{s\}, X' \cup \{q\}, \emptyset, D\rangle$. Then, one can show that s is relevant for a minimal credulous explanation for $\mathcal{P}$ iff Φ is valid. ∎

Now let us consider minimal skeptical explanations. Testing minimality of a skeptical explanation is much more involved than of a credulous explanation. While the latter has roughly the same complexity as testing the explanation property, the former is harder by one level of the polynomial hierarchy. Intuitively, this can be explained as follows. Since verifying a credulous explanation E is in Σ_2^P, it has a polynomial-size "proof" which can be checked with an NP oracle in polynomial time. Thus, if we ask for a smaller explanation $E' \subset E$, we can simultaneously guess E' and its proof, and check the proof in polynomial time with the NP oracle. However, verifying a skeptical explanation E is Π_2^P-hard, and hence E does not have such a "proof". Here, verification needs the full power of a Π_2^P oracle.

Theorem 5 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP. Deciding if a skeptical explanation E for $\mathcal{P}$ is minimal is Π_3^P-complete, with hardness holding even for normal $\mathcal{P}$.*

Proof. (Sketch) *Membership.* A guess for a smaller skeptical explanation $E' \subset E$ can be verified with two calls to a Σ_2^P oracle, and hence deciding the existence of such an E' is in Σ_3^P. Consequently, the problem is in Π_3^P.

Hardness. We describe here a reduction from deciding whether a QBF $\Phi = \forall X \exists Y \forall Z F$ is valid. Let s and q be new atoms, and define

$$D = \left\{ \frac{:\neg s}{\neg s}, \frac{s \wedge X : q}{q}, \frac{\neg s : \neg F \wedge q}{\neg F \wedge q} \right\} \cup \left\{ \frac{:\neg x}{\neg x} \mid x \in X \right\} \cup \left\{ \frac{:y}{y}, \frac{:\neg y}{\neg y} \mid y \in Y \right\}.$$

Let $\mathcal{P} = \langle X \cup \{s\}, \{q\}, \emptyset, D\rangle$. Check that $E = X \cup \{s\}$ is a skeptical explanation for $\mathcal{P}$. Moreover, E is minimal iff Φ is valid. ∎

Note that recognizing minimal skeptical explanations is in Π_3^P, since the complexity of deciding minimality (Π_3^P) dominates the complexity of the solution property ("only" Σ_2^P), and is also complete for this class.

The complexity of deciding relevance of a hypothesis increases by the same amount as testing minimality if skeptical explanations are used instead of credulous explanations. In fact, the problem resides at the fourth level of the polynomial hierarchy.

Theorem 6 *Let $\mathcal{P}$ be a PDAP based on minimal skeptical explanations. Then, problem* **Relevance** *is Σ_4^P-complete, with hardness holding even for normal $\mathcal{P}$.*

Proof. (Sketch) *Membership.* A guess for a minimal skeptical explanation E for $\mathcal{P}$ such that $h \in E$ can be verified with one call to a Σ_3^P oracle.

Hardness. We outline a reduction from deciding validity of a QBF $\Psi = \exists R \forall X \exists Y \forall Z F$, which is an extension to the reduction in the proof of Theorem 5. Let as there be s and q new atoms, and define

$$D1 = D \cup \left\{ \frac{r' : r \wedge r''}{r \wedge r''}, \frac{\neg r' : \neg r \wedge r''}{\neg r \wedge r''} \mid r \in R \right\},$$

where D is the same set of defaults as in the proof of Theorem 5. Define $\mathcal{P} = \langle H, R'' \cup \{q\}, \emptyset, D1\rangle$, where $H = R' \cup \neg R' \cup X \cup \{s\}$. (Note that if W would be empty, then $\mathcal{P}$ would be identical to the *PDAP* in the proof of Theorem 5). It holds that for each subset $R1 \subseteq R$, the set $R1' \cup \neg(R - R1)' \cup X \cup \{s\}$ is a skeptical explanation for $\mathcal{P}$. Moreover, it can be shown that s is relevant for a minimal skeptical explanation for $\mathcal{P}$ iff Ψ is valid. ∎

There is no well-known nonmonotonic logic that has similar complexity, and thus one can not take advantage of theorem provers for such logics to perform skeptical abduction from default theories.

4.3 Tractable cases

From the practical side, the results from above are discouraging, since abduction from default theories has even higher complexity than deduction, in particular for skeptical explanations. The reasoning tasks suffer from several intermingled sources of complexity, whose number is (at least) the level at the polynomial hierarchy.

For example, **Relevance** for $\mathcal{P} = \langle H, M, W, D\rangle$ using minimal skeptical explanations (complete for Σ_4^P) suffers from the following four "orthogonal" sources of complexity: (1.) classical deductive inference ($\models$), (2.) the number of extensions of $\langle W \cup E, D\rangle$, (3.) the number of candidates E for a skeptical explanation, and (4.) the number of possible smaller explanations, where each number can be exponential.

For dealing with abduction from default theories in practice, we have to find tractable cases or cases where good algorithms for handling hard problems like GSAT [Selman *et al.*, 1992] are applicable.

An example of the latter case is credulous abduction from default theories where all propositional formulas are from a tractable fragment of the propositional language, e.g. *Horn* formulas or *Krom* formulas (clauses with at most two literals). In such a case, classical inference $\models$ vanishes as source of complexity. In particular, the Σ_2^P-complete abductive reasoning tasks fall back to NP. Thus, we can use e.g. GSAT [Selman *et al.*, 1992], which provides a good heuristics for solving NP-complete problems, to solve the problems quickly.

For tractable cases of default abduction, all sources of complexity must be eliminated. In particular, the underlying default reasoning tasks must be tractable. Kautz and Selman [Kautz and Selman, 1991] and Stillman [Stillman, 1990] gave a very detailed picture of polynomial vs. intractable cases of deductive default reasoning. For the following two classes of default theories $\langle W, D\rangle$, they proved tractability of credulous inference $\langle W, D\rangle \vdash_c \ell$ of a single literal ℓ:

Literal-Horn [Kautz and Selman, 1991]: W is a set of literals and each default in D is Horn, i.e., of form $\frac{a_1\wedge\cdots\wedge a_k:\ell}{\ell}$, where the a_i's are atoms and ℓ is a literal.

Krom-pf-normal [Stillman, 1990]: W is a set of Krom formulas, and each default in D is of form $\frac{:\ell_1\wedge\cdots\wedge\ell_k}{\ell_1\wedge\cdots\wedge\ell_k}$, where all ℓ_i's are literals.

A natural generalization of the proof in [Kautz and Selman, 1991] yields the following.

Lemma 1 *Let $\langle W, D\rangle$ be a Literal-Horn default theory, and let $\ell_1,\ldots,\ell_n$ be literals. Then, deciding $\langle W, D\rangle \vdash_c \ell_1\wedge\cdots\wedge\ell_n$ is polynomial.*

For Krom-pf-normal, such a generalization is not evident as $\langle W, D\rangle \vdash_c \ell_1\wedge\cdots\wedge\ell_n$ is NP-hard. However, it is possible for a small conjunction.

In what follows, we call a set L of literals *small* iff $|L| \leq c$ for some fixed constant c.

Lemma 2 *Let $\langle W, D\rangle$ be Krom-pf-normal, and let $L = \{\ell_1,\ldots,\ell_k\}$ be a small set of literals. Then, deciding $\langle W, D\rangle \vdash_c \ell_1\wedge\cdots\wedge\ell_k$ is polynomial.*

Based on these tractable cases of credulous default reasoning, we obtain tractable cases of credulous default abduction. Similar tractability results for skeptical default abduction are unlikely, since the underlying skeptical inference $\langle W, D\rangle \vdash_s \ell$ is co-NP-complete in both cases (cf. [Kautz and Selman, 1991] for Literal-Horn).

Literal-Horn default theories

In this case, the main reasoning tasks for credulous abduction are tractable.

Theorem 7 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP based on credulous explanations and $\langle W, D\rangle$ Literal-Horn. Then,* **Consistency***,* **Relevance***, and* **Necessity** *are polynomial.*

Proof. (Sketch) Construct a Literal-Horn $T1 = \langle W, D1\rangle$, where $D1 = \left\{\frac{b_h:h}{h}, \frac{:\neg b_h}{\neg b_h}, \frac{:b_h}{b_h} \mid h \in H\right\}$, where each b_h is a new propositional atom. Then, it can be shown that $\mathcal{P}$ has an explanation iff W is consistent and $T1 \vdash_c \ell_1\wedge\cdots\wedge\ell_k$, where $M = \{\ell_1,\ldots,\ell_k\}$. By Lemma 1, this can be decided in polynomial time. Thus, **Consistency** is polynomial. **Relevance** and **Necessity** can be easily reduced to **Consistency** resp. its complement. ∎

Notice that a polynomial algorithm for *finding* a credulous explanation (even containing a given hypothesis), can be extracted from the proof.

Moreover, there is also a polynomial algorithm for finding a *minimal credulous explanation*. Indeed, an explanation E for $\mathcal{P} = \langle H, M, W, D\rangle$ is minimal iff $\langle E - \{h\}, M, W, D\rangle$ has no explanation for each $h \in E$. Thus, for $\mathcal{P}$ as above, one can check in polytime whether E is minimal and, if not, find a smaller explanation $E1 \subset E$. By repeating this test, we can minimize E.

Theorem 8 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP where $\langle W, D\rangle$ is Literal-Horn. Then, a minimal credulous explanation for $\mathcal{P}$ can be found in polynomial time.*

However, **Relevance** based on minimal credulous explanations for *PDAP*s with Literal-Horn default theories can be shown to be NP-complete.

Krom-pf-normal default theories

For this fragment, we have the following results.

Theorem 9 *Let $\mathcal{P} = \langle H, M, W, D\rangle$ be a PDAP based on credulous explanations such that $M = \{\ell_1,\ldots,\ell_k\}$ is small and $\langle W, D\rangle$ is Krom-pf-normal. Then,* **Consistency**, **Relevance***, and* **Necessity** *are polynomial.*

Proof. (Sketch) Construct a Krom-pf-normal default theory $T2 = \langle W2, D2\rangle$, where

$$W2 = \{c_h \supset h \mid h \in H\}, \quad D2 = D \cup \left\{\frac{:c_h}{c_h}, \frac{:\neg c_h}{\neg c_h} \mid h \in H\right\},$$

where each c_h is a new propositional atom. Then, $\mathcal{P}$ has an explanation iff $W2$ is consistent and $T2 \vdash_c \ell_1 \wedge \cdots\wedge\ell_k$, which are both polynomial. Consistency fo $W2$ and $T2 \vdash_c \ell_1\wedge\cdots\wedge\ell_k$ can be decided in polynomial time (cf. Lemma 2). Hence, **Consistency** is polynomial. Since **Relevance** and **Necessity** can be easily reduced to **Consistency** resp. its complement, these problems are also polynomial. ∎

Again, a polynomial time algorithm for finding an explanation can be extracted from the proof. Unfortunately, Theorem 9 can not be generalized to an arbitrary set M of literals. In fact, due to the NP-hardness of $\langle W, D\rangle \vdash_c \ell_1\wedge\cdots\wedge\ell_n$ for Krom-pf-normal $\langle W, D\rangle$, the problem is NP-hard.

Interestingly, the number of hypotheses in a minimal credulous explanation is bounded by the number of manifestations. Intuitively, this is explained by the fact that always a single hypothesis can explain a manifestation.

Proposition 2 *Let E be any minimal credulous explanation for $\mathcal{P} = \langle H, M, W, D\rangle$ where $\langle W, D\rangle$ is Krom-pf-normal and $M = \{\ell_1,\ldots,\ell_n\}$. Then, $|E| \leq |M|$.*

In particular, for a single manifestation ($M = \{\ell\}$), the minimal explanations consist of single hypotheses, if hypotheses are needed for an explanation.

A consequence of this characterization and Lemma 2 is that all minimal credulous explanations for a small set M can be computed by exhaustive testing of all subsets $E \subseteq H$ with $|E| \leq |M|$ in polynomial time.

Theorem 10 *Given a PDAP $\mathcal{P} = \langle H, M, W, D\rangle$ where $\langle W, D\rangle$ is Krom-pf-normal and M is small, all minimal credulous explanations for $\mathcal{P}$ can be computed in polynomial time.*

Consequently, also **Relevance** for minimal explanations is polynomial if M is small.

5 Conclusion and further research

We proposed a basic model of abduction from default theories, and analyzed its computational complexity. Moreover, we have shown that credulous abduction from the previously known classes of Literal-Horn and Krompf-normal default theories is tractable.

Besides identifying further tractable and manageable cases of default abduction, the following issues are currently under investigation.

The size of an explanation (cf. [Peng and Reggia, 1990]) or, more general, its cost, given by the sum of the predefined costs of its hypotheses, can be used for further pruning minimal (i.e., nonredundant) explanations. Results for abduction from classical theories [Eiter and Gottlob, 1995] suggest using such explanations, abduction from default theories yields complete problems for the classes Δ_k^P and $\Delta_k^P[O(\log n)]$ of the polynomial hierarchy.

Another issue is default logic with an underlying language richer than a plain propositional one. A generalization of our abduction model to a propositional language over atoms $p(t_1, \ldots, t_n)$ where the t_i are variables or constants, is straightforward; here, an instance of an abduction problem reduces to the propositional abduction problem obtained by replacing formulas with all ground instances. Since the grounded propositional version can be exponentially larger, this leads intuitively to an exponential increase in complexity. Thus, abduction from default theories in this nonground language is expected to be complete for the exponential analogues of Σ_k^P, Π_k^P etc.

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References

[Cadoli and Schaerf, 1993] M. Cadoli and M. Schaerf. A Survey of Complexity Results for Non-monotonic Logics. *Journal of Logic Programming*, 17:127–160, 1993.

[Charniak and McDermott, 1985] E. Charniak and P. Mc Dermott. *Introduction to Artificial Intelligence*. 1985.

[Donini *et al.*, 1995] F.M. Donini, D. Nardi, and R. Rosati. Ground Nonmonotonic Modal Logics for Knowledge Representation. In *Proc. World Congress for AI (WOCFAI-95)*, 1995. Forthcoming.

[Eiter and Gottlob, 1992] Thomas Eiter and Georg Gottlob. Reasoning with Parsimonious and Moderately Grounded Expansions. *Fundamenta Inform.*, 17(1,2):31–53, 1992.

[Eiter and Gottlob, 1995] Thomas Eiter and Georg Gottlob. The Complexity of Logic-Based Abduction. *Journal of the ACM*, January 1995. Abstract in Proc. STACS-93, LNCS 665, pp. 70–79, 1993.

[Eshghi, 1988] K. Eshghi. Abductive Planning with Event Calculus. In *Proc. 5th Int'l Conf. and Symp. on Logic Programming*, pp. 562–579. 1988.

[Gottlob, 1992] Georg Gottlob. Complexity Results for Nonmonotonic Logics. *Journal of Logic and Computation*, 2(3):397–425, June 1992.

[Hobbs *et al.*, 1988] J. R. Hobbs, M. E. Stickel, P. Martin, and D. Edwards. Interpretation as Abduction. In *Proc. 26th Annual Meeting of the Assoc. for Computational Linguistics*, pp. 95–103, Buffalo (NY), 1988.

[Johnson, 1990] D. S. Johnson. A Catalog of Complexity Classes. In *Handbook of Theoret. Comp. Sc. A*, 1990.

[Josephson *et al.*, 1987] J.R. Josephson, B. Chandrasekaran, Jr. J. W. Smith, and M.C. Tanner. A Mechanism for Forming Composite Explanatory Hypotheses. *IEEE Transactions*, TSMC-17:445–454, 1987.

[Kakas and Mancarella, 1990] A.C. Kakas and P. Mancarella. Database Updates Through Abduction. In *Proc. VLDB-90*, pp. 650–661, 1990.

[Kautz and Selman, 1991] H. Kautz and B. Selman. Hard Problems for Simple Default Logics. *Artificial Intelligence*, 49:243–279, 1991.

[Konolige, 1988] K. Konolige. On the Relationship between Default and Autoepistemic Logic. *Artificial Intelligence*, 35:343–382, 1988, + 41:115, 1989/90.

[Konolige, 1992] K. Konolige. Abduction versus closure in causal theories. *Art. Intell.*, 53:255–272, 1992.

[Marek and Truszczyński, 1993] W. Marek and M. Truszczyński. *Nonmonotonic Logics*. Springer, 1993.

[Nebel, 1994] B. Nebel. Artificial Intelligence: A Computational Perspective. November 1994. To appear in "Logic and Computation in AI", G. Brewka ed.

[Ng and Mooney, 1991] H. T. Ng and R. J. Mooney. An efficient first-order Horn clause abduction system based on the ATMS. In *Proc. AAAI-91*, pp. 494–499.

[Peng and Reggia, 1990] Y. Peng and J.A. Reggia. *Abductive Inference Models for Diagnostic Problem Solving*. 1990.

[Poole, 1989] D. Poole. Normality and Faults in Logic Based Diagnosis. In *Proc. IJCAI-89*, pp. 1304–1310.

[Reiter, 1980] R. Reiter. A Logic for Default Reasoning. *Artificial Intelligence*, 13:81–132, 1980.

[Selman and Levesque, 1990] Bart Selman and Hector J. Levesque. Abductive and Default Reasoning: A Computational Core. In *Proc. AAAI-90*, pp. 343–348, July 1990.

[Selman *et al.*, 1992] B. Selman, H. Levesque, and D. Mitchell. A New Method for Solving Hard Satisfiability Problems. In *Proc. AAAI-92*, pp. 440–446.

[Stillman, 1990] J. Stillman. It's Not My Default: The Complexity of Membership Problems in Restricted Propositional Default Logic. In *Proc. AAAI-90*, pp. 571–579, 1990.

[Stillman, 1992] J. Stillman. The Complexity of Propositional Default Logic. In *Proc. AAAI-92*, pp. 794–799.