Expressing Default Abduction Problems as Quantified Boolean Formulas

Hans Tompits
Institut für Informationssysteme 184/3,
Technische Universität Wien,
Favoritenstraße 9–11, A–1040 Vienna, Austria
E-mail: tompits@kr.tuwien.ac.at

Abduction is the process of finding explanations for observed phenomena in accord to known laws about a given application domain. This form of reasoning is an important principle of common-sense reasoning and is particularly relevant in conjunction with nonmonotonic knowledge representation formalisms. In this paper, we deal with a model for abduction in which the domain knowledge is represented in terms of a default theory. We show how the main reasoning tasks associated with this particular form of abduction can be axiomatised within the language of quantified Boolean logic. More specifically, we provide polynomial-time constructible reductions mapping a given abduction problem into a quantified Boolean formula (QBF) such that the satisfying truth assignments to the free variables of the latter determine the solutions of the original problem. Since there are now efficient QBF-solvers available, this reduction technique yields a straightforward method to implement the discussed abduction tasks. We describe a realisation of this approach by appeal to the reasoning system QUIP.

Keywords: Default logic, abduction, compilation methods, problem solving, quantified Boolean logic

1. Introduction

The philosopher C. S. Peirce identified three fundamental modes of reasoning: (i) *deduction*, an analytic process determining necessary consequences from general rules; (ii) *induction*, a synthetic form of reasoning deriving probable generalisations from factual data; and (iii) *abduction*, another form of synthetic inference, yielding relevant premisses from rules and observed consequences. Thus, following Peirce, abduction can be characterised as the "probational adoption of a hypothesis" as explanation for observed facts, according to known laws modelling a specific domain under consideration [29].

Various formalisations of abductive reasoning have been proposed in the literature, differing basically in the way the domain knowledge is represented. Besides abduction by set-covering [30] or probabilistic abduction [28; 30], an important category of abductive reasoning techniques is *logic-based abduction*, in which the application domain is modelled by means of a logical theory. Although classical logic is mainly used for this purpose [32], recent years witnessed an increasing interest in employing *nonmonotonic reasoning formalisms* as underlying language for logic-based abduction. As a case in point, *abductive logic programming* [18; 36; 11; 19] is a well-known realisation of this paradigm, and successful implementations of this particular approach exist [20; 5; 8].

Another important nonmonotonic reasoning formalism relevant for abduction is default logic [33]. A formal model for abduction from default theories has been introduced by Eiter, Gottlob, and Leone [12]. In this approach, a *default abduction problem*, $\mathcal{P}$, consists of a default theory $T = \langle W, \Delta \rangle$ (where W is a set of formulas and Δ is a set of defaults), a set M of observed phenomena, and a set H of hypotheses. An *explanation* for $\mathcal{P}$ is any set $E \subseteq H$ such that M can be inferred from the extended default theory $T_E = \langle W \cup E, \Delta \rangle$, providing T_E has a consistent extension. Inference in the context of default logic is understood either as *brave inference* (checking whether M is contained in some extension of T_E) or *skeptical inference* (checking whether M is contained in all extensions of T_E).

Although a restriced fragment of this approach has been realised by the diagnosis front-end of the DLV system [8], no general solver for default abduction has been put forth so far. In this paper, we take up this challenge and describe a general method to build a prototype reasoning system

AI Communications
ISSN 0921-7126, IOS Press.

for the default abduction model from [12], based on a reduction approach. The central idea is to translate a given abduction task into a quantified Boolean formula (QBF) and then applying some sophisticated QBF-solver to evaluate the translated QBF. The existence of efficient QBF-solvers, like, e.g., the systems developed by Cadoli *et al.* [3], Giunchiglia *et al.* [15], Rintanen [35], Letz [23], or Feldmann *et al.* [14], makes this reduction approach practicably applicable.

Concerning the particular encodings, we provide efficient (polynomial-time constructible) translations of the following reasoning tasks into QBFs:

(i) computing all explanations for a given abduction problem;
(ii) computing all hypotheses which are *relevant* for a given abduction problem, i.e., which contribute to *some* explanation; and
(iii) computing all hypotheses which are *necessary* for a given abduction problem, i.e., which contribute to *all* explanations.

Each of the above tasks is dealt with for brave and skeptical default inference, as well as for arbitrary explanations and for subset-minimal explanations.

From a theoretical point of view, the rationale of the current approach relies on the observation that the evaluation problem of quantified Boolean formulas, QSAT, is PSPACE-complete, so any decision problem in PSPACE can be polynomially reduced to QSAT. In fact, the evaluation problem for QBFs having prenex normal form with $i-1$ quantifier alternations is complete for the i-th level of the polynomial hierarchy. Since the reasoning tasks considered in this paper are located between the second and fourth level of the polynomial hierarchy, efficient translations to QBFs with one, two, or three quantifier alternations must exist.

A similar approach for solving various reasoning tasks belonging to the area of nonmonotonic reasoning has been realised in the system QUIP [6; 13; 4; 2; 27]. This prototype implementation currently handles the computation of the main reasoning tasks for default logic, several modal nonmonotonic logics, consistency-based approaches to belief revision and paraconsistent reasoning, and equilibrium logic, a generalisation of the stable model semantics for logic programs. We obtain a straightforward implementation of the translations for default abduction problems by appeal to the system QUIP.

Reduction methods to QBFs naturally generalise similar approaches for problems in NP; these latter problems can in turn be solved by translating them (in polynomial time) to SAT, the satisfiability problem of classical propositional logic (see e.g., [21] for such an application in Artificial Intelligence). Besides the implementation of different nonmonotonic reasoning tasks as realised by the system QUIP, successful applications based on reductions to QBFs have also been applied to conditional planning [34].

The paper is organised as follows. In the next section, we give some basic notation and recapitulate the relevant aspects of default logic. Section 3 introduces the default abduction framework, and Section 4 lays down the elementary facts about QBFs. Section 5 contains our main results, and Section 6 deals with implementational issues. Section 7 supplies some concluding remarks.

2. Preliminaries

2.1. Basic Notation

We deal with propositional languages and use the primitive sentential connectives $\neg$ and $\wedge$, together with the logical constant $\top$, to construct formulas in the standard way. The operators $\vee$, $\supset$, $\equiv$, as well as the symbol $\bot$, are defined from $\neg$, $\wedge$, and $\top$ as usual. We write $\mathcal{L}_{\mathcal{A}}$ to denote a language over an alphabet $\mathcal{A}$ of *propositional variables* or *atoms*. Formulas are denoted by Greek lower-case letters (possibly with subscripts). Conjunctions of form $\bigwedge_{i\in I}\phi_i$ are assumed to stand for the logical constant $\top$ whenever $I=\emptyset$. A *literal* is either an atom p or a negated atom $\neg p$. The set of all atoms occurring in a formula ϕ is denoted by $Var(\phi)$. Similarly, for a set S of formulas, $Var(S)$ is the set of all atoms occurring in elements of S, i.e., $Var(S)=\bigcup_{\phi\in S} Var(\phi)$. Furthermore, $\neg S$ denotes the set $\{\neg\phi \mid \phi\in S\}$.

The (propositional) derivability operator, $\vdash$, is defined in the usual way, and likewise its semantic counterpart $\models$. The *deductive closure* of a set $S\subseteq\mathcal{L}_{\mathcal{A}}$ of formulas is given by $Cn_{\mathcal{A}}(S)=\{\phi\in\mathcal{L}_{\mathcal{A}} \mid S\vdash\phi\}$. We say that S is *deductively closed* iff $S=Cn_{\mathcal{A}}(S)$. Furthermore, S is *consistent* providing $\bot\notin Cn_{\mathcal{A}}(S)$. If the language is clear from the context, we usually drop the index "$\mathcal{A}$" from $Cn_{\mathcal{A}}(\cdot)$ and simply write $Cn(\cdot)$.

Given an alphabet $\mathcal{A}$, we define a disjoint alphabet $\mathcal{A}'$ as $\mathcal{A}' = \{p' \mid p \in \mathcal{A}\}$. Accordingly, for $\alpha \in \mathcal{L}_{\mathcal{A}}$, we define α' as the result of replacing in α each atom p from $\mathcal{A}$ by the corresponding atom p' in $\mathcal{A}'$ (so implicitly there is an isomorphism between $\mathcal{A}$ and $\mathcal{A}'$). This is defined analogously for sets of formulas.

We assume the reader familiar with the basic concepts of complexity theory (see, e.g., [26] for a comprehensive introduction). Relevant for our purposes are the elements of the polynomial hierarchy, given by the following sequence of classes:

$$\Sigma_0^P = \Pi_0^P = \mathrm{P};$$

and, for $i > 0$,

$$\Sigma_i^P = \mathrm{NP}^{\Sigma_{i-1}^P} \quad \text{and} \quad \Pi_i^P = \text{co-NP}^{\Sigma_{i-1}^P}.$$

Here, P is the class of all problems solvable on a deterministic Turing machine in polynomial time; NP is similarly defined but using a nondeterministic Turing machine as underlying computing model; and, for complexity classes C and A, the notation C^A stands for the *relativized version* of C, consisting of all problems which can be decided by Turing machines of the same sort and time bound as in C, only that the machines have access to an oracle for problems in A. As well, co-C is the class of all problems which are complementary to the problems in C. We note that $\Sigma_1^P = \mathrm{NP}$, $\Sigma_2^P = \mathrm{NP}^{\mathrm{NP}}$, and $\Pi_2^P = \text{co-NP}^{\mathrm{NP}}$.

2.2. Default Logic

In default logic [33], knowledge about the world is represented in terms of *default theories*, which are ordered pairs of form $T = \langle W, \Delta \rangle$, where W is a set propositional formulas, called the *premisses* of T, and Δ is a set of *default rules* (or *defaults*, for short). The set W represents certain (though in general incomplete) information about the given application domain, whilst Δ represents defeasible knowledge, which may be invalidated by new, more accurate information. Formally, a default, δ, is an inference rule of form

$$\frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma},$$

where $\alpha, \beta_1, \ldots, \beta_n$, and γ are formulas. Intuitively, δ expresses that γ is asserted whenever α is believed and $\beta_1, \ldots, \beta_n$ are *consistent with what is believed* (i.e., there is no evidence that one of $\neg\beta_1, \ldots, \neg\beta_n$ holds). We call α the *prerequisite*, the formulas $\beta_1, \ldots, \beta_n$ the *justifications*, and γ the *consequent* of δ. For notational convenience, we also write $p(\delta)$ for the prerequisite α of δ, $j(\delta)$ for the set $\{\beta_1, \ldots, \beta_n\}$, and $c(\delta)$ for the consequent γ of δ. Accordingly, $p(\Delta)$ is the set of prerequisites of all default rules in Δ; $j(\Delta)$ and $c(\Delta)$ are defined analogously. Furthermore, we define $\mathit{Var}(\Delta) = \mathit{Var}(p(\Delta) \cup j(\Delta) \cup c(\Delta))$ and $\mathit{Var}(T) = \mathit{Var}(W) \cup \mathit{Var}(\Delta)$, for $T = \langle W, \Delta \rangle$. As well, we extend our priming notation introduced for (sets of) formulas to defaults and default theories in the obvious way, i.e., δ' denotes the result of replacing each atom p in default δ by p', and likewise for sets of defaults and default theories.

We say that a default theory $T = \langle W, \Delta \rangle$ is *finite* iff both W and Δ are finite.

For simplicity, defaults will also be written in the form $(\alpha : \beta_1, \ldots, \beta_n / \gamma)$.

The semantics of default theories is defined in terms of *extensions*. Formally, a set $S \subseteq \mathcal{L}_{\mathcal{A}}$ is an extension of a default theory $T = \langle W, \Delta \rangle$ iff $S = \bigcup_{i \geq 0} S_i$, where $S_0 = W$ and, for $i \geq 0$,

$$S_{i+1} = \mathit{Cn}(S_i) \cup \{c(\delta) \mid \delta \in \mathit{GD}_i\},$$

with

$$\mathit{GD}_i = \{\delta \in \Delta \mid p(\delta) \in S_i, \neg j(\delta) \cap S = \emptyset\}.$$

In general, a default theory may possess none, one, or several extensions. Intuitively, an extension characterises a possible totality of beliefs to which an agent may refer on the basis of a given default theory.

Setting $\mathit{GD}(\Delta, S) = \bigcup_{i \geq 0} \mathit{GD}_i$, it holds that $S = \mathit{Cn}(W \cup c(\mathit{GD}(\Delta, S)))$, for any extension S of $T = \langle W, \Delta \rangle$. We call the elements of $\mathit{GD}(\Delta, S)$ the *generating defaults* of S.

Following Marek and Truszczyński [24], extensions can alternatively be characterised as follows. For any set Δ of defaults and any $S \subseteq \mathcal{L}_{\mathcal{P}}$, define the *reduct of* Δ *with respect to* S as the set of classical inference rules

$$\Delta_S = \left\{ \frac{p(\delta)}{c(\delta)} \;\middle|\; \delta \in \Delta, \neg j(\delta) \cap S = \emptyset \right\}.$$

Furthermore, for any set F of propositional formulas and any set K of classical inference rules, let $\mathit{Cn}^K(F)$ be the set of all formulas derivable from F using classical logic together with the rules in K. Then, S is an extension of $T = \langle W, \Delta \rangle$ iff $S = \mathit{Cn}^{\Delta_S}(W)$.

There are two basic inference operations in the context of default logic, viz. *brave inference* and *skeptical inference*. To wit, we say that a formula ϕ is a *brave consequence* of a given default theory T, symbolically $T \mid\!\sim_b \phi$, iff ϕ belongs to some extension of T, and ϕ is a *skeptical consequence* of T, symbolically $T \mid\!\sim_s \phi$, iff ϕ belongs to all extensions of T. As shown in [16], both inference tasks are intractable in the general case. More specifically, given a default theory T and a formula ϕ, checking whether $T \mid\!\sim_b \phi$ holds is Σ_2^P-complete, whilst checking whether $T \mid\!\sim_s \phi$ holds is Π_2^P-complete. Analogously, checking whether a given default theory possesses an extension is Σ_2^P-complete as well.

Let S be a set of formulas. We write $T \mid\!\sim_b S$ to indicate that $T \mid\!\sim_b \phi$, for any $\phi \in S$; a similar notation applies for $\mid\!\sim_s$.

3. The Abduction Framework

Following Eiter, Gottlob, and Leone [12], we define a formal model for abduction from propositional default theories as follows.

Definition 1 *A* default abduction problem, *$\mathcal{P}$, is a quadruple $\langle H, M, W, \Delta\rangle$, where H and M are sets of literals, and $\langle W, \Delta\rangle$ is a default theory. We call the elements of H the* hypotheses *or* abducibles, *and the elements of M the* observations *or* manifestations *of $\mathcal{P}$.*

The default abduction problem $\mathcal{P}$ is finite *iff each constituting member H, M, W, and Δ of $\mathcal{P}$ is finite.*

Informally, in a default abduction problem $\mathcal{P} = \langle H, M, W, \Delta\rangle$, H specifies the universe of admissible explanations, the elements of M are the observed phenomena, and $\langle W, \Delta\rangle$ represents the current world knowledge. A set $E \subseteq H$ serves as an explanation for the observed manifestations in M if the adjunction of E to the world knowledge $\langle W, \Delta\rangle$ entails all propositions in M. Given that there are two different inference operations associated with default theories, there are, accordingly, two fundamental kinds of explanations for observed phenomena, as defined next.

Definition 2 *Let $\mathcal{P} = \langle H, M, W, \Delta\rangle$ be a default abduction problem, and let $E \subseteq H$.*

We call E a brave explanation *for $\mathcal{P}$ iff (i) $\langle W \cup E, \Delta\rangle \mid\!\sim_b M$ and (ii) $\langle W \cup E, \Delta\rangle$ has a consistent extension. Similarly, E is a* skeptical explanation *for $\mathcal{P}$ iff (i) $\langle W \cup E, \Delta\rangle \mid\!\sim_s M$ and (ii) $\langle W \cup E, \Delta\rangle$ has a consistent extension.*

The stipulation that $\langle W \cup E, \Delta\rangle$ possesses a consistent extension assures that explanations are consistent with the knowledge represented in the given default theory $\langle W, \Delta\rangle$. This requirement is similar to the usual consistency condition in abduction from theories in classical logic.

We note that brave explanations for a given abduction problem $\mathcal{P} = \langle H, M, W, \Delta\rangle$ can be equivalently characterised as sets $E \subseteq H$ such that $\langle W \cup E, \Delta\rangle$ has a consistent extension containing M. This is a simple consequence of the well-known fact that if a default theory T has some consistent extension, then *all* extensions of T must be consistent.

The assumption that hypotheses and manifestations in a default abduction problem $\mathcal{P} = \langle H, M, W, \Delta\rangle$ are given by literals instead of arbitrary formulas is no restriction since for each non-literal hypothesis or manifestation ϕ a new propositional atom p_ϕ can be introduced, and after adding the formula $p_\phi \equiv \phi$ to W, ϕ can be equivalently replaced by p_ϕ.

We extend our notation $Var(\cdot)$ and our priming convention to default abduction problems in the obvious way, i.e., for $\mathcal{P} = \langle H, M, W, \Delta\rangle$, we define $Var(\mathcal{P}) = Var(H \cup M \cup W) \cup Var(\Delta)$ and $\mathcal{P}' = \langle H', M', W', \Delta'\rangle$.

Following Occam's principle of parsimony, one usually prefers simpler explanations over more complicated ones. In the present context, this idea is realised by accepting only those explanations which are *minimal* among the class of explanations. More formally, we call $E \subseteq H$ a *minimal brave* (resp., *minimal skeptical*) *explanation for $\mathcal{P}$* iff E is a brave (resp., skeptical) explanation for $\mathcal{P}$ and there is no $F \subset E$ such that F is also a brave (resp., skeptical) explanation for $\mathcal{P}$.

Another interesting property of hypotheses is whether they contribute to *some* explanation or to *all* explanations for a given abduction problem. More specifically, we call a set $H_0 \subseteq H$ *relevant for $\mathcal{P} = \langle H, M, W, \Delta\rangle$ under brave* (resp., *skeptical*) *explanations* iff $H_0 \subseteq E$ for some brave (resp., skeptical) explanation E for $\mathcal{P}$. Dually, $H_0 \subseteq H$ is *necessary for $\mathcal{P}$ under brave* (resp., *skeptical*) *ex-*

Table 1
Complexity results of abduction from default theories.

	Abduction Problem $\mathcal{P}$			
	arbitrary explanations		minimal explanations	
Decision Problem	brave	skeptical	brave	skeptical
CONSISTENCY	Σ_2^P	Σ_3^P	Σ_2^P	Σ_3^P
RELEVANCE	Σ_2^P	Σ_3^P	Σ_3^P	Σ_4^P
NECESSITY	Π_2^P	Π_3^P	Π_2^P	Π_3^P

planations iff $H_0 \subseteq E$ for each brave (resp., skeptical) explanation E for $\mathcal{P}$. Both concepts are similarly defined for abduction problems under minimal explanations.

For illustration, let us consider the following simple example.

Example 1 Let $\langle W, \Delta\rangle$ be the default theory consisting of the following items:

$$W = \big\{\big(l \wedge (s \vee (c \wedge r))\big) \supset m,\\ (\neg s \vee \neg c)\big\};$$

$$D = \big\{(w : \neg j/r),\ (w : \neg r/j)\big\}.$$

Furthermore, assume we are interested in finding explanations for the manifestation $M = \{m\}$ using $H = \{l, s, c, w\}$ as the set of hypotheses. Then, by straightforward calculations, it can be shown that the brave explanations for $\mathcal{P} = \langle H, M, W, \Delta\rangle$ are given by $\{l, s\}$, $\{l, s, w\}$, and $\{l, c, w\}$, whilst only $\{l, s\}$ and $\{l, s, w\}$ represent the skeptical explanations for $\mathcal{P}$. The reason for this difference is that m is not contained in all extensions of $\langle W \cup \{l, c, w\}, \Delta\rangle$, which are given by

$$S_1 = Cn(\{l, c, w, r, m\}) \qquad \text{and}$$

$$S_2 = Cn(\{l, c, w, j\}).$$

Observe that $\{l\}$ is the only set necessary for $\mathcal{P}$ under brave explanations, whilst $\{l, s\}$ is necessary for $\mathcal{P}$ under skeptical explanations. Moreover, $\{c\}$ is relevant for $\mathcal{P}$ under brave explanations but not under skeptical ones. Also, $\{l, s\}$ and $\{l, c, w\}$ are the minimal brave explanations for $\mathcal{P}$, whereas $\{l, s\}$ is the single minimal skeptical explanation for $\mathcal{P}$. □

Abstracting from the particular kind of explanation, the main decision problems in the context of abductive reasoning are the following:

CONSISTENCY: Does a given finite default abduction problem possess an explanation?

RELEVANCE: Given some finite default abduction problem $\mathcal{P} = \langle H, M, W, \Delta\rangle$ and a set $H_0 \subseteq H$, is H_0 relevant for $\mathcal{P}$?

NECESSITY: Given some finite default abduction problem $\mathcal{P} = \langle H, M, W, \Delta\rangle$ and a set $H_0 \subseteq H$, is H_0 necessary for $\mathcal{P}$?

Concerning the computational complexity of these tasks, as shown in [12], these problems are located between the second and fourth level of the polynomial hierarchy. Table 1 gives the specific results; each entry C represents completeness of the corresponding problem for the class C.[1] As can be seen from these results, skeptical abduction is always one level harder than brave abduction. This contrasts with complexity results for usual nonmonotonic reasoning formalisms, where skeptical reasoning has complementary complexity than brave reasoning. Also, minimality of explanations is a source of complexity for RELEVANCE but neither for CONSISTENCY nor for NECESSITY. The reason for this phenomenon is the fact that an abduction problem $\mathcal{P}$ has a minimal brave (resp., minimal skeptical) explanation iff it has a brave (resp., skeptical) explanation; likewise, a set is necessary for $\mathcal{P}$ under minimal brave (resp., minimal skeptical) explanations iff it is necessary for $\mathcal{P}$ under brave (resp., skeptical) explanations. Finally, the

[1] Strictly speaking, [12] defines relevant and necessary hypotheses as *single formulas* (actually, literals) whereas we allow here *sets* for this purpose; but this difference is immaterial.

results for RELEVANCE contrast with corresponding results for abduction from theories in classical logic where minimality is not a source of complexity [10].

The main contribution of this paper is to show how each of the above decision problems, as well as each corresponding *search problem*, can be mapped in polynomial time into a quantified Boolean formula such that the models of the latter determine the answers of the former. In fact, the translations are realised in such a way that the structure of the constructed formulas precisely matches the complexity of the encoded reasoning tasks. In the next section, we briefly recapitulate the relevant aspects of quantified Boolean logic.

4. Quantified Boolean Logic

Quantified Boolean logic is an extension of classical propositional logic in which formulas are permitted to contain quantifications over propositional variables. More formally, the language of quantified Boolean logic comprises the symbols of propositional logic, together with unary operators of form $\forall p$ (where p is some atom), called *universal quantifiers*. Similar to first-order logic, $\exists p$ is defined as the operator $\neg\forall p\neg$ and is called an *existential quantifier*. Formulas of this language are referred to as *quantified Boolean formulas* (QBFs) and are denoted by Greek upper-case letters.

Informally, a QBF of form $\forall p\,\exists q\,\Phi$ means that for all truth assignments of p there is a truth assignment of q such that Φ is true. For instance, under this reading, it is easily seen that the QBF $\exists p\,\exists q\,((p \supset q) \wedge \forall r(r \supset q))$ evaluates to true.

The precise semantical meaning of QBFs is defined as follows. First, some ancillary notation. An occurrence of a propositional variable p in a QBF Φ is *free* iff it does not appear in the scope of a quantifier $\mathsf{Q}p$ ($\mathsf{Q} \in \{\forall, \exists\}$), otherwise the occurrence of p is *bound*. If Φ contains no free variable occurrences, then Φ is *closed*, otherwise Φ is *open*. Furthermore, $\Phi[p_1/\phi_1, \ldots, p_n/\phi_n]$ denotes the result of uniformly substituting in Φ each free occurrence of a variable p_i by a formula ϕ_i, for $1 \leq i \leq n$.

By an *interpretation*, I, we understand a set of atoms. Informally, an atom p is true under I iff $p \in I$. In general, the truth value, $\nu_I(\Phi)$, of a QBF Φ under an interpretation I is recursively defined as follows:

1. if $\Phi = \top$, then $\nu_I(\Phi) = 1$;
2. if $\Phi = p$, for some atom p, then $\nu_I(\Phi) = 1$ if $p \in I$, otherwise $\nu_I(\Phi) = 0$;
3. if $\Phi = \neg\Psi$, then $\nu_I(\Phi) = 1$ if $\nu_I(\Psi) = 0$, otherwise $\nu_I(\Phi) = 0$;
4. if $\Phi = (\Phi_1 \wedge \Phi_2)$, then $\nu_I(\Phi) = 1$ if $\nu_I(\Phi_1) = \nu_I(\Phi_2) = 1$, otherwise $\nu_I(\Phi) = 0$;
5. if $\Phi = \forall p\,\Psi$, then $\nu_I(\Phi) = 1$ if $\nu_I(\Psi[p/\top]) = \nu_I(\Psi[p/\bot]) = 1$, otherwise $\nu_I(\Phi) = 0$.

The truth conditions for $\bot$, $\vee$, $\supset$, $\equiv$, and $\exists$ follow from the above in the usual way. Obviously, we have that

$$\nu_I(\forall p\Psi) = \nu_I(\Psi[p/\top] \wedge \Psi[p/\bot]) \quad \text{and}$$
$$\nu_I(\exists p\Psi) = \nu_I(\Psi[p/\top] \vee \Psi[p/\bot]).$$

We say that Φ is *true under I* iff $\nu_I(\Phi) = 1$, otherwise Φ is *false under I*. If $\nu_I(\Phi) = 1$, then I is a *model* of Φ. The set of all models of Φ is denoted by $Mod(\Phi)$. If $Mod(\Phi) \neq \emptyset$, then Φ is *satisfiable*. If Φ is true under every interpretation, then Φ is *valid*. As usual, we write $\models \Phi$ to express that Φ is valid.

It is easily seen that the truth value of a QBF Φ under interpretation I depends only on the free variables in Φ. Hence, without loss of generality, for determining the truth value of QBFs, we may restrict our attention to interpretations which contain only atoms occurring free in the given QBF. In particular, closed QBFs are either true under every interpretation or false under every interpretation, i.e., they are either valid or unsatisfiable. So, for closed QBFs there is no need to refer to particular interpretations. As well, if a closed QBF Φ is valid, we say that Φ *evaluates to true*, and, correspondingly, if Φ is unsatisfiable, we say that Φ *evaluates to false*. Two sets of formulas (i.e., ordinary propositional formulas or QBFs) are *logically equivalent* iff they possess the same models. Thus, formulas Φ and Ψ are logically equivalent iff $\Phi \equiv \Psi$ is valid.

In the sequel, we use the following abbreviations in the context of QBFs: Let $R = \{\phi_1, \ldots, \phi_n\}$ and $S = \{\psi_1, \ldots, \psi_n\}$ be indexed sets of formulas. Then, $R \leq S$ abbreviates $\bigwedge_{i=1}^{n}(\phi_i \supset \psi_i)$, and $R < S$ is a shorthand for $(R \leq S) \wedge \neg(S \leq R)$. Furthermore, for an indexed set $P = \{p_1, \ldots, p_n\}$ of variables and $\mathsf{Q} \in \{\forall, \exists\}$, we let $\mathsf{Q}P\,\Phi$ stand for the formula $\mathsf{Q}p_1\mathsf{Q}p_2 \cdots \mathsf{Q}p_n\,\Phi$. To ease notation, we usually identify a finite set of formulas with the conjunction of its elements. Finally, the logical complexity of a formula Φ, or the *degree* of Φ, symbolically $d(\Phi)$, is the number of occurrences of the primitive operators $\neg$, $\wedge$, and $\forall p$ (where p is some variable) occurring in Φ.

The operators $\leq$ and $<$ are fundamental tools for expressing certain tests on sets of atoms. In particular, the following properties hold: Let $P = \{p_1, \ldots, p_n\}$ be a set of indexed atoms, and let $I_1, I_2 \subseteq P$ be two interpretations. Then,

(i) $I_1 \subseteq I_2$ iff $I_1' \cup I_2$ is a model of $P' \leq P$; and
(ii) $I_1 \subset I_2$ iff $I_1' \cup I_2$ is a model of $P' < P$.

We say that QBF Φ is in *prenex form* if $\Phi = \mathsf{Q}_1 p_1 \ldots \mathsf{Q}_n p_n \phi$, where $\mathsf{Q}_i \in \{\forall, \exists\}$, for $1 \leq i \leq n$, and ϕ is some propositional formula. Using quantifier-shifting rules similar to those of classical first-order logic, any QBF can be effectively transformed into an equivalent QBF in prenex form. In fact, this transformation can be done in polynomial time.

QBFs are closely related to the constituting members of the polynomial hierarchy, as described by the following well-known result:

Proposition 1 ([40]) *Given a propositional formula ϕ whose atoms are partitioned into $i \geq 1$ sets $V_1, \ldots, V_i$, deciding whether $\exists V_1 \forall V_2 \exists V_3 \ldots \mathsf{Q} V_i \phi$ evaluates to true is Σ_i^P-complete, where $\mathsf{Q} = \exists$ if i is odd and $\mathsf{Q} = \forall$ if i is even. Dually, deciding whether $\forall V_1 \exists V_2 \forall V_3 \ldots \bar{\mathsf{Q}} V_i \phi$ evaluates to true is Π_i^P-complete, where $\bar{\mathsf{Q}} = \forall$ if i is odd and $\bar{\mathsf{Q}} = \exists$ if i is even.*

From this proposition it follows that any decision problem in Σ_i^P can be efficiently reduced to the evaluation problem of prenex QBFs of form $\exists V_1 \forall V_2 \exists V_3 \ldots \mathsf{Q} V_i \phi$, with Q as above, and similarly for problems in Π_i^P. In the next section, we describe polynomial-time constructible translations from the main abductive reasoning tasks into QBFs such that the quantifier structure of the resultant formulas precisely matches the inherent complexity of the encoded problem.

5. Translations

In this section, we present efficient reductions of the main reasoning tasks in the context of abduction from default theories into QBFs. These reductions are constructed in such a way that the solutions of the given abduction problems are determined by the models of the corresponding QBFs.

In what follows, we tacitly assume that all considered default abduction problems are finite.

5.1. *Preparatory Characterisations*

We start with some basic results concerning the operator $\leq$. To wit, we first describe a particular substitution theorem involving $\leq$, and afterwards we discuss two basic QBF modules required subsequently.

Proposition 2 *Let $H = \{\phi_1, \ldots, \phi_n\}$ and $G = \{g_1, \ldots, g_n\}$ be indexed sets of formulas and atoms, respectively, such that $G \cap Var(H) = \emptyset$, and let $\Phi_{G \leq H}$ be a QBF containing one or more designated occurrences of $G \leq H$ such that the elements of G occur free in $\Phi_{G \leq H}$ and, moreover, only in the designated subformulas $G \leq H$ of $\Phi_{G \leq H}$. Furthermore, let $I \subseteq Var(\Phi_{G \leq H}) \setminus G$ and $J \subseteq G$, and let $H_J = \bigwedge_{g_i \in J} \phi_i$.*

Then, $\Phi_{G \leq H}$ is true under $I \cup J$ iff Φ_{H_J} is true under I, where Φ_{H_J} comes from $\Phi_{G \leq H}$ by replacing all designated occurrences of $G \leq H$ by H_J.

Proof. For a formula Ψ_α containing designated occurrences of a formula α as consecutive parts, define $d'(\Psi_\alpha)$ like $d(\Psi_\alpha)$, but count the specific occurrences of α as atomic formulas. The proof of the proposition proceeds by induction on $d'(\Phi_{G \leq H})$.

Induction Base. Assume $d'(\Phi_{G \leq H}) = 0$. Then, $\Phi_{G \leq H} = G \leq H$ and $\Phi_{H_J} = H_J$. Since, for each $1 \leq i \leq n$, the truth value of $g_i \supset \phi_i$ under interpretation $I \cup J$ coincides with the truth value of ϕ_i under I if $g_i \in J$, and $g_i \supset \phi_i$ is trivially true under $I \cup J$ if $g_i \notin J$, we have that

$$\nu_{I \cup J}\Big(\bigwedge_{i=1}^{n} (g_i \supset \phi_i)\Big) = \nu_I\Big(\bigwedge_{g_i \in J} \phi_i\Big),$$

which proves that $\Phi_{G \leq H}$ is true under $I \cup J$ iff H_J is true under I.

Induction Step. Assume $d'(\Phi_{G \leq H}) > 0$, and let the statement hold for all formulas $\Psi_{G \leq H}$, containing specified occurrences of $G \leq H$, such that $d'(\Psi_{G \leq H}) < d'(\Phi_{G \leq H})$.

Since $d'(\Phi_{G \leq H}) > 0$, $\Phi_{G \leq H}$ is a compound formula having as its main operator either $\neg$, $\wedge$, or a universal quantifier (recall that $\vee$, $\supset$, $\equiv$, and existential quantifiers are defined operators). We only deal with the case where $\Phi_{G \leq H}$ is a universally quantified formula; the remaining cases follow by similar arguments.

Assume therefore that $\Phi_{G\leq H} = \forall p\, \Psi_{G\leq H}$. Recall that $\nu_{I\cup J}(\forall p\, \Psi_{G\leq H})$ is given by

$$\nu_{I\cup J}(\Psi_{G\leq H}[p/\top] \wedge \Psi_{G\leq H}[p/\bot]).$$

Since both $d'(\Psi_{G\leq H}[p/\top])$ and $d'(\Psi_{G\leq H}[p/\bot])$ are strictly less than $d'(\Phi_{G\leq H})$, by induction hypothesis it follows that

$$\nu_{I\cup J}(\Psi_{G\leq H}[p/\top]) = \nu_I(\Psi_{H_J}[p/\top]) \quad \text{and}$$
$$\nu_{I\cup J}(\Psi_{G\leq H}[p/\bot]) = \nu_I(\Psi_{H_J}[p/\bot]),$$

from which we immediately get, by the semantics of conjunction and universal quantification, that

$$\nu_{I\cup J}(\forall p\, \Psi_{G\leq H}) = \nu_I(\forall p\, \Psi_{H_J}),$$

i.e., $\Phi_{G\leq H}$ is true under $I\cup J$ iff Φ_{H_J} is true under I. ∎

Next, we describe two fundamental QBF modules, capturing the following tasks, respectively:

1. Given finite sets R and S of propositional formulas, compute all $Q\subseteq S$ such that $R\cup Q$ is satisfiable.
2. Given finite sets R and S of propositional formulas, and some propositional formula ψ, compute all $Q\subseteq S$ such that $R\cup Q\models\psi$.

These tasks can be characterised as follows.

Proposition 3 *Let R and $S=\{\phi_1,\ldots,\phi_n\}$ be finite sets of propositional formulas, let ψ be a propositional formula, and let $G=\{g_1,\ldots,g_n\}$ be a set of new variables. Furthermore, consider any $Q\subseteq S$ and any $I\subseteq G$ such that $\phi_i\in Q$ iff $g_i\in I$, for $1\leq i\leq n$.*

Then,

1. *$R\cup Q$ is satisfiable iff I is a model of the QBF*
$$\mathcal{C}[R,S;G] = \exists U\big(R\wedge(G\leq S)\big),$$
where $U=Var(R\cup S)$ and such that $G\cap U=\emptyset$; and
2. *$R\cup Q\models\psi$ iff I is a model of the QBF*
$$\mathcal{D}[R,S,\psi;G] = \forall V\big((R\wedge(G\leq S))\supset\psi\big),$$
where $V=Var(R\cup S\cup\{\psi\})$ and such that $G\cap V=\emptyset$.

Proof. Consider $Q\subseteq S$ and $I\subseteq G$ such that $\phi_i\in Q$ iff $g_i\in I$, for $1\leq i\leq n$. For proving Part 1, according to Proposition 2, we have that I is a model of $\exists U\big(R\wedge(G\leq S)\big)$ iff

$$\exists U\Big(R\wedge\bigwedge_{g_i\in I}\phi_i\Big) \tag{1}$$

is valid. But $Q=\bigwedge_{g_i\in I}\phi_i$, hence, by the semantics of existential quantification, we get that (1) is valid iff $R\cup Q$ is satisfiable. It follows that I is a model of $\mathcal{C}[R,S;G]$ iff $R\cup Q$ is satisfiable.

Part 2 is an immediate consequence of Part 1, by observing that $R\cup Q\models\psi$ iff $(R\cup\{\neg\psi\})\cup Q$ is unsatisfiable, and since

$$\neg\exists V\big((R\wedge\neg\psi)\wedge(G\leq S)\big)$$

is equivalent to

$$\forall V\big((R\wedge(G\leq S))\supset\psi\big).$$ ∎

Note that $\mathcal{C}[R,S;G]$ is an open QBF having G as its set of free variables. These variables facilitate the selection of those elements of S which determine the sets $Q\subseteq S$ such that $R\cup Q$ is satisfiable. In fact, $\mathcal{C}[R,S;G]$ is designed to express *all* potential subsets $Q\subseteq S$ such that $R\cup Q$ is satisfiable. Similar considerations apply to $\mathcal{D}[R,S,\psi;G]$.

5.2. Encodings for Default Logic

In order to express default abduction problems in terms of QBFs, we need suitable QBF encodings capturing the underlying default inference mechanism. To this end, we use results from [6], where QBF reductions for default reasoning are given. More specifically, that paper contains two different kinds of encodings for default logic: one translation is based on the characterisation of extensions following the method of Marek and Truszczyński [24], and the other translation exploits the so-called *full-set* characterisation of extensions, due to Niemelä [25]. Here, we adopt the former method because it is more suitable for our purposes.

The following result contains the required reductions.

Proposition 4 ([6; 39]) *Let $T=\langle W,\Delta\rangle$ be a default theory with $\Delta=\{\delta_1,\ldots,\delta_n\}$, let ϕ be a propositional formula, and let $C=\{c_1,\ldots,c_n\}$, $D=\{d_1,\ldots,d_n\}$, and $D'=\{d'_1,\ldots,d'_n\}$ be sets of pairwise distinct new variables.*

Furthermore, consider the QBFs from Fig. 1 with $V=Var(W\cup c(\Delta))$, $P=Var(\phi)\setminus V$, $U_i=Var(p(\delta_i))\setminus V$, and $Z_{i,j}=Var(\beta_j)\setminus V$, for $\beta_j\in j(\delta_i)$ and $1\leq i\leq n$.

Then, for any $F\subseteq\Delta$ and any $I\subseteq D$ such that $\delta_i\in F$ iff $d_i\in I$, for $1\leq i\leq n$, the following conditions hold:

$$\mathcal{E}[W,\Delta;D] = \exists D'((D \leq D') \wedge \Phi_1 \wedge \Phi_2 \wedge \Phi_3);$$

$$\mathcal{M}_b[W,\Delta,\phi;D] = \mathcal{E}[W,\Delta;D] \wedge \forall V\,\forall P\Big(\big(W \wedge (D \leq c(\Delta))\big) \supset \phi\Big);$$

$$\mathcal{M}_s[W,\Delta,\phi;D] = \mathcal{E}[W,\Delta;D] \wedge \exists V\,\exists P\Big(W \wedge (D \leq c(\Delta)) \wedge \neg\phi\Big);$$

$$\Phi_1 = \bigwedge_{i=1}^{n}\Big[d'_i \equiv \bigwedge_{\beta_j \in j(\delta_i)} \exists V\,\exists Z_{i,j}\Big(W \wedge \big(D \leq c(\Delta)\big) \wedge \beta_j\Big)\Big];$$

$$\Phi_2 = \bigwedge_{i=1}^{n}\Big[(d'_i \wedge \neg d_i) \supset \exists V\,\exists U_i\Big(W \wedge \big(D \leq c(\Delta)\big) \wedge \neg p(\delta_i))\Big)\Big];$$

$$\Phi_3 = \forall C\Big\{(C \leq D') \supset \Big[\forall V\Big(\big(W \wedge (C \leq c(\Delta))\big) \supset (D \leq c(\Delta))\Big) \vee \bigvee_{i=1}^{n}\Big(d'_i \wedge \neg c_i \wedge \forall V\,\forall U_i\Big(\big(W \wedge (C \leq c(\Delta))\big) \supset p(\delta_i)\Big)\Big)\Big]\Big\}.$$

Fig. 1. QBF modules for expressing default reasoning.

1. *$Cn(W \cup c(F))$ is an extension of T iff I is a model of $\mathcal{E}[W,\Delta;D]$, providing $(C \cup D \cup D') \cap Var(T) = \emptyset$;*
2. *$Cn(W \cup c(F))$ is an extension of T containing ϕ iff I is a model of $\mathcal{M}_b[W,\Delta,\phi;D]$, providing $D \cap (Var(T) \cup Var(\phi)) = (C \cup D') \cap Var(T) = \emptyset$; and*
3. *$Cn(W \cup c(F))$ is an extension of T not containing ϕ iff I is a model of $\mathcal{M}_s[W,\Delta,\phi;D]$, assuming the same proviso as in 2.*

Here, the module $\mathcal{E}[W,\Delta;D]$ characterises the extensions of the given default theory $T = \langle W,\Delta\rangle$, and

$$\mathcal{D}[W,c(\Delta),\phi;D] = \forall V\,\forall P\Big(\big(W \wedge (D \leq c(\Delta))\big) \supset \phi\Big)$$

checks whether a given formula ϕ is contained in a selected extension. Note that, dually,

$$\mathcal{C}[W \cup \{\neg\phi\}, c(\Delta);D] = \exists V\,\exists P\Big(W \wedge (D \leq c(\Delta)) \wedge \neg\phi\Big),$$

which is equivalent to $\neg\mathcal{D}[W,c(\Delta),\phi;D]$, checks whether ϕ is *not* contained in the selected extension. The selection of an extension S, in turn, is done by means of the variables in D, which constitute the set of free variables of each of the three main modules $\mathcal{E}[W,\Delta;D]$, $\mathcal{M}_b[W,\Delta,\phi;D]$, and $\mathcal{M}_s[W,\Delta,\phi;D]$. More specifically, the variables in D' take care of selecting the members in the reduct Δ_S, and the variables in D check whether the consequent of a selected rule is contained in $S = Cn^{\Delta_S}(W)$. To that end, the following tests are performed, expressed by the submodules Φ_1, Φ_2, and Φ_3:

- Φ_1 tests whether each default in the guessed set Δ_S is consistent with the guess for the extension S;
- Φ_2 tests whether no applicable default in Δ_S is missing with respect to the guessed assignment for D; and
- Φ_3 utilises the test whether, for each default, it holds that its consequent is actually contained in $Cn^{\Delta_S}(W)$.

The latter formula encodes a result due to Gottlob [17] stating that a formula $\psi \notin Cn^{\Delta_S}(W)$ iff there exists a subset $B \subseteq \{\gamma_i \mid \alpha_i/\gamma_i \in \Delta_S\}$ such that (i) $W \cup B \not\models \psi$, and (ii) for each $\alpha_i/\gamma_i \in \Delta_S$ with $\gamma_i \notin B$, we have that $W \cup B \not\models \alpha_i$.

Encodings for brave and skeptical default inference are obtained from Proposition 4 in the following way.

Corollary 1 *Let T, ϕ, and D be as in Proposition 4.*

Then,

1. *$T \mathrel{|\!\!\sim}_b \phi$ iff $\models \exists D\,\mathcal{M}_b[W,\Delta,\phi;D]$; and*

2. $T \mathrel{\mid\!\sim}_s \phi$ *iff* $\models \neg\exists D\, \mathcal{M}_s[W, \Delta, \phi; D]$.

We note that $\exists D\, \mathcal{M}_b[W, \Delta, \phi; D]$ can be transformed in polynomial time into a QBF of prenex form $\exists X_1 \forall X_2 \psi$, whilst $\neg\exists D\, \mathcal{M}_s[W, \Delta, \phi; D]$ can be transformed, in the same manner, into a QBF of prenex form $\forall Y_1 \exists Y_2 \varphi$. Hence, in view of Proposition 1, these encodings precisely match the inherent computational complexity of brave and skeptical default reasoning, respectively.

5.3. Encodings of the Basic Abduction Tasks

The following definition gives the core modules for our encodings.

Definition 3 *Let $\mathcal{P} = \langle H, M, W, \Delta \rangle$ be a default abduction problem with $\Delta = \{\delta_1, \ldots, \delta_n\}$, and let $D = \{d_1, \ldots, d_n\}$ and $U_H = \{u_h \mid h \in H\}$ be sets of pairwise distinct variables with $(D \cup U_H) \cap Var(\mathcal{P}) = \emptyset$. Furthermore, let $\mathcal{E}[\cdot, \cdot; \cdot]$ and $\mathcal{M}_\sigma[\cdot, \cdot, \cdot; \cdot]$ be the translations from Proposition 4, for $\sigma \in \{b, s\}$, and consider the values $\mathcal{E}[W \wedge (U_H \leq H), \Delta; D]$ and $\mathcal{M}_\sigma[W \wedge (U_H \leq H), \Delta, M; D]$.*

Then,

$$\mathcal{E}^+[W, \Delta; D, U_H]$$

is the result of dropping in $\mathcal{E}[W \wedge (U_H \leq H), \Delta; D]$ all quantifiers which bind variables from U_H. Likewise,

$$\mathcal{M}^+_\sigma[W, \Delta, M; D, U_H]$$

is the result of dropping in formula $\mathcal{M}_\sigma[W \wedge (U_H \leq H), \Delta, M; D]$ all quantifiers which bind variables from U_H.

Observe that the modules $\mathcal{E}^+[W, \Delta; D, U_H]$ and $\mathcal{M}^+_\sigma[W, \Delta, M; D, U_H]$ have $D \cup U_H$ as their sets of free variables. In fact, the variables in U_H will be used for selecting those elements from H which represent a solution to the given abduction problem.

The next results gives the specific encodings for brave and skeptical explanations.

Theorem 1 *Under the circumstances of Definition 3, and supposing further that $V = Var(H \cup W \cup c(\Delta))$, the following conditions hold, for any $E \subseteq H$ and any $I \subseteq U_H$ such that $h \in E$ iff $u_h \in I$:*

1. *E is a brave explanation for $\mathcal{P}$ iff I is a model of the QBF $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$, given by*

$$\exists D \Big(\mathcal{M}^+_b[W, \Delta, M; D, U_H] \wedge \exists V \big(W \wedge (U_H \leq H) \wedge (D \leq c(\Delta)) \big) \Big);$$

and

2. *E is a skeptical explanation for $\mathcal{P}$ iff I is a model of the QBF $\mathcal{T}^s_{cons}[\mathcal{P}; U_H]$, given by*

$$\exists D \Big(\mathcal{E}^+[W, \Delta; D, U_H] \wedge \exists V \big(W \wedge (U_H \leq H) \wedge (D \leq c(\Delta)) \big) \Big) \wedge \neg\exists D\, \mathcal{M}^+_s[W, \Delta, M; D, U_H].$$

Proof. For Part 1, recall that E is a brave explanation for $\mathcal{P}$ iff there is a consistent extension of $\langle W \cup E, \Delta \rangle$ containing M. Consider any $J \subseteq D$ and any $F \subseteq \Delta$ such that $d_i \in J$ iff $\delta_i \in F$, for $1 \leq i \leq n$. We show that $Cn(W \cup E \cup c(F))$ is a consistent extension of $\langle W \cup E, \Delta \rangle$ containing M iff $I \cup J$ is a model of

$$\mathcal{M}^+_b[W, \Delta, M; D, U_H] \wedge \exists V \big(W \wedge (U_H \leq H) \wedge (D \leq c(\Delta)) \big).$$

From this, by the semantics of existential quantification, Part 1 follows.

First of all, according to Part 1 of Proposition 3, we obviously have that $Cn(W \cup E \cup c(F))$ is consistent iff $I \cup J$ is a model of

$$\exists V \big(W \wedge (U_H \leq H) \wedge (D \leq c(\Delta)) \big).$$

It remains to show that $Cn(W \cup E \cup c(F))$ is an extension of $\langle W \cup E, \Delta \rangle$ containing M iff $I \cup J$ is a model of

$$\mathcal{M}^+_b[W, \Delta, M; D, U_H]. \tag{2}$$

This can be seen as follows.

To begin with, observe that, by construction, (2) is a QBF of form $\Phi_{W \wedge (U_H \leq H)}$, resulting from $\mathcal{M}_b[W \wedge (U_H \leq H), \Delta, M; D]$ by dropping all quantifiers which bind variables from U_H. Since E and I are chosen in such a way that $E = \bigwedge_{u_h \in I} h$, Proposition 2 implies that $I \cup J$ is a model of (2) iff J is a model of $\Phi_{W \wedge E}$, where the latter formula is the result of replacing all occurrences of $W \wedge (U_H \leq H)$ in $\Phi_{W \wedge (U_H \leq H)}$ by $W \wedge E$. Now, $\Phi_{W \wedge E}$ is a QBF differing from $\mathcal{M}_b[W \wedge E, \Delta, M; D]$ only by the presence of quantifiers binding variables from $Var(H \setminus E)$. But such quantifiers clearly have no influence on the semantics of the respective formulas, i.e., we have that

$$\nu_J(\Phi_{W \wedge E}) = \nu_J(\mathcal{M}_b[W \wedge E, \Delta, M; D]).$$

Hence, $I \cup J$ is a model of (2) iff J is a model of $\mathcal{M}_b[W \wedge E, \Delta, M; D]$. Consequently, in view of Proposition 4(2), it follows that $I \cup J$ is a model of (2) iff $Cn(W \cup E \cup c(F))$ is an extension of $\langle W \cup E, \Delta\rangle$ containing M. This concludes the proof of Part 1.

Concerning Part 2, consider again $J \subseteq D$ and $F \subseteq \Delta$ as above, satisfying $d_i \in J$ iff $\delta_i \in F$, for $1 \leq i \leq n$. By an analogous argumentation as in Part 1, Proposition 2 implies that $I \cup J$ is a model of $\mathcal{E}^+[W, \Delta; D, U_H]$ iff J is a model of $\mathcal{E}[W \wedge E, \Delta; D]$. Hence, in view of Propositions 3(1) and 4(1), we have that $Cn(W \cup E \cup c(F))$ is a consistent extension of $\langle W \cup E, \Delta\rangle$ iff $I \cup J$ is a model of

$$\Psi = \mathcal{E}^+[W, \Delta; D, U_H] \wedge \exists V\big(W \wedge (U_H \leq H) \wedge (D \leq c(\Delta))\big).$$

Therefore, the following condition holds:

(α) $\langle W \cup E, \Delta\rangle$ has a consistent extension iff $\exists D\, \Psi$ is true under I.

Likewise, using Proposition 2 and Part 2 of Corollary 1, one can show as above that

(β) $\langle W \cup E, \Delta\rangle \ \vdash_s \phi$ just in case that I is a model of

$$\neg\exists D\, \mathcal{M}_s^+[W, \Delta, M; D, U_H].$$

Combining (α) and (β), we obtain that E is a skeptical explanation for $\mathcal{P}$ iff I is a model of $\mathcal{T}^s_{cons}[\mathcal{P}; U_H]$. ■

We note that, besides $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$, another QBF encoding for brave explanations could be obtained by replacing in $\mathcal{T}^s_{cons}[\mathcal{P}; U_H]$ the second conjunct

$$\neg\exists D\, \mathcal{M}_s^+[W, \Delta, M; D, U_H]$$

by

$$\exists D\, \mathcal{M}_b^+[W, \Delta, M; D, U_H].$$

Albeit such a transformation would result in a more uniform axiomatics for the considered kinds of explanations, the present reduction $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ is clearly more concise than that version.

By taking the existential closure of the reductions $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ and $\mathcal{T}^s_{cons}[\mathcal{P}; U_H]$, we immediately obtain encodings for CONSISTENCY under brave and skeptical explanations, respectively.

Corollary 2 *Let $\mathcal{P}$ be a default abduction problem. Then,*

1. *there is at least one brave explanation for $\mathcal{P}$ iff $\models \exists U_H\, \mathcal{T}^b_{cons}[\mathcal{P}; U_H]$; and*
2. *there is at least one skeptical explanation for $\mathcal{P}$ iff $\models \exists U_H\, \mathcal{T}^s_{cons}[\mathcal{P}; U_H]$.*

It is easy to check that $\exists U_H\, \mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ can be transformed in polynomial time into an equivalent QBF of prenex form $\exists X_1 \forall X_2 \phi$, where X_1, X_2 are disjoint sets of variables and ϕ is some propositional formula, matching the Σ^P_2 complexity of the decision problem CONSISTENCY under brave explanations. However, $\exists U_H\, \mathcal{T}^s_{cons}[\mathcal{P}; U_H]$ is equivalent to a QBF of prenex form $\exists Y_1 \forall Y_2 \exists Y_3 \psi$, containing an additional quantifier alternation, stemming from module $\neg\exists D\, \mathcal{M}_s^+[W, \Delta, M; D, U_H]$ and reflecting the Σ^P_3 complexity of CONSISTENCY under skeptical explanations.

Let us illustrate the mechanism of these encodings with our example from Section 3.

Example 2 Consider the default abduction problem $\mathcal{P}$ from Example 1, and suppose we want to compute the brave explanations for $\mathcal{P}$ by means of translation $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$. Recall that $\mathcal{P}$ has three brave explanations, viz. $\{l, s\}$, $\{l, s, w\}$, and $\{l, c, w\}$, so we expect that $\{u_l, u_s\}$, $\{u_l, u_s, u_w\}$, and $\{u_l, u_c, u_w\}$ are the models of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$.

In the present case, $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ is given by

$$\exists d_1 \exists d_2 \Big(\mathcal{E}^+[W, \Delta; D, U_H] \wedge \Psi_1 \wedge \Psi_2\Big),$$

where d_1 corresponds to default $\delta_1 = (w : \neg j / r)$, d_2 corresponds to default $\delta_2 = (w : \neg r / j)$, $D = \{d_1, d_2\}$, $U_H = \{u_l, u_s, u_c, u_w\}$, and Ψ_1 and Ψ_2 are the following QBFs:

$$\Psi_1 = \forall V\Big(\big(\phi_1 \wedge \phi_2 \wedge \phi_3\big) \supset m\Big);$$

$$\Psi_2 = \exists V\big(\phi_1 \wedge \phi_2 \wedge \phi_3\big),$$

with $V = Var(\mathcal{P}) = \{l, s, c, r, m, j, w\}$ and ϕ_i ($i = 1, 2, 3$) given as

$$\phi_1 = \big(\big[l \wedge (s \vee (c \wedge r))\big] \supset m\big) \wedge \big(\neg s \vee \neg c\big);$$

$$\phi_2 = \big(u_l \supset l\big) \wedge \big(u_s \supset s\big) \wedge \big(u_c \supset c\big) \wedge \big(u_w \supset w\big);$$

$$\phi_3 = \big(d_1 \supset r\big) \wedge \big(d_2 \supset j\big).$$

It is easy to check that the models of Ψ_1 are given by

$$Mod(\Psi_1) = \{I \subseteq U_H \cup D \mid u_l, u_s \in I\} \cup \{I \subseteq U_H \cup D \mid u_l, u_c, d_1 \in I\}.$$

In view of the conjunct $\big(\neg s \vee \neg c\big)$ of ϕ_1, only those $I \in Mod(\Psi_1)$ are models of Ψ_2 which satisfy the condition that $u_s \in I$ iff $u_c \notin I$. Hence, $Mod(\Psi_1 \wedge \Psi_2) = S_1 \cup S_2$, where

$$S_1 = \big\{\{u_l, u_s\} \cup I \mid I \subseteq \{u_w, d_1, d_2\}\big\} \quad \text{and}$$

$$S_2 = \big\{\{u_l, u_c, d_1\} \cup I \mid I \subseteq \{u_w, d_2\}\big\}.$$

Therefore, the potential models of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ are given by $\{u_l, u_s\}$, $\{u_l, u_s, u_w\}$, $\{u_l, u_c\}$, and $\{u_l, u_c, u_w\}$. First of all, $\{u_l, u_c\}$ is not a model of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$, because, as easily seen, there is no $J \subseteq D$ such that $\{u_l, u_c\} \cup J$ is a model of

$$\mathcal{E}^+[W, \Delta; D, U_H] \wedge \Psi_1 \wedge \Psi_2. \tag{3}$$

Indeed, both $\{u_l, u_c\}$ and $\{u_l, u_c, d_2\}$ are not models of Ψ_1, and, by the basic mechanism of translation $\mathcal{E}^+[W, \Delta; D, U_H]$, neither $\{u_l, u_c, d_1\}$ nor $\{u_l, u_c, d_1, d_2\}$ are models of the latter QBF.

Now consider $\{u_l, u_s\}$. It is easy to verify that $\{u_l, u_s\}$ is a model of $\mathcal{E}^+[W, \Delta; D, U_H]$ (mainly because, under this interpretation, both d_1 and d_2 are false, so no default is applied), hence there is some $J \subseteq D$ (namely $J = \emptyset$) such that $\{u_l, u_s\} \cup J$ is a model of (3). By the semantics of existential quantification, it follows that $\{u_l, u_s\}$ is a model of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$.

Similarly, $\{u_l, u_s, u_w\}$ is a model of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ because choosing, e.g., $J_0 = \{d_1\}$, it follows that $\{u_l, u_s, u_w\} \cup J_0$ is a model of $\mathcal{E}^+[W, \Delta; D, U_H]$, from which we get that $\{u_l, u_s, u_w\} \cup J_0$ is also a model of (3). Therefore, there is some $J \subseteq D$ such that $\{u_l, u_s, u_w\} \cup J$ is a model of (3).

Finally, $\{u_l, u_c, u_w\}$ is a model of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ because for $J_0 = \{d_1\}$, we have that $\{u_l, u_c, u_w\} \cup J_0$ is a model $\mathcal{E}^+[W, \Delta; D, U_H]$, implying again that there is some $J \subseteq D$ such that $\{u_l, u_c, u_w\} \cup J$ is a model of (3). □

We continue with the translations for expressing relevant and necessary explanations.

Theorem 2 *Let $\mathcal{P} = \langle H, M, W, \Delta\rangle$ be a default abduction problem, let $H_0 \subseteq H$, and let $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ and $\mathcal{T}^s_{cons}[\mathcal{P}; U_H]$ be the encodings from Theorem 1, for $U_H = \{u_h \mid h \in H\}$.*

Consider the QBFs

$$\mathcal{T}^\sigma_{rel}[\mathcal{P}, H_0; U_H] = \Big(\bigwedge_{h \in H_0} u_h\Big) \wedge \mathcal{T}^\sigma_{cons}[\mathcal{P}; U_H]$$

and

$$\mathcal{T}^\sigma_{nec}[\mathcal{P}, H_0; U_H] = \neg\Big(\bigwedge_{h \in H_0} u_h\Big) \wedge \mathcal{T}^\sigma_{cons}[\mathcal{P}; U_H],$$

for $\sigma \in \{b, s\}$.

Then, for any any $E \subseteq H$ and any $I \subseteq U_H$ such that $h \in E$ iff $u_h \in I$, the following conditions hold:

1. *E is a brave (resp., skeptical) explanation for $\mathcal{P}$ containing H_0 iff I is a model of $\mathcal{T}^b_{rel}[\mathcal{P}, H_0; U_H]$ (resp., $\mathcal{T}^s_{rel}[\mathcal{P}, H_0; U_H]$); and*
2. *E is a brave (resp., skeptical) explanation for $\mathcal{P}$ not containing H_0 iff I is a model of $\mathcal{T}^b_{nec}[\mathcal{P}, H_0; U_H]$ (resp., $\mathcal{T}^s_{nec}[\mathcal{P}, H_0; U_H]$).*

Proof. Consider sets $E \subseteq H$ and $I \subseteq U_H$ such that $h \in E$ iff $u_h \in I$. According to Theorem 1, we have that

(α) E is a brave (resp., skeptical) explanation for $\mathcal{P}$ iff I is a model of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ (resp., $\mathcal{T}^s_{cons}[\mathcal{P}; U_H]$).

Furthermore, by construction of E and I, the following condition holds:

(β) $H_0 \subseteq E$ iff $\bigwedge_{h \in H_0} u_h$ is true under I.

Obviously, Part 1 is an immediate consequence of (α) and (β). But Part 2 follows from these two conditions as well, since (β) is clearly equivalent to

(γ) $H_0 \not\subseteq E$ iff $\neg\Big(\bigwedge_{h \in H_0} u_h\Big)$ is true under I. ■

Similar to Corollary 2, taking the existential closure of the above translations yields encodings for the decision problems RELEVANCE and NECESSITY.

Corollary 3 *Let $\mathcal{P} = \langle H, M, W, \Delta\rangle$ be a default abduction problem and let $H_0 \subseteq H$. Then,*

1. *H_0 is relevant for $\mathcal{P}$ under brave explanations iff*

   $$\models \exists U_H\, \mathcal{T}^b_{rel}[\mathcal{P}, H_0; U_H];$$

   likewise, H_0 is relevant for $\mathcal{P}$ under skeptical explanations iff

   $$\models \exists U_H\, \mathcal{T}^s_{rel}[\mathcal{P}, H_0; U_H];$$

   and

2. *H_0 is necessary for $\mathcal{P}$ under brave explanations iff*

$$\models \neg\exists U_H\, \mathcal{T}^b_{nec}[\mathcal{P}, H_0; U_H];$$

likewise, H_0 is necessary for $\mathcal{P}$ under skeptical explanations iff

$$\models \neg\exists U_H\, \mathcal{T}^s_{nec}[\mathcal{P}, H_0; U_H].$$

Since $\mathcal{T}^\sigma_{rel}[\mathcal{P}, H_0; U_H]$ possesses the same quantifier structure than $\mathcal{T}^\sigma_{cons}[\mathcal{P}; U_H]$ (for $\sigma \in \{b, s\}$), we have that $\exists U_H\, \mathcal{T}^b_{rel}[\mathcal{P}, H_0; U_H]$ is equivalent to a QBF of prenex form $\exists X_1 \forall X_2 \phi$, whilst encoding $\exists U_H\, \mathcal{T}^s_{rel}[\mathcal{P}, H_0; U_H]$ is equivalent to a QBF of form $\exists Y_1 \forall Y_2 \exists Y_3 \psi$, corresponding to the Σ^P_2 and Σ^P_3 complexity of RELEVANCE under brave and skeptical explanations, respectively. Accordingly, $\neg\exists U_H\, \mathcal{T}^b_{nec}[\mathcal{P}, H_0; U_H]$ and $\neg\exists U_H\, \mathcal{T}^s_{nec}[\mathcal{P}, H_0; U_H]$ are equivalent to QBFs of form $\forall Z_1 \exists Z_2 \varphi$ and $\forall Q_1 \exists Q_2 \forall Q_3 \xi$, respectively, reflecting the Π^P_2 and Π^P_3 complexity of NECESSITY under brave and skeptical explanations.

Example 3 Continuing Examples 1 and 2, suppose we want to check whether c is relevant for $\mathcal{P}$ under brave explanations. In view of Corollary 3, we thus have to check whether $\exists U_H\, \mathcal{T}^b_{rel}[\mathcal{P}, \{u_c\}; U_H]$ evaluates to true, which is given by

$$\exists U_H \big(u_c \wedge \mathcal{T}^b_{cons}[\mathcal{P}; U_H]\big), \tag{4}$$

for $U_H = \{u_l, u_s, u_c, u_w\}$ and $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ as in Example 2. We already know that the models of $\mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ are given by $\{u_l, u_s\}$, $\{u_l, u_s, u_w\}$, and $\{u_l, u_c, u_w\}$. Due to the last of these models, we have that there is an interpretation $I \subseteq U_H$ such that $u_c \wedge \mathcal{T}^b_{cons}[\mathcal{P}; U_H]$ is true under I. Hence, by the semantics of $\exists U_H$, (4) evaluates to true. $\Box$

Based on results shown in [11], there are alternative methods to decide relevance and necessity of hypotheses. First of all, for a given abduction problem $\mathcal{P} = \langle H, M, W, \Delta\rangle$, it holds that $H_0 \subseteq H$ is relevant for $\mathcal{P}$ under brave (resp., skeptical) explanations iff $\mathcal{P}_{rel}(H_0) = \langle H \setminus H_0, M, W \cup H_0, \Delta\rangle$ has a brave (resp., skeptical) explanation, and H_0 is necessary for $\mathcal{P}$ under brave (resp., skeptical) explanations iff $\mathcal{P}_{nec}(H_0) = \langle H \setminus H_0, M, W, \Delta\rangle$ has no brave (resp., skeptical) explanation. Hence, $\exists U_H\, \mathcal{T}^\sigma_{cons}[\mathcal{P}_{rel}(H_0); U_H]$ and $\neg\exists U_H\, \mathcal{T}^\sigma_{cons}[\mathcal{P}_{nec}(H_0); U_H]$ check relevance and necessity of H_0, respectively (for $\sigma \in \{b, s\}$).

Also, checking relevance and necessity for brave explanations can be efficiently reduced to default reasoning, by means of the following construction [11]: given $\mathcal{P} = \langle H, M, W, \Delta\rangle$, define

$$W_\mathcal{P} = W \cup \{a_h \supset h \mid h \in H\}; \quad \text{and}$$

$$\Delta_\mathcal{P} = \Delta \cup \left\{ \tfrac{:\neg m}{\bot} \mid m \in M \right\} \cup \left\{ \tfrac{:a_h}{a_h}, \tfrac{:\neg a_h}{\neg a_h} \mid h \in H \right\},$$

where, for each $h \in H$, a_h is a new atom. Then, $H_0 \subseteq H$ is relevant for $\mathcal{P}$ under brave explanations iff there is a consistent extension of $\langle W_\mathcal{P}, \Delta_\mathcal{P}\rangle$ containing $\bigwedge_{h \in H_0} a_h$, and H_0 is necessary for $\mathcal{P}$ under brave explanations iff $\bigwedge_{h \in H_0} a_h$ belongs to all extensions of $\langle W_\mathcal{P}, \Delta_\mathcal{P}\rangle$. Therefore, relevance of H_0 can be checked by QBF

$$\exists D \Big(\mathcal{M}_b[W_\mathcal{P}, \Delta_P, \textstyle\bigwedge_{h \in H_0} a_h; D] \wedge \exists V \left(W_\mathcal{P} \wedge (D \leq c(\Delta_\mathcal{P}))\right)\Big),$$

and necessity of H_0 can be checked by

$$\neg\exists D\, \mathcal{M}_s[W_\mathcal{P}, \Delta_P, \textstyle\bigwedge_{h \in H_0} a_h; D],$$

where D is a set of new variables corresponding to the defaults in $\Delta_\mathcal{P}$ and $V = Var(W_\mathcal{P} \cup c(\Delta_\mathcal{P}))$. However, the above two QBFs have in general a higher logical complexity (but of course the same quantifier structure) than the encodings $\exists U_H \mathcal{T}^b_{rel}[\mathcal{P}, H_0; U_H]$ and $\neg\exists U_H \mathcal{T}^b_{nec}[\mathcal{P}, H_0; U_H]$, as the module $\mathcal{M}_b[\cdot, \cdot, \cdot; \cdot]$ grows quadratically in the number of defaults but only linearly in the number of premisses of the associated default theory of a given default abduction problem—and the definition of the module $\mathcal{M}^+_b[\cdot, \cdot, \cdot; \cdot]$, required for all abduction encodings, involves only a modification of the premisses of the associated default theory, but no new defaults are introduced.

5.4. Minimal Explanations

Now we turn our attention to abductive reasoning under minimal explanations.

Theorem 3 *Let $\mathcal{P} = \langle H, M, W, \Delta\rangle$ be a default abduction problem, let $H_0 \subseteq H$, and let $\mathcal{T}^\sigma_{cons}[\mathcal{P}; U_H]$, $\mathcal{T}^\sigma_{rel}[\mathcal{P}, H_0; U_H]$, and $\mathcal{T}^\sigma_{nec}[\mathcal{P}, H_0; U_H]$ be the encodings from Theorems 1 and 2, respectively, for $\sigma \in \{b, s\}$.*

Furthermore, let $\mathcal{M}_{min}[\mathcal{P}; U_H]$ be the QBF

$$\neg\exists U'_H \Big((U'_H < U_H) \wedge \mathcal{T}^\sigma_{cons}[\mathcal{P}'; U'_H]\Big),$$

where $\mathcal{P}' = \langle H', M', W', \Delta'\rangle$, and consider

$$\mathcal{T}^{\sigma,min}_{cons}[\mathcal{P}; U_H] = \mathcal{T}^{\sigma}_{cons}[\mathcal{P}; U_H] \wedge \mathcal{M}_{min}[\mathcal{P}; U_H],$$

as well as

$$\mathcal{T}^{\sigma,min}_{\mu}[\mathcal{P}, H_0; U_H] = \mathcal{T}^{\sigma}_{\mu}[\mathcal{P}, H_0; U_H] \wedge \mathcal{M}_{min}[\mathcal{P}; U_H],$$

for $\mu \in \{rel, nec\}$.

Then, for any any $E \subseteq H$ *and any* $I \subseteq U_H$ *such that* $h \in E$ *iff* $u_h \in I$, *the following conditions hold:*

1. *E is a minimal brave (resp., minimal skeptical) explanation for $\mathcal{P}$ iff I is a model of $\mathcal{T}^{b,min}_{cons}[\mathcal{P}; U_H]$ (resp., $\mathcal{T}^{s,min}_{cons}[\mathcal{P}; U_H]$);*
2. *E is a minimal brave (resp., minimal skeptical) explanation for $\mathcal{P}$ such that $H_0 \subseteq E$ iff I is a model of $\mathcal{T}^{b,min}_{rel}[\mathcal{P}, H_0; U_H]$ (resp., $\mathcal{T}^{s,min}_{rel}[\mathcal{P}, H_0; U_H]$); and*
3. *E is a minimal brave (resp., minimal skeptical) explanation for $\mathcal{P}$ such that $H_0 \not\subseteq E$ iff I is a model of $\mathcal{T}^{b,min}_{nec}[\mathcal{P}, H_0; U_H]$ (resp., $\mathcal{T}^{s,min}_{nec}[\mathcal{P}, H_0; U_H]$).*

Proof. We only show Part 2; the remaining cases follow in essentially the same way.

Consider sets $E \subseteq H$ and $I \subseteq U_H$ such that $h \in E$ iff $u_h \in I$. By definition, E is a minimal brave (resp., minimal skeptical) explanation for $\mathcal{P}$ containing H_0 iff

(α) E is a brave (resp., skeptical) explanation for $\mathcal{P}$ containing H_0; and

(β) there is no $F \subset E$ such that F is a brave (resp., skeptical) explanation for $\mathcal{P}$.

According to Theorem 2(1), (α) is equivalent to the condition that I is a model of $\mathcal{T}^{b}_{rel}[\mathcal{P}, H_0; U_H]$ (resp., $\mathcal{T}^{s}_{rel}[\mathcal{P}, H_0; U_H]$). It remains to show that Condition (β) holds iff I is a model of

$$\mathcal{M}_{min}[\mathcal{P}; U_H] = \neg \exists U'_H \Big((U'_H < U_H) \wedge \mathcal{T}^{\sigma}_{cons}[\mathcal{P}'; U'_H] \Big).$$

This can be seen as follows.

Consider any $F \subseteq H$ and any $J \subseteq U_H$ such that $h \in F$ iff $u_h \in J$. Obviously, we have that $F \subset E$ iff $J \subset I$. Furthermore, by Theorem 1, F is a brave (resp., skeptical) explanation for $\mathcal{P}$ iff J is a model of $\mathcal{T}^{b}_{cons}[\mathcal{P}; U_H]$ (resp., $\mathcal{T}^{s}_{cons}[\mathcal{P}; U_H]$). By a simple renaming, we have that J is a model of $\mathcal{T}^{\sigma}_{cons}[\mathcal{P}; U_H]$ iff J' is a model of $\mathcal{T}^{\sigma}_{cons}[\mathcal{P}'; U'_H]$ (for $\sigma \in \{b, s\}$). Since the latter formula has no free occurrences of elements from U_H, and since $U_H \cap U'_H = \emptyset$, we thus obtain that

(γ) F is a brave (resp., skeptical) explanation for $\mathcal{P}$ iff $I \cup J'$ is a model of $\mathcal{T}^{b}_{cons}[\mathcal{P}'; U'_H]$ (resp., $\mathcal{T}^{s}_{cons}[\mathcal{P}'; U'_H]$).

Observing that $J \subset I$ iff $I \cup J'$ is a model of $(U'_H < U_H)$, it follows that

(δ) $F \subset E$ iff $I \cup J'$ is a model of $(U'_H < U_H)$.

Combining (γ) and (δ), by construction of F and J, and by the semantics of existential quantification, we obtain that (β) holds precisely in case when I is a model of $\mathcal{M}_{min}[\mathcal{P}; U_H]$. ■

In passing, we note that the minimisation principle expressed by the QBF

$$\neg \exists U'_H \Big((U'_H < U_H) \wedge \mathcal{T}^{\sigma}_{cons}[\mathcal{P}'; U'_H] \Big)$$

is precisely the same as that used in the propositional circumscription of a set of atoms in a propositional theory. More specifically, given disjoint sets P and S of variables and a propositional formula $\phi(P, S)$ possibly containing elements from P and S, the *circumscription of P in $\phi(P, S)$ with varying S* is the QBF $CIRC(\phi; P, S)$, given by

$$\phi(P, S) \wedge \neg \exists P' \exists S' \Big((P' < P) \wedge \phi(P', S') \Big),$$

where $\phi(P', S')$ results from $\phi(P, S)$ by replacing uniformly all occurrences of atoms in P and S by their primed counterparts. The salient difference, however, between the above QBF and the encodings in Theorem 3 is that in the latter formulas the occurrences of $\mathcal{T}^{\sigma}_{cons}[\mathcal{P}; U_H]$, $\mathcal{T}^{\sigma}_{cons}[\mathcal{P}'; U'_H]$, and $\mathcal{T}^{\sigma}_{\mu}[\mathcal{P}, H_0; U_H]$ ($\mu \in \{rel, nec\}$) represent *arbitrary QBFs*, whilst the corresponding occurrences of $\phi(P, S)$ and $\phi(P', S')$ in $CIRC(\phi; P, S)$ are ordinary propositional formulas. From a complexity-theoretical point of view, this difference can be seen as the primary source of the additional complexity of the current abduction problems compared to reasoning from circumscriptive propositional theories (cf. [9] for results about the computational complexity of propositional circumscription).

Concerning the encodings for the main *decision problems* for abduction under minimal explanations, in principle it is possible to obtain such characterisations in the same manner as done in Corollaries 2 and 3, i.e., by taking the existential closure of the encodings of the respective search problems. However, it turns out that in the present case the consistency and necessity problem can be de-

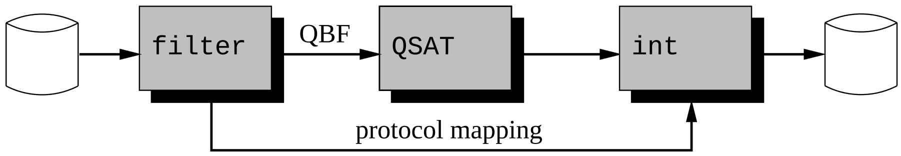

Fig. 2. Architecture to use different QBF-solvers.

scribed by somewhat simpler encodings. To wit, deciding the existence of a minimal brave (resp., minimal skeptical) explanation for a given abduction problem $\mathcal{P} = \langle H, M, W, \Delta \rangle$ is obviously equivalent to checking whether $\mathcal{P}$ has some brave (resp., skeptical) explanation at all. Similarly, checking whether some set $H_0 \subseteq H$ is necessary for $\mathcal{P}$ under minimal brave (resp., minimal skeptical) explanations is equivalent to checking whether H_0 is necessary for $\mathcal{P}$ under brave (resp., skeptical) explanations (cf. Proposition 4.1 in [12]). Hence, we have the following result:

Theorem 4 *Let $\mathcal{P} = \langle H, M, W, \Delta \rangle$ be a default abduction problem and let $H_0 \subseteq H$. Then,*

1. *$\mathcal{P}$ has some minimal brave (resp., skeptical) explanation iff $\models \exists U_H\, \mathcal{T}^{b}_{cons}[\mathcal{P}; U_H]$ (resp., $\models \exists U_H\, \mathcal{T}^{s}_{cons}[\mathcal{P}; U_H]$);*
2. *the set of hypotheses H_0 is relevant for $\mathcal{P}$ under minimal brave (resp., minimal skeptical) explanations iff $\models \exists U_H\, \mathcal{T}^{b,min}_{rel}[\mathcal{P}, H_0; U_H]$ (resp., $\models \exists U_H\, \mathcal{T}^{s,min}_{rel}[\mathcal{P}, H_0; U_H]$); and*
3. *H_0 is necessary for default abduction problem $\mathcal{P}$ under brave (resp., skeptical) explanations iff $\models \neg\exists U_H\, \mathcal{T}^{b}_{nec}[\mathcal{P}, H_0; U_H]$ (resp., $\models \neg\exists U_H\, \mathcal{T}^{s}_{nec}[\mathcal{P}, H_0; U_H]$).*

Concerning the quantifier order of the encodings in Part 2 of the above theorem, the presence of the minimisation module $\mathcal{M}_{min}[\mathcal{P}; U_H]$ in $\exists U_H\, \mathcal{T}^{\sigma,min}_{rel}[\mathcal{P}, H_0; U_H]$ (for $\sigma \in \{b, s\}$) yields an additional quantifier alternation compared to $\exists U_H \mathcal{T}^{\sigma}_{rel}[\mathcal{P}, H_0; U_H]$, reflecting the higher complexity of checking relevance under minimal explanations.

6. Implementation

Our methodology for expressing default abduction tasks in terms of quantified Boolean formulas is motivated by the availability of several practicably efficient QBF-solvers. Among the different tools, there is a propositional theorem prover, `boole`,[2] based on *binary decision diagrams*, a system using a generalised resolution principle [22], several provers implementing an extended Davis-Putnam procedure [3; 14; 15; 23; 35], as well as a distributed algorithm running on a PC-cluster [14]. With the exception of `boole`, these tools do not accept arbitrary QBFs, but require the input formula to be in *prenex conjunctive normal form*.[3] To avoid an exponential increase of formula size, *structure-preserving normal-form translations* [38; 31] can be used to translate a general QBF into the required normal form. In contrast to the usual normal-form translations based on distributivity laws, structure-preserving normal-form translations introduce new labels for subformula occurrences and are polynomial in the length of the input formula.

The translations discussed in the previous section can be easily incorporated as a special module of the reasoning system QUIP [6; 13; 4; 2; 27], which is a prototype tool for solving several nonmonotonic reasoning tasks based on reductions to QBFs.

The general architecture of QUIP is depicted in Fig. 2. QUIP consists of three parts, namely the `filter` program, a QBF-evaluator, and the interpreter `int`. The input filter translates the given problem description (in our case, a default abduction problem and a specified reasoning task) into the corresponding quantified Boolean formula, which is then fed into the QBF-evaluator. The current version of QUIP provides interfaces to most of the sequential QBF-solvers mentioned above. For the solvers requiring prenex conjunctive normal form, the QBFs are translated into structure-preserving normal form. The result of the QBF-evaluator is interpreted by `int`. Depending on the

[2] The system can be downloaded from the Web at URL http://www.cs.cmu.edu/∼modelcheck/bdd.html.

[3] A QBF is in prenex conjunctive normal form iff it is in prenex form and its quantifier-free part is a propositional formula in conjunctive normal form.

capabilities of the employed QBF-evaluator, `int` provides an explanation in terms of the underlying problem instance (e.g., listing all explanations for a given abduction problem). This task relies on a protocol mapping of internal variables of the generated QBF into concepts of the problem description which is provided by `filter`.

7. Conclusion and Discussion

In this paper, we have shown how the main reasoning tasks associated with the default abduction model from [12] can be axiomatised by means of quantified Boolean formulas. The general mechanism of our approach is to translate (in polynomial time) a reasoning problem into the evaluation problem for QBFs such that the satisfying truth-assignments of the latter determine the solutions of the original problem. Thus, in effect, we reduced abduction to satisfiability.

The present framework is a natural generalisation of a similar method successfully applied to problems in NP. In general, the use of QBFs for knowledge-representation purposes has been advocated in the literature [3; 35], and, besides the current framework, reductions of other reasoning tasks to QBFs have been discussed in [34; 6; 13; 4; 2; 27].

Our approach has several benefits. First, by employing off-the-shelf QBF-solvers, we easily obtain a prototype reasoning system for the considered abduction tasks. We discussed an architecture of such an implementation by appeal to the system QUIP. Second, the given axiomatics provides us with further insight into the mechanism of abduction from default theories, as we obtain an object-level description of this kind of reasoning without the need of meta-logical constraints like certain fixed-point conditions. Third, the approach is flexible and easily extensible. A change or refinement of a specific reasoning task is reflected by replacing or adding a corresponding QBF module within the overall translation schema. For instance, a simple modification of our current translations would suffice to allow for general *default rules* as abducibles instead of literals only. Also, other forms of logic-based abduction can be expressed through QBFs, by utilising suitable QBF modules for the underlying inference mechanism. For instance, a QBF encoding for abductive logic programming has been described in [7].

The current translations leave room for several optimisations. On the one hand, simpler encodings for syntactically restricted classes of default theories can be found. For instance, [1] describes efficient mappings from certain classes of default theories into formulas of propositional logic such that each model of the latter corresponds to an extension of the former. Thus, adaptions of these translations yield more direct QBF encodings for default abduction problems whose associated default theories belong to the respective syntax classes. On the other hand, as regards the actual implementation of the considered tasks, the modular architecture of our approach enables a straightforward *parallelisation* of the overall evaluation process. Indeed, this can be achieved either by using different provers in parallel or by applying the distributed QBF-solver `PQsolve` [14]. We note that, in general, it is a non-trivial task to design an efficient distributed variant of a given special-purpose algorithm.

Acknowledgements

This work was partially supported by the Austrian Science Fund under grant P15068. The author would like to thank Stefan Woltran for his useful comments on an earlier version of this paper.

References

[1] R. Ben-Eliyahu and R. Dechter. Default Reasoning Using Classical Logic. *Artificial Intelligence*, 84(1–2):113–150, 1996.

[2] P. Besnard, T. Schaub, H. Tompits, and S. Woltran. Paraconsistent Reasoning via Quantified Boolean Formulas, I: Axiomatising Signed Systems. In *Proceedings of the Eighth European Conference on Logics in Artificial Intelligence (JELIA 2002)*, volume 2424 of *Lecture Notes in Artificial Intelligence*, pages 320–331. Springer-Verlag, 2002.

[3] M. Cadoli, A. Giovanardi, and M. Schaerf. An Algorithm to Evaluate Quantified Boolean Formulae. In *Proceedings of the Fifteenth National Conference on Artificial Intelligence (AAAI-98) and the Tenth Conference on Innovative Applications of Artificial Intelligence (IAAI-98)*, pages 262–267, 1998.

[4] J. Delgrande, T. Schaub, H. Tompits, and S. Woltran. On Computing Solutions to Belief Change Scenarios. In *Proceedings of the Sixth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty* (*ECSQARU-2001*), volume 2143 of *Lecture Notes in Artificial Intelligence*, pages 510–521. Springer-Verlag, 2001.

[5] M. Denecker, A. Kakas, and B. Van Nuffelen. A-System: Problem Solving through Abduction. In *Proceedings of the Seventeenth International Conference on Artificial Intelligence* (*IJCAI-01*), pages 591–596. Morgan Kaufmann Publishers, Inc., 2001.

[6] U. Egly, T. Eiter, H. Tompits, and S. Woltran. Solving Advanced Reasoning Tasks Using Quantified Boolean Formulas. In *Proceedings of the Seventeenth National Conference on Artificial Intelligence* (*AAAI-2000*) *and the Twelfth Innovative Applications of Artificial Intelligence Conference* (*IAAI-2000*), pages 417–422. AAAI Press/The MIT Press, 2000.

[7] U. Egly, V. Klotz, H. Tompits, and S. Woltran. A Toolbox for Abduction: Preliminary Report. In *Proceedings of the IJCAR Workshop on Theory and Applications of Quantified Boolean Formulas*, pages 29–39, 2001.

[8] T. Eiter, W. Faber, N. Leone, and G. Pfeifer. The Diagnosis Frontend of the DLV System. *AI Communications – The European Journal on Artificial Intelligence*, 12(1–2):99–111, 1999.

[9] T. Eiter and G. Gottlob. Propositional Circumscription and Extended Closed World Reasoning are Π_2^P-complete. *Theoretical Computer Science*, 144(2):231–245, Addendum: vol. 118, p. 315, 1993, 1993.

[10] T. Eiter and G. Gottlob. The Complexity of Logic-Based Abduction. *Journal of the Association of Computing Machinery*, 42(1):3–42, 1995.

[11] T. Eiter, G. Gottlob, and N. Leone. Abduction from Logic Programs: Semantics and Complexity. *Theoretical Computer Science*, 189(1–2):129–177, 1997.

[12] T. Eiter, G. Gottlob, and N. Leone. Semantics and Complexity of Abduction from Default Theories. *Artificial Intelligence*, 90(1–2):177–233, 1997.

[13] T. Eiter, V. Klotz, H. Tompits, and S. Woltran. Modal Nonmonotonic Logics Revisited: Efficient Encodings for the Basic Reasoning Tasks. In *Proceedings of the Eleventh Conference on Automated Reasoning with Analytic Tableaux and Related Methods* (*TABLEAUX-2002*), volume 2381 of *Lecture Notes in Artificial Intelligence*, pages 100–114. Springer-Verlag, 2002.

[14] R. Feldmann, B. Monien, and S. Schamberger. A Distributed Algorithm to Evaluate Quantified Boolean Formulas. In *Proceedings of the Seventeenth National Conference on Artificial Intelligence* (*AAAI-2000*) *and the Twelfth Innovative Applications of Artificial Intelligence Conference* (*IAAI-2000*), pages 285–290. AAAI Press/The MIT Press, 2000.

[15] E. Giunchiglia, M. Narizzano, and A. Tacchella. QuBE: A System for Deciding Quantified Boolean Formulas Satisfiability. In *Proceedings of the International Joint Conference on Automated Reasoning* (*IJCAR-2001*), volume 2083 of *Lecture Notes in Computer Science*, pages 364–369. Springer-Verlag, 2001.

[16] G. Gottlob. Complexity Results for Nonmonotonic Logics. *Journal of Logic and Computation*, 2(3):397–425, 1992.

[17] G. Gottlob. The Complexity of Propositional Default Reasoning Under the Stationary Fixed Point Semantics. *Information and Computation*, 121(1):81–92, 1995.

[18] A. C. Kakas, R. A. Kowalski, and F. Toni. Abductive Logic Programming. *Journal of Logic and Computation*, 2(6):719–770, 1993.

[19] A. C. Kakas, R. A. Kowalski, and F. Toni. The Role of Abduction in Logic Programming. In *Handbook of Artificial Intelligence and Logic Programming*, volume 5, pages 235–324. Clarendon Press, 1998.

[20] A. C. Kakas and C. Mourlas. ACLP: Flexible Solutions to Complex Problems. In *Proceedings of the Fourth International Conference on Logic Programming and Nonmonotonic Reasoning* (*LPNMR-97*), volume 1265 of *Lecture Notes in Artificial Intelligence*, pages 387–398. Springer-Verlag, 1997.

[21] H. Kautz and B. Selman. Planning as Satisfiability. In *Proceedings of the Tenth European Conference on Artificial Intelligence* (*ECAI-92*), pages 359–363. John Wiley and Sons, 1992.

[22] H. Kleine-Büning, M. Karpinski, and A. Flögel. Resolution for Quantified Boolean Formulas. *Information and Computation*, 117(1):12–18, 1995.

[23] R. Letz. Advances in Decision Procedures for Quantified Boolean Formulas. In *Proceedings of the IJCAR Workshop on Theory and Applications of Quantified Boolean Formulas*, pages 55–64, 2001.

[24] V. Marek and M. Truszczyński. *Nonmonotonic Logic: Context-Dependent Reasoning*. Artifical Intelligence. Springer-Verlag, 1993.

[25] I. Niemelä. Towards Efficient Default Reasoning. In Chris Mellish, editor, *Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence* (*IJCAI-95*), pages 312–318, San Francisco, 1995. Morgan Kaufmann.

[26] C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley, Reading, Mass., 1994.

[27] D. Pearce, H. Tompits, and S. Woltran. Encodings for Equilibrium Logic and Logic Programs with Nested Expressions. In *Proceedings of the Tenth Portuguese Conference on Artificial Intelligence* (*EPIA-2001*), volume 2258 of *Lecture Notes in Artificial Intelligence*, pages 306–320. Springer-Verlag, 2001.

[28] J. Pearl. *Probabilistic Inference in Intelligent Systems. Networks of Plausible Inference*. Morgan Kaufmann, San Mateo, Canada, 1988.

[29] C. S. Peirce. Abduction and Induction. In J. Buchler, editor, *Philosophical Writings of Peirce*, pages 150–156. Dover Books, New York, 1955.

[30] Y. Peng and J. A. Reggia. *Abductive Inference Models for Diagnostic Problem-Solving*. Springer-Verlag, 1990.

[31] D. A. Plaisted and S. Greenbaum. A Structure Preserving Clause Form Translation. *Journal of Symbolic Computation*, 2(3):293–304, 1986.

[32] D. Poole. Explanation and Prediction: An Architecture for Default and Abductive Reasoning. *Computational Intelligence*, 5(2):97–110, 1989.

[33] R. Reiter. A Logic for Default Reasoning. *Artificial Intelligence*, 13:81–132, 1980.

[34] J. Rintanen. Constructing Conditional Plans by a Theorem Prover. *Journal of Artificial Intelligence Research*, 10:323–352, 1999.

[35] J. Rintanen. Improvements to the Evaluation of Quantified Boolean Formulae. In *Proceedings of the Sixteenth International Joint Conference on Artificial Intelligence* (*IJCAI-99*), pages 1192–1197. Morgan Kaufmann, 1999.

[36] C. Sakama and K. Inoue. Transforming Abductive Logic Programs to Disjunctive Programs. In *Proceedings of the Tenth International Conference on Logic Programming* (*ICLP-93*), pages 335–353. The MIT Press, 1993.

[37] J. Siekmann and G. Wrightson, editors. *Automation of Reasoning: Classical Papers in Computational Logic 1967–1970*, volume 2. Springer-Verlag, 1983.

[38] G. Tseitin. On the Complexity of Proofs in Propositional Logics. *Seminars in Mathematics*, 8, 1970. Reprinted in [37].

[39] S. Woltran. A Framework for Solving Advanced Reasoning Tasks. Master Thesis, Technische Universität Wien, Institut für Informationssysteme, 2000.

[40] C. Wrathall. Complete Sets and the Polynomial-Time Hierarchy. *Theoretical Computer Science*, 3(1):23–33, 1976.