Q-Resolution with Generalized Axioms

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Introduction (1/4)

Q-Resolution Calculus (QRES): [KBKF95]

- Prenex CNF $\psi = \hat{Q}.\phi$ unsatisfiable iff empty clause derivable from ψ.
- Completeness: resolution on $\exists$ pivots and universal reduction.
- Resolution on $\forall$ pivots (QU-resolution) [VG12], long-distance resolution [ZM02a, BJ12], combinations thereof [BWJ14].
- QRES-based calculi vs. expansion/instantiation [BCJ15, JM15].
- Traditional QRES used for clause learning in QCDCL.

Introduction (1/4)

Problem:

- Problem (cf. previous talk [Jan16]): current implementations of QCDCL do not harness the full power of QRES in clause learning.
- Resolution in QCDCL guided by assignments.
- Prefix ordering restricts assignment generation in QCDCL.
- Axioms of QRES are weak.

Introduction (2/4): QRES Clause Derivations

Definition (Clause Axiom of QRES)

$$\frac{}{C}$$ Given a PCNF $\psi = \hat{Q}.\phi$, $C \in \phi$ and for all $x \in \hat{Q}$: $\{x, \bar{x}\} \not\subseteq C$.

Example

- Traditional Q-resolution [KBKF95].
- Only $\exists$ pivots.
- No tautologies.

$$\dfrac{(x) \qquad \dfrac{\dfrac{(\bar{x} \vee u \vee y) \qquad (\bar{x} \vee u \vee \bar{y})}{(\bar{x} \vee u)}}{(\bar{x})}}{\emptyset}$$

$\psi = \exists x \forall u \exists y \forall v \exists z.$
$(x) \wedge$
$(\bar{x} \vee u \vee y) \wedge$
$(\bar{x} \vee u \vee \bar{y}) \wedge$
$(x \vee \bar{u} \vee \bar{z})$
$(x \vee u \vee \bar{z})$

- QRES variants: different power but same clause axiom.
- Clause axiom derives only input clauses (falsified in QCDCL).

Introduction (2/4): QRES Clause Derivations

Definition (Clause Axiom of QRES)

$$\frac{}{C} \qquad \text{Given a PCNF } \psi = \hat{Q}.\phi,\ C \in \phi \text{ and for all } x \in \hat{Q}\text{: } \{x, \bar{x}\} \not\subseteq C.$$

Example

- Long distance Q-resolution [ZM02a, BJ12].
- Only $\exists$ pivots.
- Tautologies over $\forall$ variables (pivot level!).

$$\dfrac{(\bar{x} \vee u \vee \bar{y}) \qquad (x \vee \bar{u} \vee \bar{z})}{(u \vee \bar{u} \vee \bar{y} \vee \bar{z})}$$

$$\begin{aligned}\psi = \exists x \forall u \exists y \forall v \exists z.\ & (x) \wedge \\ & (\bar{x} \vee u \vee y) \wedge \\ & (\bar{x} \vee u \vee \bar{y}) \wedge \\ & (x \vee \bar{u} \vee \bar{z}) \\ & (x \vee u \vee \bar{z})\end{aligned}$$

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- Clause axiom derives only input clauses (falsified in QCDCL).

Introduction (2/4): QRES Clause Derivations

Definition (Clause Axiom of QRES)

$$\overline{C}$$ Given a PCNF $\psi = \hat{Q}.\phi$, $C \in \phi$ and for all $x \in \hat{Q}$: $\{x, \bar{x}\} \not\subseteq C$.

Example

- QU-resolution [VG12].
- $\forall/\exists$ pivots.
- No tautologies.
- LQU^{+}-Res: $\forall/\exists$ pivots and tautologies [BWJ14].

$$\frac{(x \vee \bar{u} \vee \bar{z}) \quad (x \vee u \vee \bar{z})}{(x \vee \bar{z})}$$

$$\psi = \exists x \forall u \exists y \forall v \exists z. \\ (x) \wedge \\ (\bar{x} \vee u \vee y) \wedge \\ (\bar{x} \vee u \vee \bar{y}) \wedge \\ (x \vee \bar{u} \vee \bar{z}) \\ (x \vee u \vee \bar{z})$$

- QRES variants: different power but same clause axiom.
- Clause axiom derives only input clauses (falsified in QCDCL).

Introduction (3/4)

QRES for Satisfiable QBFs: [GNT06, Let02, ZM02b]

- Operates on cubes (conjunctions of literals).
- Dual to QRES for clauses: cube resolution and existential reduction.
- Prenex CNF $\psi = \hat{Q}.\phi$ satisfiable iff empty cube derivable from ψ.
- Traditional QRES used for cube learning in QCDCL.

Introduction (4/4): QRES Cube Derivations

Definition (Cube Axiom of QRES [GNT06, Let02, ZM02b])

$$\frac{}{C}$$

Given a PCNF $\psi = \hat{Q}.\phi$ and an assignment A with $\{x, \bar{x}\} \not\subseteq A$ and $\psi[A] = \top$, $C = (\bigwedge_{l \in A})$ is a cube.

Example

- Axiom: model generation.
- Cubes at leaves are part of DNF of ϕ.
- Existential reduction.
- Cube resolution.

$$\dfrac{\dfrac{\dfrac{(\bar{x} \wedge u \wedge \bar{y})}{(\bar{x} \wedge u)} \qquad \dfrac{(\bar{x} \wedge \bar{u} \wedge y)}{(\bar{x} \wedge \bar{u})}}{(\bar{x})}}{\emptyset}$$

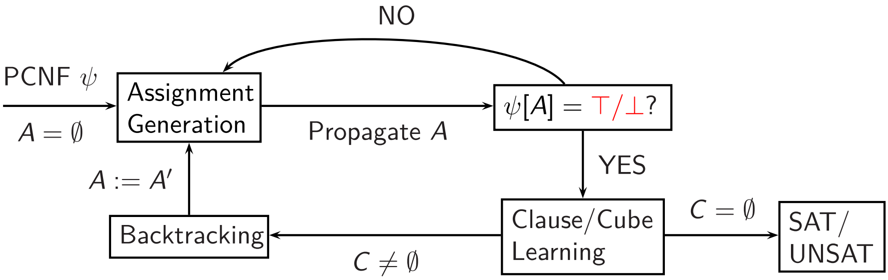
$\psi = \exists x \forall u \exists y.$
$(\bar{x} \vee u \vee \bar{y}) \wedge$
$(\bar{x} \vee \bar{u} \vee y) \wedge$
$(x \vee u \vee y) \wedge$
$(x \vee \bar{u} \vee \bar{y})$

- Cube axiom allows to derive only CNF models of ψ.
- Trivial formulas with exponential cube proofs: [RBM97, Let02]
  $\Psi(n) = \forall u_1 \exists x_1 \ldots \forall u_n \exists x_n. \bigwedge_{i=1}^{n} \left[(u_i \vee \bar{x}_i) \wedge (\bar{u}_i \vee x_i)\right]$

Contributions

- Generalized axioms: stronger clauses/cubes at leaves of derivations.
- Idea: check satisfiability of PCNF ψ under assignment A in QCDCL.
- Integration of arbitrary QBF proof system in QRES via axioms.
- Stronger QRES variants by integrating orthogonal proof systems.
- Tight integration in QCDCL by learning asserting clauses/cubes.
- Implementation in DepQBF, experimental study.
- Formula class CR_n from previous talk [Jan16]: short QRES proofs by QCDCL based on stronger axiom.

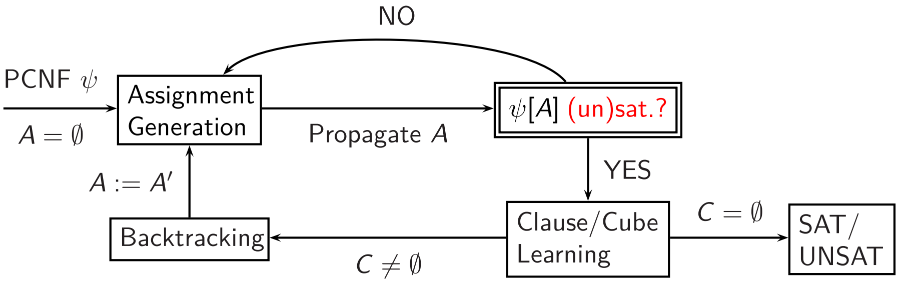
QCDCL (1/2)



Traditional Axioms:

- *QCDCL assignments*: select decision variables from left end of prefix of $\psi[A]$, unit and pure literal detection out of prefix order.
- $\psi[A] = \bot$: CNF ϕ contains a falsified clause.
- $\psi[A] = \top$: all clauses in CNF ϕ satisfied.
- Asserting clause (cube) C: $C[A']$ unit for some $A' \subseteq A$.

QCDCL (2/2)



Generalized Axioms:

- Check satisfiability of $\psi[A]$ in QBCP by incomplete approaches.
- QBF is hard: spend more time on reasoning before assigning decision variables (similar argument as in, e.g., [SB06]).
- $\psi[A]$ (un)sat.: derive asserting clause (cube) C from a start clause (cube) generated based on A.

Generalized Axioms: Theory

Definition (Generalized Clause Axiom)

$$\frac{}{C}$$

Given a PCNF $\psi = \hat{Q}.\phi$ and a QCDCL assignment A, $\psi[A]$ is unsatisfiable, and $C = (\bigvee_{l \in A} \overline{l})$ is a clause.

Definition (Generalized Cube Axiom)

$$\frac{}{C}$$

Given a PCNF $\psi = \hat{Q}.\phi$ and a QCDCL assignment A, $\psi[A]$ is satisfiable, and $C = (\bigwedge_{l \in A} l)$ is a cube.

Proposition (Soundness)

For a clause (cube) C derived by the generalized clause (cube) axiom: $\hat{Q}.\phi \equiv_{sat} \hat{Q}.(\phi \wedge C)$, respectively $\hat{Q}.\phi \equiv_{sat} \hat{Q}.(\phi \vee C)$.

Generalized Axioms: Practice

Axiom Applications in QCDCL:

- *Any* QBF proof system can be used to check satisfiability of $\psi[A]$.
- Combinations of proof systems within QRES via generalized axioms.
- Clauses (cubes) by generalized axioms used as usual in learning.
- Checking satisfiability of PCNF $\psi[A]$ is PSPACE-complete.

Incomplete QBF Satisfiability Checks:

- E.g. bounded variable expansion [Bie04, BB07]: QBF preprocessing.
- E.g. SAT-based techniques in early QDPLL [CGS98]:
  - Trivial truth: check ψ' obtained by discarding all $\forall$ literals in ψ.
  - Trivial falsity: check ψ' obtained by treating every variable in ψ as $\exists$.
- If unsuccessful: extend A to A' by decisions and QBCP, check $\psi[A']$.

Formalizing the Use of SAT Solving

Definition (Abstraction-Based Clause Axiom)

For PCNF $\psi = \hat{Q}.\phi$, $Abs_{\exists}(\psi) := \exists(X_1 \cup \ldots \cup X_n).\phi$.

$$\frac{}{C} \quad \text{For a PCNF } \psi = \hat{Q}.\phi \text{, and a (non-)QCDCL assignment } A\text{, } Abs_{\exists}(\psi)[A] \text{ is unsatisfiable, and } C = (\bigvee_{l \in A} \bar{l}) \text{ is a clause.}$$

Proposition (cf. appendix of [LES16])

QRES with the abstraction-based clause axiom p-simulates QU-resolution.

Example

$$\frac{C' \cup \{p\} \qquad C'' \cup \{\bar{p}\}}{C' \cup C''}$$

Let $q(p) = \forall$ and clause $C = C' \cup C''$ be derived by QU-resolution.

- For $A = \{\bar{l} \mid l \in C\}$, $Abs_{\exists}(\psi)[A]$ unsatisfiable: $(p), (\bar{p}) \in Abs_{\exists}(\psi)[A]$.
- Hence C derivable by QRES with the abstraction-based clause axiom.

Short Proof of CR_n by QCDCL

Definition ([Jan16])

For $i, j \in \{1, \ldots, n\}$,
$CR_n := \exists x_{ij} \forall z \exists a_i, b_i.(x_{ij} \vee z \vee a_i) \wedge (\bar{x}_{ij} \vee \bar{z} \vee b_j) \wedge (\bigvee \bar{a}_i) \wedge (\bigvee \bar{b}_i)$

1. We assume a "perfect" restart and assignment strategy in QCDCL.
2. By QCDCL, derive $(x_{1,j} \vee \cdots \vee x_{n,j})$ and $(x_{i,1} \vee \cdots \vee x_{i,n})$ for all i, j by QCDCL assignments $A := \{\bar{x}_{1,j} \cdots \bar{x}_{n,j}\}$, $A = \{\bar{x}_{i,1} \cdots \bar{x}_{i,n}\}$.
3. By abstraction-based axiom, derive clauses $(x_{i,j} \vee a_i)$ and $(x_{i,j} \vee b_j)$ by *non-QCDCL* assignments $A := \{\bar{x}_{i,j}, \bar{a}_i\}$ and $A := \{\bar{x}_{i,j}, \bar{b}_j\}$, respectively. (The SAT solver needs the clauses derived in step 2).
4. Derive unit clauses (x_{ij}) in QCDCL using QCDCL assignments $A := \{\bar{x}_{ij}\}$. (By clauses from step 3, we get propagations on a_i, b_i).
5. Get $\emptyset$ by unit resolution with (x_{ij}) clauses, resolution on $(\bigvee \bar{b}_i)$.

All resolution derivations above are polynomial, also inside the SAT solver.

Implementing Generalized Axioms in QCDCL

QBF Preprocessing:

- Incomplete solving.
- QBF preprocessors may have considerable solving power [LSVG16].
- Integration of Bloqqer: `http://fmv.jku.at/bloqqer/`.
- Clause and cube learning wrt. SAT/UNSAT result.
- Nonincremental, $\psi[A]$ added to Bloqqer always from scratch.

Implementing Generalized Axioms in QCDCL

SAT Solving:

- Integration of PicoSAT: abstraction-based axiom, trivial truth.
- Incremental solving under QCDCL assignment A (assumptions).
- Failed assumptions $A' \subseteq A$: already $Abs_{\exists}(\psi)[A']$ unsatisfiable.
- Clauses learned based on possible *non-QCDCL* assignments A'.
- Effects of QU-resolution in QCDCL.

Implementing Generalized Axioms in QCDCL

Dynamic Blocked Clause Elimination (QBCE): [LBB+15]

- Apply QBCE incrementally in QBCP by watched data structures.
- Empty formula: cube learning by generalized axiom.

Example ([LBB+15])

$\exists z,z' \forall u \exists y.$
$(u \vee \bar{y}) \wedge (\bar{u} \vee y) \wedge (z \vee u \vee \bar{y}) \wedge (z' \vee \bar{u} \vee y) \wedge (\bar{z} \vee \bar{u} \vee \bar{y}) \wedge (\bar{z}' \vee u \vee y)$

- Initially $A = \emptyset$ and no clause blocked in $\psi[A] = \psi$.
- For $A = \{\bar{z}, \bar{z}'\}$ all clauses blocked in $\psi[A] = \forall u \exists y.(u \vee \bar{y}) \wedge (\bar{u} \vee y)$.
- Derive $C = (\bar{z} \wedge \bar{z}')$ by generalized cube axiom and immediately $\emptyset$.
- Exponential cube proofs with traditional axiom:
  $\Phi(n) = \exists z_1,z_1' \forall u_1 \exists y_1, \ldots, \exists z_n,z_n' \forall u_n \exists y_n. \bigwedge_{i=1}^{n} \left[\mathcal{C}_0(i) \wedge \mathcal{C}_1(i) \wedge \mathcal{C}_2(i)\right]$,
  $\mathcal{C}_0(i) = (u_i \vee \bar{y}_i) \wedge (\bar{u}_i \vee y_i)$,
  $\mathcal{C}_1(i) = (z_i \vee u_i \vee \bar{y}_i) \wedge (z_i' \vee \bar{u}_i \vee y_i)$,
  $\mathcal{C}_2(i) = (\bar{z}_i \vee \bar{u}_i \vee \bar{y}_i) \wedge (\bar{z}_i' \vee u_i \vee y_i)$.

Experiments

Variants of DepQBF:

- DQ: only dynamic QBCE.
- DQ-T: + trivial truth.
- DQ-A: + abs. clause axiom.
- DQ-B: + Bloqqer.
- DQ-BAT: all listed above.

Solver	$\#T$	$\#U$	$\#S$	*Time*
DQ-BAT	466	236	230	553K
DQ-AT	461	234	227	555K
DQ-A	459	237	222	561K
DQ-B	449	222	227	563K
DQ-T	441	220	221	571K
DQ	441	224	217	575K
QELL-nc	434	302	132	563K
RAReQS	414	272	142	611K
CAQE	370	192	178	708K
GhostQ	347	166	181	752K
QESTO	331	188	143	767K

- QBF Gallery 2014 application benchmark set (735 formulas).
- Total solved ($\#T$), solved unsatisfiable ($\#U$), and satisfiable ($\#S$).
- No preprocessing by Bloqqer.

Experiments

Variants of DepQBF:

- DQ: only dynamic QBCE.
- DQ-T: + trivial truth.
- DQ-A: + abs. clause axiom.
- DQ-B: + Bloqqer.
- DQ-BAT: all listed above.

Solver	$\#T$	$\#U$	$\#S$	*Time*
QELL-nc	483	306	177	480K
DQ-AT	483	260	223	509K
DQ-A	481	262	219	528K
DQ-BAT	480	257	223	516K
RAReQS	471	272	199	509K
CAQE	465	248	217	534K
DQ-T	464	243	221	526K
DQ	456	242	214	542K
DQ-B	450	245	205	550K
QESTO	401	212	189	662K
GhostQ	306	148	158	823K

- QBF Gallery 2014 application benchmark set (735 formulas).
- Total solved ($\#T$), solved unsatisfiable ($\#U$), and satisfiable ($\#S$).
- Restricted preprocessing by Bloqqer (only QBCE and $\forall$ expansion).

Experiments

Variants of DepQBF:

- DQ: only dynamic QBCE.
- DQ-T: + trivial truth.
- DQ-A: + abs. clause axiom.
- DQ-B: + Bloqqer.
- DQ-BAT: all listed above.

Solver	$\#T$	$\#U$	$\#S$	*Time*
RAReQS	547	314	233	379K
QELL-nc	501	301	200	445K
QESTO	463	248	215	558K
DQ-AT	434	209	225	579K
DQ-BAT	432	209	223	585K
DQ-T	426	200	226	586K
DQ-A	418	207	211	623K
DQ-B	409	201	208	622K
DQ	407	200	207	623K
CAQE	401	193	208	640K
GhostQ	350	176	174	739K

- QBF Gallery 2014 application benchmark set (735 formulas).
- Total solved ($\#T$), solved unsatisfiable ($\#U$), and satisfiable ($\#S$).
- Full preprocessing by Bloqqer.

Experiments

Solver rankings: no (n), restricted (r), full preprocessing (f)			
Solver	*n*	*r*	*f*
CAQE	9	6	10
DQ	6	8	9
DQ-A	3	3	7
DQ-AT	2	2	4
DQ-B	4	9	8
DQ-BAT	1	4	5
DQ-T	5	7	6
GhostQ	10	11	11
QELL-nc	7	1	2
QESTO	11	10	3
RAReQS	8	5	1

- Three different winning solvers/approaches.
- Preprocessing may be harmful to the performance of certain solvers.

Conclusion and Outlook

Generalized Axioms in QRES:

- Derive clauses (cubes) other than input clauses (CNF models).
- Interface to combining QRES with orthogonal QBF proof systems.
- In QCDCL: incomplete QBF satisfiability check of $\psi[A]$.
- Applicable to any variant of QRES (long-distance, QU-, . . .).
- Proof search more complex due to additional proof rules (CR_n class).

Proof Generation:

- A clause (cube) C obtained by generalized axioms has a proof P (perhaps) in a proof system other than $QRES$.
- P is part of the final proof P' produced by QRES e.g. in QCDCL.
- Checking P' requires to check subproofs P in different proof systems.

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