I N F S Y S

R E S E A R C H

R E P O R T

INSTITUT FÜR INFORMATIONSSYSTEME

ABTEILUNG WISSENSBASIERTE SYSTEME

PROBABILISTIC LOGIC UNDER COHERENCE: COMPLEXITY AND ALGORITHMS

Veronica BIAZZO Angelo GILIO
Thomas LUKASIEWICZ Giuseppe SANFILIPPO

INFSYS RESEARCH REPORT 1843-01-04

APRIL 2001 & OCTOBER 2002

Institut für Informationssysteme
Abtg. Wissensbasierte Systeme
Technische Universität Wien
Favoritenstraße 9-11
A-1040 Wien, Austria
Tel: +43-1-58801-18405
Fax: +43-1-58801-18493
sek@kr.tuwien.ac.at
www.kr.tuwien.ac.at

TU

TECHNISCHE UNIVERSITÄT WIEN

INFSYS RESEARCH REPORT

INFSYS RESEARCH REPORT 1843-01-04, APRIL 2001 & OCTOBER 2002

PROBABILISTIC LOGIC UNDER COHERENCE: COMPLEXITY AND ALGORITHMS

(REVISED VERSION, 12 OCTOBER 2002)

Veronica Biazzo [1], Angelo Gilio [2], Thomas Lukasiewicz [3], and Giuseppe Sanfilippo[1]

Abstract. In previous work [5], we have explored the relationship between probabilistic reasoning under coherence and model-theoretic probabilistic reasoning. In particular, we have shown that the notions of g-coherence and of g-coherent entailment in probabilistic reasoning under coherence can be expressed by combining notions in model-theoretic probabilistic reasoning with concepts from default reasoning. In this paper, we continue this line of research. Based on the above semantic results, we draw a precise picture of the computational complexity of probabilistic reasoning under coherence. Moreover, we introduce transformations for probabilistic reasoning under coherence, which reduce an instance of deciding g-coherence or of computing tight intervals under g-coherent entailment to a smaller problem instance, and which can be done very efficiently. Furthermore, we present new algorithms for deciding g-coherence and for computing tight intervals under g-coherent entailment, which reduce these tasks to standard problems in model-theoretic probabilistic reasoning, which can be reduced to linear optimization problems. Thus, efficient techniques for model-theoretic probabilistic reasoning can immediately be applied for probabilistic reasoning under coherence (for example, column generation techniques). We describe several such techniques, which transform problem instances in model-theoretic probabilistic reasoning into smaller problem instances. We also describe a technique for obtaining a reduced set of variables for the associated linear optimization problems in the conjunctive case, and give new characterizations of this reduced set as a set of non-decomposable variables, and using the concept of random gain.

[1]Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Città Universitaria, Viale A. Doria 6, 95152 Catania, Italy; e-mail: {vbiazzo, gsanfilippo}@dmi.unict.it.

[2]Dipartimento di Metodi e Modelli Matematici, Università "La Sapienza", Via A. Scarpa 16, 00161 Rome, Italy; e-mail: gilio@dmmm.uniroma1.it.

[3]Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", Via Salaria 113, 00198 Rome, Italy; e-mail: lukasiewicz@dis.uniroma1.it. Alternate address: Institut für Informationssysteme, Technische Universität Wien, Favoritenstraße 9-11, 1040 Vienna, Austria; e-mail: lukasiewicz@kr.tuwien.ac.at.

Acknowledgements: This work has been partially supported by the Austrian Science Fund under project N Z29-INF, by a DFG grant, and by a Marie Curie Individual Fellowship of the European Community (Disclaimer: The authors are solely responsible for information communicated and the European Commission is not responsible for any views or results expressed). We are very grateful to the reviewers of the ISIPTA '01 abstract of this paper whose constructive comments were very helpful to improve our work.

Contents

1 Introduction

The probabilistic treatment of uncertainty plays an important role in many applications of knowledge representation and reasoning. Often, we need to reason with uncertain information under partial knowledge and then the use of precise probabilistic assessments seems unrealistic. Moreover, the family of uncertain quantities at hand has often no particular algebraic structure.

In such cases, a general approach is obtained by using (conditional and/or unconditional) probabilistic constraints, based on the coherence principle of de Finetti and suitable generalizations of it (Biazzo and Gilio [3], Coletti [11], Coletti and Scozzafava [12, 15, 14], Gilio [25, 26], Gilio and Scozzafava [27], and Scozzafava [43]), or on similar principles which have been adopted for lower and upper probabilities (Pelessoni and Vicig [42], Vicig [45], and Walley [47]). Two important aspects in dealing with uncertainty are: (i) checking the consistency of a probabilistic assessment; and (ii) the propagation of a given assessment to further uncertain quantities. The problem of reducing or eliminating the computational difficulties of (i) and (ii) has been recently investigated by Biazzo *et al.* [6, 7, 4], Capotorti and Vantaggi [10], Capotorti *et al.* [9], and Coletti and Scozzafava [13].

Another approach to handle constraints for probabilities is model-theoretic probabilistic logic, whose roots go back to Boole's book of 1854 "The Laws of Thought" [8]. There is a wide spectrum of formal languages that have been explored in probabilistic logic, which ranges from constraints for unconditional and conditional events (see especially the work by Nilsson [40], Dubois *et al.* [17], Amarger *et al.* [1], and Frisch and Haddawy [20]) to rich languages that specify linear inequalities over events (Fagin *et al.* [19]). Furthermore, new notions of entailment in model-theoretic probabilistic logic, which are based on ideas from default reasoning from conditional knowledge bases [21], have been recently presented in [38, 37]. The reasoning methods in model-theoretic probabilistic logic can be roughly divided into local approaches based on local inference rules and global ones using linear optimization techniques (see especially [35, 34] on the issue of local versus global approaches). As shown by Georgakopoulos *et al.* [23], deciding satisfiability and logical consequence in probabilistic logic is NP- and co-NP-complete, and thus intractable. As recently shown in [36], deciding and computing tight logical consequences is complete for the classes D^{P} and $\mathrm{FP}^{\mathrm{NP}}$, respectively. Substantial research efforts were directed towards efficient reasoning in probabilistic logic. In particular, column generation techniques from the area of linear optimization have been successfully used to solve large problem instances (see especially the work by Jaumard *et al.* [30] and Hansen *et al.* [29]). Other techniques, which may be described as problem transformations on the language level, have been successfully applied in probabilistic logic programming [36]. Moreover, a global approach for the conjunctive case, which characterizes a reduced set of variables, has been presented in [33].

We point out that in model-theoretic probabilistic logic, for every conditional constraint $\alpha \leq Pr(\psi|\phi) \leq \beta$ in the given probabilistic knowledge base, the conditional probability $Pr(\psi|\phi)$ is looked at as the ratio of $Pr(\psi \wedge \phi)$ and $Pr(\phi)$, so that it is defined only if $Pr(\phi) > 0$. On the contrary, as well known, within the approach of de Finetti: (i) one can directly assess conditional probabilities, with no need of defining them as ratios; (ii) no theoretical problem arises when the probabilities of some (or possibly all) *conditioning* events are (judged equal to) zero; and (iii) by exploiting the zero probabilities, the computational difficulties may be reduced (or even eliminated).

Coherence-based and model-theoretic probabilistic reasoning have been explored quite independently from each other by two different research communities. For this reason, the relationship between the two areas has not been studied in depth so far. Our work in [5] and the current paper aim at filling this gap. Our research is essentially guided by the following two questions:

- Which is the semantic relationship between probabilistic reasoning under coherence and model-theoretic probabilistic reasoning?

- Is it possible to use algorithms that have been developed for efficient reasoning in one area also in the other area?

Interestingly, it turns out that the answers to these questions are closely related to default reasoning from conditional knowledge bases [21]. Roughly speaking, coherence-based probabilistic reasoning can be understood as a combination of model-theoretic probabilistic logic with concepts from default reasoning. That is, deciding coherence and computing tight intervals under coherence can be reduced to standard reasoning tasks in model-theoretic probabilistic logic, using concepts from default reasoning. Thus, in particular, efficient techniques for model-theoretic probabilistic reasoning can immediately be applied for probabilistic reasoning under coherence.

Our work in [5] explores the semantic aspects of this relationship. In this paper, we focus on its computational implications for coherence-based probabilistic reasoning. The main contributions can be summarized as follows:

- We recall from [5] our coherence-based probabilistic logic, which is a formal language of logical and conditional constraints, for which we have introduced the notions of generalized coherence (or g-coherence), g-coherent consequence, and tight g-coherent consequence. We then recall from [5] the relationship between probabilistic reasoning under coherence and model-theoretic probabilistic reasoning.

- We draw a precise picture of the computational complexity of probabilistic reasoning under coherence. It turns out that deciding g-coherence, g-coherent consequence, and tight g-coherent consequence are complete for NP, co-NP, and D^{P}, respectively. Moreover, computing tight intervals under g-coherent entailment is complete for the class $\mathrm{FP}^{\mathrm{NP}}$. This shows that deciding g-coherence, deciding g-coherent and tight g-coherent consequence, and computing tight intervals under g-coherent entailment have the same complexity as deciding satisfiability, deciding logical and tight logical consequence, and computing tight intervals under logical entailment, respectively, in model-theoretic probabilistic logic.

- We newly introduce transformations of problem instances in probabilistic reasoning under coherence into smaller problem instances. They are (i) decomposing a problem instance, and (ii) removing inactive logical and conditional constraints from a problem instance. Here, the former applies to the general case, while the latter works only for the conjunctive case. These transformations are inspired by similar techniques from model-theoretic probabilistic and default reasoning in [18, 39] and [36, 18], respectively. In particular, we prove that these two transformations can be done very efficiently (that is, in linear time).

- We present new algorithms for deciding g-coherence and for computing tight intervals under g-coherent entailment. They are based on concepts from default reasoning, and reduce checking g-coherence and computing tight g-coherent intervals to standard tasks in model-theoretic probabilistic logic, which can be reduced to linear optimization problems. As a consequence of this result, all the techniques that have been developed for efficient model-theoretic probabilistic reasoning can now also be applied in probabilistic reasoning under g-coherence (for example, column generation techniques [23, 30, 29]).

- We describe some transformations of problem instances in model-theoretic probabilistic reasoning into smaller problem instances. These are (i) adding logical constraints to a problem instance, (ii) removing vacuous conditional constraints from a problem instance, (iii) decomposing a problem instance, and (iv) removing inactive constraints from a problem instance. Here, (i)–(iii) apply to the general case, while (iv) applies only to the conjunctive case. The techniques (i), (ii), and (iv) are adapted from [36], while (iii) is inspired by [39]. We show that (i) and (ii) can be done in polynomial time in the conjunctive case, and that (iii) (resp., (iv)) can be done in linear time in the general (resp., conjunctive) case.

- In the conjunctive case, a technique from [33] can be used to produce a reduced set of variables for the linear optimization problems in model-theoretic probabilistic reasoning. We present the new result that this reduced set of variables can be characterized as a set of non-decomposable variables. Furthermore, we show that it can be characterized using the concept of random gain. These results give new insights into the above technique from [33] of producing a reduced set of variables.

The rest of this paper is organized as follows. Sections 2 and 3 recall our coherence-based probabilistic logic and its semantic relationship to model-theoretic probabilistic logic. In Section 4, we present our results on the complexity of probabilistic reasoning under coherence. Section 5 describes transformations of problem instances in coherence-based probabilistic reasoning. In Section 6, we present our new algorithms, which reduce probabilistic reasoning under coherence to standard tasks in model-theoretic probabilistic reasoning. In Sections 7 and 8, we focus on transformations of problem instances in model-theoretic probabilistic reasoning and on a reduced set of variables for their linear optimization problems. In Section 9, we summarize the main results and give an outlook on future research.

Note that detailed proofs of all results are given in Appendices A–D.

2 Probabilistic logic under coherence

In this section, we first give some technical preliminaries. We then recall precise and imprecise probability assessments under coherence. We finally describe our probabilistic logic under coherence and give an example.

2.1 Preliminaries

We first define the syntax and semantics of events. We assume a nonempty set of *basic events* Φ. We use $\perp$ and $\top$ to denote the constant events *false* and *true*, respectively. The set of *events* is the closure of $\Phi \cup \{\perp, \top\}$ under the Boolean operators $\neg$ and $\wedge$. That is, each element of $\Phi \cup \{\perp, \top\}$ is an event, and if ϕ and ψ are events, then also $(\phi \wedge \psi)$ and $\neg\phi$. We use $(\phi \vee \psi)$ and $(\psi \Leftarrow \phi)$ to abbreviate $\neg(\neg\phi \wedge \neg\psi)$ and $\neg(\phi \wedge \neg\psi)$, respectively, and adopt the usual conventions to eliminate parentheses. We often use $\bar{\phi}$ and $\phi\psi$ to denote the negation $\neg\phi$ and the conjunction $\phi \wedge \phi$, respectively. We use $bas(\phi)$ to denote the set of all basic events $b \in \Phi$ that occur in ϕ. A *logical constraint* is an event of the form $\psi \Leftarrow \phi$.

A *world* I is a truth assignment to the basic events in Φ (that is, a mapping $I\colon \Phi \to \{\mathbf{false}, \mathbf{true}\}$), which is extended to all events in the usual way (that is, $(\phi \wedge \psi)$ is true in I iff ϕ and ψ are true in I, and $\neg\phi$ is true in I iff ϕ is not true in I). We use $\mathcal{I}_\Phi$ to denote the set of all worlds for Φ. We often identify the truth values $\mathbf{false}$ and $\mathbf{true}$ with the real numbers 0 and 1, respectively. Moreover, we often identify the event ϕ with the real number 0 (resp., 1), when $I(\phi) = \mathbf{false}$ (resp., $I(\phi) = \mathbf{true}$). A world I *satisfies* an event ϕ, or I is a *model* of ϕ, denoted $I \models \phi$, iff $I(\phi) = \mathbf{true}$. We say I *satisfies* a set of events L, or I is a *model* of L, denoted $I \models L$, iff I is a model of all $\phi \in L$. An event ϕ (resp., a set of events L) is *satisfiable* iff a model of ϕ (resp., L) exists. An event ψ is a *logical consequence* of ϕ (resp., L), denoted $\phi \models \psi$ (resp., $L \models \psi$), iff each model of ϕ (resp., L) is also a model of ψ. We use $\phi \not\models \psi$ (resp., $L \not\models \psi$) to denote that $\phi \models \psi$ (resp., $L \models \psi$) does not hold. We say ϕ is *logically equivalent* to ψ, denoted $\phi \equiv \psi$, iff $\phi \models \psi$ and $\psi \models \phi$ hold. Notice in particular that $\bot \Leftarrow \alpha$ is logically equivalent to $\neg\alpha$.

2.2 Probability assessments

We now recall conditional events, (precise) probability assessments on conditional events, and the concept of coherence for such assessments.

A *conditional event* is an expression of the kind $\psi|\phi$ with events ψ and ϕ. It can be looked at as a three-valued logical entity, with values $\mathbf{true}$, or $\mathbf{false}$, or $\mathbf{indeterminate}$, according to whether ψ and ϕ are true, or ψ is false and ϕ is true, or ϕ is false, respectively. That is, we extend worlds I to conditional events $\psi|\phi$ by $I(\psi|\phi) = \mathbf{true}$ iff $I \models \psi \wedge \phi$, $I(\psi|\phi) = \mathbf{false}$ iff $I \models \neg\psi \wedge \phi$, and $I(\psi|\phi) = \mathbf{indeterminate}$ iff $I \models \neg\phi$. Two conditional events $\psi_1|\phi_1$ and $\psi_2|\phi_2$ are *logically equivalent*, denoted $\psi_1|\phi_1 \equiv \psi_2|\phi_2$, iff $I(\psi_1|\phi_1) = I(\psi_2|\phi_2)$ for all worlds I. Observe especially that $\psi|\phi \equiv \psi \wedge \phi|\phi$. More generally, $\psi_1|\phi_1 \equiv \psi_2|\phi_2$ iff $\psi_1 \wedge \phi_1 \equiv \psi_2 \wedge \phi_2$ and $\phi_1 \equiv \phi_2$.

We recall that in the framework of subjective probability, given an event ψ, the evaluation $Pr(\psi) = \alpha$, given by someone, is a numerical representation of his *degree of belief* on ψ being true. Using the betting criterion, if such individual evaluates $Pr(\psi) = \alpha$, then he / she will pay an amount of money $\alpha\, S$, where $S \neq 0$ is arbitrary, and he / she will get back the amount $S\, \psi$. Then, the associated random gain is $G = S\, (\psi - \alpha)$. Coherence requires that, for every $S \neq 0$, it must be $\max\, G \geq 0$, which implies $0 \leq \alpha \leq 1$.

For a conditional event $\psi|\phi$, if the individual evaluates $Pr(\psi|\phi) = \alpha$, then he / she will pay an amount of money $\alpha\, S$ getting back the amount $S\, \psi$ (resp., $\alpha\, S$) if ϕ is true (resp., false). Then, the associated random gain is $G = S\, \phi\, (\psi - \alpha)$. Coherence requires that, for every $S \neq 0$, it must be $\max\, G|\phi \geq 0$, which (again) implies $0 \leq \alpha \leq 1$.

We remark that the coherence-based probabilistic approach is more general than the usual one, because to assess conditional probabilities we do not rely on unconditional probabilities. In fact, in our framework the probability assessment $Pr(\psi|\phi) = \alpha$ has a natural meaning in all cases, including the one in which $Pr(\phi) = 0$. In other words, the quantity $Pr(\phi)$ plays *no role* and the only relevant thing is the assumption "*ϕ true*". Moreover, given a function A on a family of conditional events $\mathcal{E}$, if A is coherent, then A satisfies all the usual axioms of a conditional probability, while the converse is not true (for a counterexample see [24], Example 8).

More formally, a *probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L, and a mapping A that assigns to each $\varepsilon \in \mathcal{E}$ a real number in $[0, 1]$. Informally, L describes logical relationships, while A represents probabilistic knowledge. For $\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$ and n real numbers $s_1, \ldots, s_n$, let the mapping $G\colon \mathcal{I}_\Phi \to \mathbf{R}$ be defined as follows. For all $I \in \mathcal{I}_\Phi$:

$$G(I) \;=\; \sum_{i=1}^{n} s_i \cdot I(\phi_i) \cdot (I(\psi_i) - A(\psi_i|\phi_i))\,.$$

In the framework of betting criterion, G can be interpreted as the random gain corresponding to a combination of n bets of amounts $s_1 \cdot A(\psi_1|\phi_1), \ldots, s_n \cdot A(\psi_n|\phi_n)$ on $\psi_1|\phi_1, \ldots, \psi_n|\phi_n$ with stakes $s_1, \ldots, s_n$. In detail, to bet on $\psi_i|\phi_i$, one pays an amount of $s_i \cdot A(\psi_i|\phi_i)$, and one gets back the amount of s_i, 0, and $s_i \cdot A(\psi_i|\phi_i)$, when $\psi_i \wedge \phi_i$, $\neg\psi_i \wedge \phi_i$, and $\neg\phi_i$, respectively, turns out to be true. The following notion of *coherence* now assures that it is impossible (for both the gambler and the bookmaker) to have uniform loss. A probability assessment (L, A) on a set of conditional events $\mathcal{E}$ is *coherent* iff for every

$\{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$ and for all real numbers $s_1, \ldots, s_n$, it holds that $\max\{G(I) \mid I \in \mathcal{I}_\Phi, I \models L \cup \{\phi_1 \vee \cdots \vee \phi_n\}\} \geq 0$.

2.3 Imprecise probability assessments

We next recall imprecise probability assessments on conditional events and the concepts of g-coherence and g-coherent entailment for such assessments.

An *imprecise probability assessment* (L, A) on a set of conditional events $\mathcal{E}$ consists of a set of logical constraints L and a mapping A that assigns to each $\varepsilon \in \mathcal{E}$ an interval $[l, u] \subseteq [0, 1]$ with $l \leq u$. We say (L, A) is *g-coherent* iff there exists a coherent precise probability assessment $(L, A^\star)$ on $\mathcal{E}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$.

We recall a characterization of g-coherence due to Gilio [25]; equivalent results have been obtained by Coletti [11]. Given a set of logical constraints L and a set of conditional events $\mathcal{E} = \{\varepsilon_1, \ldots, \varepsilon_n\}$, denote by $R_L(\mathcal{E})$ the set of all mappings r that associate with each $\varepsilon_i = \psi_i|\phi_i \in \mathcal{E}$ a member of $\{\psi_i \wedge \phi_i, \neg\psi_i \wedge \phi_i, \neg\phi_i\}$ such that (i) $L \cup \{r(\varepsilon_i) \mid \varepsilon_i \in \mathcal{E}\}$ is satisfiable, and (ii) $r(\varepsilon_i) \neq \neg\phi_i$ for some $i \in \{1, \ldots, n\}$. For such mappings r and events ϕ, we abbreviate $r(\varepsilon_1) \wedge \cdots \wedge r(\varepsilon_n)$ by $\wedge r$, and $\wedge r \models \phi$ by $r \models \phi$.

Theorem 2.1 (Gilio [25]) *An imprecise probability assessment (L, A) on a set of conditional events $\mathcal{E}$ is g-coherent iff for every $\mathcal{E}_n = \{\psi_1|\phi_1, \ldots, \psi_n|\phi_n\} \subseteq \mathcal{E}$ with $n \geq 1$, the following system of linear constraints over the variables λ_r $(r \in R)$, where $R = R_L(\mathcal{E}_n)$, is solvable:*

$$\begin{aligned} \sum_{r \in R} \lambda_r \cdot p_{ri} &\geq l_i \quad (\textit{for all } i \in \{1, \ldots, n\}) \\ \sum_{r \in R} \lambda_r \cdot q_{ri} &\leq u_i \quad (\textit{for all } i \in \{1, \ldots, n\}) \\ \sum_{r \in R} \lambda_r &= 1 \\ \lambda_r &\geq 0 \quad (\textit{for all } r \in R)\,, \end{aligned} \tag{1}$$

where l_i and u_i are defined by $A(\psi_i|\phi_i) = [l_i, u_i]$, for all $i \in \{1, \ldots, n\}$, and p_{ri} and q_{ri} are defined as follows, for all $r \in R$ and $i \in \{1, \ldots, n\}$:

$$p_{ri} \ \ (\textit{resp., } q_{ri}) = \begin{cases} 1, & \textit{if } r \models \psi_i \wedge \phi_i \\ 0, & \textit{if } r \models \neg\psi_i \wedge \phi_i \\ l_i \ \ (\textit{resp., } u_i), & \textit{if } r \models \neg\phi_i\,. \end{cases} \tag{2}$$

We next define the notion of *g-coherent entailment* for imprecise probability assessments. Let (L, A) be a g-coherent imprecise probability assessment on a set of conditional events $\mathcal{E}$. The imprecise probability assessment $[l, u]$ on a conditional event γ, denoted $(\gamma, [l, u])$, is called a *g-coherent consequence* of (L, A) iff $A^\star(\gamma) \in [l, u]$ for every g-coherent precise probability assessment $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. It is a *tight g-coherent consequence* of (L, A) iff l (resp., u) is the infimum (resp., supremum) of $A^\star(\gamma)$ subject to all g-coherent precise probability assessments $A^\star$ on $\mathcal{E} \cup \{\gamma\}$ such that $A^\star(\varepsilon) \in A(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. Observe that for $\varepsilon = \beta|\alpha$ such that $L \models \neg\alpha$, every $\{(\varepsilon, [l, u])\}$ with $l, u \in [0, 1]$ is a g-coherent consequence of (L, A), and $\{(\varepsilon, [1, 0])\}$ is the unique tight g-coherent consequence of (L, A). Note that here we identify $[1, 0]$ with the empty set.

2.4 Probabilistic logic under coherence

We now define conditional constraints, probabilistic knowledge bases, and the concepts of g-coherence and of g-coherent entailment for probabilistic knowledge bases. In the rest of this paper, we assume Φ is finite.

A *conditional constraint* is an expression of the form $(\psi|\phi)[l,u]$ with real numbers $l,u \in [0,1]$ and events ψ, ϕ. We call ϕ its *antecedent* and ψ its *consequent*. A *probabilistic knowledge base* $KB = (L,P)$ consists of a finite set of logical constraints L, and a finite set of conditional constraints P such that (i) $l \leq u$ for all $(\psi|\phi)[l,u] \in P$, and (ii) $\psi_1|\phi_1 \not\equiv \psi_2|\phi_2$ for any two distinct $(\psi_1|\phi_1)[l_1,u_1]$, $(\psi_2|\phi_2)[l_2,u_2] \in P$. We use $bas(KB)$ to denote the set of all basic events $b \in \Phi$ that occur in KB.

Every imprecise probability assessment $IP = (L,A)$, where L is finite, and A is defined on a finite set of conditional events $\mathcal{E}$, can be represented by the following probabilistic knowledge base:

$$KB_{IP} \;=\; (L, \{(\psi|\phi)[l,u] \mid \psi|\phi \in \mathcal{E},\ A(\psi|\phi) = [l,u]\})\,.$$

Conversely, each probabilistic knowledge base $KB{=}(L,P)$ can be expressed by the following imprecise probability assessment $IP_{KB}{=}(L,A_{KB})$ on $\mathcal{E}_{KB}$:

$$\begin{aligned} A_{KB} &= \{(\psi|\phi,[l,u]) \mid (\psi|\phi)[l,u] \in KB\}\,,\\ \mathcal{E}_{KB} &= \{\psi|\phi \mid \exists\, l,u \in [0,1]\colon (\psi|\phi)[l,u] \in KB\}\,. \end{aligned}$$

A probabilistic knowledge base KB is *g-coherent* iff IP_{KB} is g-coherent. For g-coherent probabilistic knowledge bases KB and conditional constraints $(\psi|\phi)[l,u]$, we say $(\psi|\phi)[l,u]$ is a *g-coherent consequence* of KB, denoted $KB \mid\!\sim (\psi|\phi)[l,u]$, iff $\{(\psi|\phi,[l,u])\}$ is a g-coherent consequence of IP_{KB}. We say $(\psi|\phi)[l,u]$ is a *tight g-coherent consequence* of KB, denoted $KB \mid\!\sim_{tight} (\psi|\phi)[l,u]$, iff $\{(\psi|\phi,[l,u])\}$ is a tight g-coherent consequence of IP_{KB}.

The following example illustrates the concept of a probabilistic knowledge base and the notions of g-coherence and of g-coherent entailment.

Example 2.2 Let the probabilistic knowledge base $KB = (L,P)$ be given by:

$$\begin{aligned} L &= \{\mathsf{bird} \Leftarrow \mathsf{penguin}\}\,,\\ P &= \{(\mathsf{legs}|\mathsf{bird})[1,1], (\mathsf{fly}|\mathsf{bird})[1,1], (\mathsf{fly}|\mathsf{penguin})[0,0.05]\}\,. \end{aligned}$$

In probabilistic logic under coherence, KB may represent the *logical knowledge* "all penguins are birds", the *logical default knowledge* "generally, birds have legs" and "generally, birds fly", and the *probabilistic default knowledge* "generally, penguins fly with a probability of at most 0.05".

It is not difficult to verify that KB is g-coherent, and that some tight g-coherent consequences of KB are given as follows:

$$\begin{aligned} &KB \mid\!\sim_{tight} (\mathsf{legs}|\mathsf{bird})[1,1],\quad KB \mid\!\sim_{tight} (\mathsf{fly}|\mathsf{bird})[1,1]\,,\\ &KB \mid\!\sim_{tight} (\mathsf{legs}|\mathsf{penguin})[0,1],\quad KB \mid\!\sim_{tight} (\mathsf{fly}|\mathsf{penguin})[0,0.05]\,. \end{aligned}$$

Here, the interval "$[0,1]$" is due to the fact that the logical property of having legs is not inherited from birds down to penguins. □

The following characterization of g-coherent consequence is an immediate implication of the definitions of g-coherence and g-coherent entailment.

Theorem 2.3 *Let* $KB = (L, P)$ *be a g-coherent probabilistic knowledge base, and let* $(\beta|\alpha)[l,u]$ *be a conditional constraint. Then,* $KB \mathrel{|\!\sim} (\beta|\alpha)[l,u]$ *iff* $(L, P \cup \{(\beta|\alpha)[p,p]\})$ *is not g-coherent for all* $p \in [0,l) \cup (u,1]$.

The next result, from [5], shows that a similar characterization holds for tight g-coherent consequence.

Theorem 2.4 *Let* $KB = (L, P)$ *be a g-coherent probabilistic knowledge base, and let* $(\beta|\alpha)[l,u]$ *be a conditional constraint. Then,* $KB \mathrel{|\!\sim}_{\mathit{tight}} (\beta|\alpha)[l,u]$ *iff (i)* $(L, P \cup \{(\beta|\alpha)[p,p]\})$ *is not g-coherent for all* $p \in [0,l) \cup (u,1]$*, and (ii)* $(L, P \cup \{(\beta|\alpha)[p,p]\})$ *is g-coherent for all* $p \in [l,u]$.

3 Relationship to model-theoretic probabilistic logic

In this section, we recall some of our results from [5], which concern the relationship between coherence-based and model-theoretic probabilistic logic. We first recall the main concepts of model-theoretic probabilistic logic. We then describe how the notion of g-coherence can be reduced to the existence of satisfying probabilistic interpretations in model-theoretic probabilistic logic. Furthermore, we show that g-coherent entailment can be reduced to logical entailment in model-theoretic probabilistic logic.

3.1 Model-theoretic probabilistic logic

We now recall the main notions of model-theoretic probabilistic logic. In particular, we define probabilistic interpretations and the semantics of events, logical constraints, and conditional constraints under probabilistic interpretations. We then define the notions of satisfiability and logical entailment.

A *probabilistic interpretation* Pr is a probability function on $\mathcal{I}_\Phi$ (that is, a mapping $Pr\colon \mathcal{I}_\Phi \to [0,1]$ such that all $Pr(I)$ with $I \in \mathcal{I}_\Phi$ sum up to 1). The *probability* of an event ϕ in the probabilistic interpretation Pr, denoted $Pr(\phi)$, is defined as the sum of all $Pr(I)$ such that $I \in \mathcal{I}_\Phi$ and $I \models \phi$. For events ϕ and ψ with $Pr(\phi) > 0$, we use $Pr(\psi|\phi)$ to abbreviate $Pr(\psi \wedge \phi) \,/\, Pr(\phi)$. The *truth* of logical and conditional constraints F in a probabilistic interpretation Pr, denoted $Pr \models F$, is defined as follows:

- $Pr \models \psi \Leftarrow \phi$ iff $Pr(\psi \wedge \phi) = Pr(\phi)$,
- $Pr \models (\psi|\phi)[l,u]$ iff $Pr(\phi) = 0$ or $Pr(\psi|\phi) \in [l,u]$.

We say Pr *satisfies* a logical or conditional constraint F, or Pr is a *model* of F, iff $Pr \models F$. We say Pr *satisfies* a set of logical and conditional constraints $\mathcal{F}$, or Pr is a *model* of $\mathcal{F}$, denoted $Pr \models \mathcal{F}$, iff Pr is a model of all $F \in \mathcal{F}$. We say $\mathcal{F}$ is *satisfiable* iff a model of $\mathcal{F}$ exists. A probabilistic interpretation Pr *satisfies* a probabilistic knowledge base $KB = (L, P)$, or Pr is a *model* of KB, denoted $Pr \models KB$, iff Pr is a model of $L \cup P$. We say KB is *satisfiable* iff $L \cup P$ is satisfiable.

We next define the notion of *logical entailment*. A conditional constraint $(\psi|\phi)[l,u]$ is a *logical consequence* of a set of logical and conditional constraints $\mathcal{F}$, denoted $\mathcal{F} \models (\psi|\phi)[l,u]$, iff each model of $\mathcal{F}$ is also a model of $(\psi|\phi)[l,u]$. We say $(\psi|\phi)[l,u]$ is a *tight logical consequence* of $\mathcal{F}$, denoted $\mathcal{F} \models_{\mathit{tight}} (\psi|\phi)[l,u]$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all models Pr of $\mathcal{F}$ with $Pr(\phi) > 0$. Note that we define $l = 1$ and $u = 0$, when $\mathcal{F} \models (\phi|\top)[0,0]$. A conditional constraint $(\psi|\phi)[l,u]$ is a *logical consequence* of a probabilistic knowledge base $KB = (L, P)$, denoted $KB \models (\psi|\phi)[l,u]$, iff $L \cup P \models (\psi|\phi)[l,u]$. It is a *tight logical consequence* of KB, denoted $KB \models_{\mathit{tight}} (\psi|\phi)[l,u]$, iff $L \cup P \models_{\mathit{tight}} (\psi|\phi)[l,u]$.

Example 3.1 Consider again the probabilistic knowledge base $KB = (L, P)$ of Example 2.2. In model-theoretic probabilistic logic, KB may represent the *logical knowledge* "all penguins are birds", "all birds have legs", and "all birds fly" (that is, in model-theoretic probabilistic logic, a logical constraint $\psi \Leftarrow \phi$ means the same as the conditional constraint $(\psi|\phi)[1,1]$), and the *probabilistic knowledge* "penguins fly with a probability of at most 0.05".

It is easy to verify that KB is satisfiable. Some tight logical consequences of KB are given as follows:

$$KB \models_{tight} (\mathsf{legs}|\mathsf{bird})[1,1], \;\; KB \models_{tight} (\mathsf{fly}|\mathsf{bird})[1,1]\,,$$
$$KB \models_{tight} (\mathsf{legs}|\mathsf{penguin})[1,0], \;\; KB \models_{tight} (\mathsf{fly}|\mathsf{penguin})[1,0]\,.$$

Here, we have the empty set "$[1,0]$" in the last two conditional constraints, as the logical property of being able to fly is inherited from birds down to penguins, and is incompatible there with the probabilistic knowledge "penguins fly with a probability of at most 0.05". □

3.2 G-coherence

The following theorem describes how the notion of g-coherence can be expressed through the existence of satisfying probabilistic interpretations.

Theorem 3.2 *Let $KB = (L, P)$ be a probabilistic knowledge base. Then, KB is g-coherent iff for every nonempty $P_n = \{(\psi_i|\phi_i)[l_i, u_i] \mid i \in \{1, \ldots, n\}\} \subseteq P$, there exists a model Pr of $L \cup P_n$ such that $Pr(\phi_1 \vee \cdots \vee \phi_n) > 0$.*

Thus, if a probabilistic knowledge base $KB = (L, P)$ with $P \neq \emptyset$ is g-coherent, then it is also satisfiable (as shown in [5], g-coherence is in fact strictly stronger than satisfiability). We get the following corollary.

Corollary 3.3 *Every g-coherent probabilistic knowledge base $KB = (L, P)$ with $P \neq \emptyset$ is satisfiable.*

The next result shows that g-coherence has a characterization similar to the one of p-consistency in default reasoning by Goldszmidt and Pearl [28]. To formulate this result, we adopt the following terminology from default reasoning [21, 2]. A probabilistic interpretation Pr *verifies* a conditional constraint $(\psi|\phi)[l, u]$ iff $Pr(\phi) > 0$ and $Pr \models (\psi|\phi)[l, u]$. A set of conditional constraints P *tolerates* a conditional constraint F under a set of logical constraints L iff there exists a model of $L \cup P$ that verifies F. We say P is *under L in conflict* with F iff no model of $L \cup P$ verifies F.

Theorem 3.4 *A probabilistic knowledge base $KB = (L, P)$ is g-coherent iff there exists an ordered partition $(P_0, \ldots, P_k)$ of P such that either*

(a) every P_i, $0 \leq i \leq k$, is the set of all $F \in P_i \cup P_{i+1} \cup \cdots \cup P_k$ tolerated under L by $P_i \cup P_{i+1} \cup \cdots \cup P_k$, or

(b) for all i, $0 \leq i \leq k$, each $F \in P_i$ is tolerated under L by $P_i \cup P_{i+1} \cup \cdots \cup P_k$.

3.3 G-coherent entailment

We next show how g-coherent entailment can be reduced to logical entailment. We first give some preparative definitions as follows. For probabilistic knowledge bases $KB = (L, P)$ and events α such that $L \not\models \neg\alpha$, let $P_\alpha(KB)$ denote the set of all subsets $P_n = \{(\psi_1|\phi_1)[l_1, u_1], \ldots, (\psi_n|\phi_n)[l_n, u_n]\}$ of P such that every model Pr of $L \cup P_n$ with $Pr(\phi_1 \vee \cdots \vee \phi_n \vee \alpha) > 0$ satisfies $Pr(\alpha) > 0$. For $KB = (L, P)$ and α with $L \models \neg\alpha$, let $P_\alpha(KB) = \{\emptyset\}$.

The following theorem shows that the tight interval entailed under g-coherence can be expressed as the intersection of some logically entailed tight intervals. More precisely, it says that $(\beta|\alpha)[l,u]$ is a tight g-coherent consequence of $KB = (L,P)$ iff $[l,u]$ is the intersection of all $[c,d]$ such that $L \cup P' \models_{tight} (\beta|\alpha)[c,d]$ for some $P' \in P_\alpha(KB)$.

Theorem 3.5 *Let $KB{=}(L,P)$ be a g-coherent probabilistic knowledge base, and let $\beta|\alpha$ be a conditional event. Then, $KB \mathrel{|\!\!\sim}_{tight} (\beta|\alpha)[l,u]$, where*

$$[l,u] \quad = \quad \bigcap\{[c,d] \mid L \cup P' \models_{tight} (\beta|\alpha)[c,d] \textit{ for some } P' \in P_\alpha(KB)\}\,.$$

Clearly, this reduction of g-coherent entailment to logical entailment is computationally expensive, as we have to compute a tight logically entailed interval for each member of $P_\alpha(KB)$. In the following, we show that we can restrict our attention to the unique greatest element in $P_\alpha(KB)$ with respect to set inclusion. The following lemma shows that $P_\alpha(KB)$ contains indeed such a unique greatest element.

Lemma 3.6 *Let $KB = (L,P)$ be a g-coherent probabilistic knowledge base, and let α be an event. Then, $P_\alpha(KB)$ contains a unique greatest element.*

The next theorem shows the important result that g-coherent entailment of $(\beta|\alpha)[l,u]$ from $KB = (L,P)$ can be reduced to logical entailment of $(\beta|\alpha)[l,u]$ from L and the greatest element in $P_\alpha(KB)$.

Theorem 3.7 *Let $KB{=}(L,P)$ be a g-coherent probabilistic knowledge base, and let $(\beta|\alpha)[l,u]$ be a conditional constraint. Let $KB^\star = (L,P^\star)$, where $P^\star$ is the greatest element in $P_\alpha(KB)$. Then,*

(a) $KB \mathrel{|\!\!\sim} (\beta|\alpha)[l,u]$ *iff* $KB^\star \models (\beta|\alpha)[l,u]$.

(b) $KB \mathrel{|\!\!\sim}_{tight} (\beta|\alpha)[l,u]$ *iff* $KB^\star \models_{tight} (\beta|\alpha)[l,u]$.

Thus, g-coherent (resp., tight g-coherent) consequences from KB can be reduced to logical (resp., tight logical) consequences from $KB^\star = (L,P^\star)$, where $P^\star$ is the greatest element in $P_\alpha(KB)$. The following theorem shows how $P^\star$ can be characterized and thus computed.

Theorem 3.8 *Let $KB{=}(L,P)$ be a g-coherent probabilistic knowledge base, and let α be an event. Let $P^\star \subseteq P$, and let $(P_0,\ldots,P_k)$ be an ordered partition of $P \setminus P^\star$ such that the following two conditions hold:*

(i) Every P_i, $0 \leq i \leq k$, is the set of all elements in $P_i \cup \cdots \cup P_k \cup P^\star$ that are tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $P_i \cup \cdots \cup P_k \cup P^\star$.

(ii) No member of $P^\star$ is tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $P^\star$.

Then, $P^\star$ is the greatest element in $P_\alpha(KB)$.

Hence, by Theorems 3.7 and 3.8, deciding g-coherent (resp., tight g-coherent) consequence from $KB = (L,P)$ can be reduced to first computing $P^\star$ by a g-coherence check, and then deciding logical (resp., tight logical) consequence from $KB^\star = (L,P^\star)$. Similarly, computing the tight interval under g-coherent entailment from KB can be reduced to computing first $P^\star$ and then the tight interval under logical entailment from $KB^\star$.

Theorem 3.8 also shows that logical entailment is stronger than g-coherent entailment (as proved in [5], logical entailment is in fact strictly stronger than g-coherent entailment). Furthermore, if a g-coherent probabilistic knowledge base KB does not have any model Pr with $Pr(\alpha) = 0$, then g-coherent entailment of $(\beta|\alpha)[l, u]$ from KB coincides with logical entailment of $(\beta|\alpha)[l, u]$ from KB. This immediate result is expressed by the following corollary.

Corollary 3.9 *Let $KB = (L, P)$ be a g-coherent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Assume that $L \cup P \cup \{\bot \Leftarrow \alpha\}$ is not satisfiable. Then,*

(a) $KB \mathrel{|\!\!\sim} (\beta|\alpha)[l, u]$ *iff* $KB \models (\beta|\alpha)[l, u]$.

(b) $KB \mathrel{|\!\!\sim}_{tight} (\beta|\alpha)[l, u]$ *iff* $KB \models_{tight} (\beta|\alpha)[l, u]$.

4 Computational complexity

In this section, we give a precise picture of the complexity of deciding g-coherence, of deciding g-coherent consequence, of deciding tight g-coherent consequence, and of computing tight intervals under g-coherent entailment.

4.1 Complexity classes

We assume some basic knowledge about complexity theory. We now briefly describe the complexity classes that we encounter in our complexity results (see especially [22, 32, 41, 44, 31] for further background).

The class P (resp., NP) contains all decision problems that can be solved in deterministic (resp., non-deterministic) polynomial time. The class co-NP is the complementary class of NP, which has Yes- and No-instances interchanged. The class $\mathrm{D^P}$ contains all decision problems that can be expressed as a logical conjunction of a problem in NP and a problem in co-NP. The class $\mathrm{P^{NP}}$ contains all decision problems that can be solved in deterministic polynomial time with an oracle for NP. The following inclusion hierarchy describes the relationship between the above complexity classes (note that all inclusions are currently believed to be strict):

$$\mathrm{P} \subseteq \mathrm{NP}, \text{co-NP} \subseteq \mathrm{D^P} \subseteq \mathrm{P^{NP}}.$$

To classify problems that compute an output value, rather than a Yes / No-answer, function classes have been introduced. In particular, the class FP (resp., $\mathrm{FP^{NP}}$) is the functional analog to P (resp., $\mathrm{P^{NP}}$).

In this paper, completeness for a decision class is with respect to standard polynomial time transformations. Furthermore, completeness for a function class is understood in terms of a natural generalization of polynomial time transformations: The problem P_1 reduces to P_2, if there are polynomial time functions f and g such that for each instance I_1 of P_1, the output for I_1 is given by $g(I_1, P_2(f(I_1)))$; see [44, 31] for formal details.

In the sequel, we consider presumably intractable problems (it has not been proved so far that $\mathrm{P} \neq \mathrm{NP}$) as intractable.

4.2 Problem statements and overview of results

We analyze the complexity of the following important decision and optimization problems in probabilistic reasoning under coherence:

G-COHERENCE: Given a probabilistic knowledge base KB, decide whether KB is g-coherent.

G-COHERENT ENTAILMENT: Given a g-coherent probabilistic knowledge base KB and a conditional conditional constraint $(\beta|\alpha)[l,u]$, decide whether $KB \mid\!\sim (\beta|\alpha)[l,u]$.

TIGHT G-COHERENT ENTAILMENT (D): Given a g-coherent probabilistic knowledge base KB and a conditional conditional constraint $(\beta|\alpha)[l,u]$, decide whether $KB \mid\!\sim_{\mathit{tight}} (\beta|\alpha)[l,u]$.

TIGHT G-COHERENT ENTAILMENT (OPT): Given a g-coherent probabilistic knowledge base KB and a conditional event $\beta|\alpha$, compute the unique $l,u \in [0,1]$ such that $KB \mid\!\sim_{\mathit{tight}} (\beta|\alpha)[l,u]$.

In our complexity analysis, we assume that all lower and upper bounds in the probabilistic knowledge base $KB = (L,P)$ are rational numbers. A conditional constraint $(\psi|\phi)[l,u]$ (resp., logical constraint $\psi \Leftarrow \phi$) is *atomic* iff ψ is a basic event (resp., ψ is either $\bot$ or a basic event) and ϕ is either $\top$ or a basic event. We say $(\psi|\phi)[l,u]$ (resp., $\psi \Leftarrow \phi$) is *1-conjunctive* iff ψ is a basic event (resp., ψ is either $\bot$ or a basic event) and ϕ is either $\top$ or a conjunction of basic events. We say $KB = (L,P)$ is *atomic* (resp., *1-conjunctive*) iff all members of $L \cup P$ are atomic (resp., 1-conjunctive).

Our complexity results are compactly summarized in Table 1. It turns out that deciding g-coherence, g-coherent consequence, and tight g-coherent consequence are complete for NP, co-NP, and D^{P}, respectively, while computing tight intervals under g-coherent entailment is $\mathrm{FP}^{\mathrm{NP}}$-complete. Note that hardness holds even in very restricted cases.

Problem	general case	1-conjunctive case
G-COHERENCE	NP-complete	NP-complete
G-COHERENT ENTAILMENT	co-NP-complete	co-NP-complete
TIGHT G-COHERENT ENTAILMENT (D)	D^{P}-complete	D^{P}-complete
TIGHT G-COHERENT ENTAILMENT (OPT)	$\mathrm{FP}^{\mathrm{NP}}$-complete	$\mathrm{FP}^{\mathrm{NP}}$-complete

Table 1: Complexity of probabilistic reasoning under coherence.

4.3 Detailed complexity results

The following theorem shows that deciding g-coherence is NP-complete. Here, membership in NP follows from the characterization of g-coherence given in Theorem 3.4 (b) and a small-model theorem from model-theoretic probabilistic reasoning. Hardness for NP is obtained by a polynomial reduction from the NP-complete graph 3-colorability problem.

Theorem 4.1 *Given a probabilistic knowledge base $KB = (L,P)$, deciding whether KB is g-coherent is* NP-*complete. Hardness holds even if L is empty and P is atomic.*

The next theorem shows that deciding g-coherent consequence is co-NP-complete. Membership in co-NP holds by the reduction to the complement of g-coherence (Theorem 2.3) and the NP-completeness of g-coherence (Theorem 4.1). Hardness for co-NP is proved by a polynomial reduction from the complement of the NP-complete graph 3-colorability problem.

Theorem 4.2 *Given a g-coherent probabilistic knowledge base $KB{=}(L,P)$ and a conditional constraint $(\beta|\alpha)[l,u]$, deciding whether $KB \mid\!\sim (\beta|\alpha)[l,u]$ is* co-NP-*complete. Hardness holds even if L is empty, P is 1-conjunctive, and $(\beta|\alpha)[l,u]$ is of the form $(b|\top)[l,1]$ with $b \in \Phi$.*

The next result shows that deciding tight g-coherent consequence is complete for $\mathrm{D^P}$. We obtain membership in $\mathrm{D^P}$, as tight g-coherent consequence can be expressed as the logical conjunction of g-coherence and g-coherent consequence, by Theorems 2.1, 2.3, and 2.4. Hardness for $\mathrm{D^P}$ holds by a polynomial reduction from the exact traveling salesman problem.

Theorem 4.3 *Given a g-coherent probabilistic knowledge base* $KB = (L, P)$ *and a conditional constraint* $(\beta|\alpha)[l, u]$*, deciding if* $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$ *is* $\mathrm{D^P}$*-complete. Hardness holds even if* L *is empty,* P *is 1-conjunctive, and* $(\beta|\alpha)[l, u]$ *is of the form* $(b|\top)[l, 1]$ *with* $b \in \Phi$.

The next result shows that computing tight g-coherent intervals is complete for $\mathrm{FP^{NP}}$. As for membership in $\mathrm{FP^{NP}}$, by Theorems 3.7 and 3.8, the tight g-coherent interval can be computed by first making a polynomial number of toleration checks, each of which is in NP, and then computing the tight logically entailed interval, which is in $\mathrm{FP^{NP}}$. Hardness for $\mathrm{FP^{NP}}$ holds by a polynomial reduction from the traveling salesman cost problem.

Theorem 4.4 *Given a g-coherent probabilistic knowledge base* $KB = (L, P)$ *and a conditional event* $\beta|\alpha$*, computing the unique* $l, u \in [0, 1]$ *such that* $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$ *is* $\mathrm{FP^{NP}}$*-complete. Hardness holds even if* L *is empty,* P *is 1-conjunctive, and* $(\beta|\alpha)[l, u]$ *is of the form* $(b|\top)[l, 1]$ *with* $b \in \Phi$.

4.4 Implications for computation

As shown in the previous section, all four analyzed problems in probabilistic reasoning under coherence are intractable, where deciding g-coherence and g-coherent consequence has the lowest complexity, and computing tight g-coherent bounds has the highest complexity.

As far as implementations are concerned, we thus cannot hope for algorithms that efficiently solve every problem instance. But we can still hope for efficient special-case, average-case, and approximation algorithms.

Furthermore, our results about the precise complexity of the analyzed problems give insight into which kinds of algorithms can be used for probabilistic reasoning under coherence. In particular, they show that there are polynomial-time translations of probabilistic reasoning under coherence into other reasoning problems, and thus existing algorithms and theorem provers might be used for its implementation. For example, deciding g-coherence can be polynomially translated into a classical SAT instance, which can then be solved using any of the sophisticated SAT packages (see e.g. [16]). Moreover, deciding g-coherent (resp., tight g-coherent) consequence can be polynomially reduced to one call (resp., two calls) of a SAT procedure. Finally, computing tight g-coherent intervals can be polynomially translated into, for example, computing a minimum nonnegative integer solution for a system of linear integer inequalities, which is known to be $\mathrm{FP^{NP}}$-hard.

5 Very efficient transformations

In this section, we present two transformations of problem instances of deciding g-coherence and of computing tight intervals under g-coherent entailment into smaller problem instances. In detail, these two transformations are (i) decomposing a problem instance, and (ii) removing inactive logical and conditional constraints from a problem instance (which only applies to the conjunctive case). In particular, we show that these two transformations can be done very efficiently (that is, in linear time).

5.1 Decomposition

Our first transformation is to decompose problem instances of deciding g-coherence and of computing tight intervals under g-coherent entailment into smaller problem instances. It is inspired by similar ideas from [18, 39].

We start by formally defining the notion of a decomposition of a probabilistic knowledge base. Let $KB = (L, P)$ be a probabilistic knowledge base, and let α be an event. Let $\{\Phi_1, \ldots, \Phi_k\}$ be the unique partition of $bas(KB) \cup bas(\alpha)$ such that (i) each member of $L \cup P$ is defined over some Φ_i, (ii) α is defined over some Φ_i, and (iii) $k \geq 1$ is maximal. For $i \in \{1, \ldots, k\}$, let KB_i denote the probabilistic knowledge base (L_i, P_i), where L_i (resp., P_i) is the set of all members of L (resp., P) that are defined over Φ_i. We then call $\{KB_1, \ldots, KB_k\}$ the *decomposition* of KB w.r.t. α, denoted $dec_\alpha(KB)$. We call the unique KB_i such that (i) $bas(\alpha) \subseteq \Phi_i$ and (ii) $i \in \{1, \ldots, k\}$ is minimal, the *relevant subbase* of KB w.r.t. α, denoted $rel_\alpha(KB)$.

The following theorem shows that $KB = (L, P)$ with $P \neq \emptyset$ is g-coherent iff every member $KB_i = (L_i, P_i)$ of the decomposition of KB w.r.t. $\top$ is g-coherent (resp., satisfiable) if $P_i \neq \emptyset$ (resp., $P_i = \emptyset$). Here, the g-coherence of KB immediately implies the g-coherence (resp., satisfiability) of the KB_i's. The converse follows from the fact that a model Pr of KB can be constructed from models Pr_i of the KB_i's by assuming probabilistic independence.

Theorem 5.1 *Let $KB = (L, P)$ be a probabilistic knowledge base with $P \neq \emptyset$. Then, KB is g-coherent iff every $KB_i = (L_i, P_i) \in dec_\top(KB)$ with $P_i \neq \emptyset$ (resp., $P_i = \emptyset$) is g-coherent (resp., satisfiable).*

The following example illustrates the above result.

Example 5.2 Let the probabilistic knowledge base $KB = (L, P)$ over the set of basic events $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{H}, \mathsf{K}\}$ be given as follows:

$$\begin{aligned} L &= \{\mathsf{A} \Leftarrow \mathsf{B}, \mathsf{H} \Leftarrow \mathsf{D}\}\,, \\ P &= \{(\mathsf{B}|\mathsf{A}\wedge\mathsf{C})[\tfrac{3}{4}, 1], (\neg\mathsf{A}\vee\mathsf{B}|\mathsf{C})[\tfrac{3}{5}, 1], (\mathsf{D}|\mathsf{H}\wedge\mathsf{K})[\tfrac{2}{3}, 1], (\mathsf{H}|\mathsf{D}\wedge\mathsf{K})[\tfrac{5}{8}, 1]\}\,. \end{aligned}$$

Then, the decomposition of KB w.r.t. $\top$ is given by $dec_\top(KB) = \{KB_1, KB_2\}$, where $KB_1 = (L_1, P_1)$ and $KB_2 = (L_2, P_2)$ are defined over $\Phi_1 = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ and $\Phi_2 = \{\mathsf{D}, \mathsf{H}, \mathsf{K}\}$, respectively, and given as follows:

$$\begin{aligned} L_1 &= \{\mathsf{A} \Leftarrow \mathsf{B}\}\,, \\ P_1 &= \{(\mathsf{B}|\mathsf{A}\wedge\mathsf{C})[\tfrac{3}{4}, 1], (\neg\mathsf{A}\vee\mathsf{B}|\mathsf{C})[\tfrac{3}{5}, 1]\}\,, \\ L_2 &= \{\mathsf{H} \Leftarrow \mathsf{D}\}\,, \\ P_2 &= \{(\mathsf{D}|\mathsf{H}\wedge\mathsf{K})[\tfrac{2}{3}, 1], (\mathsf{H}|\mathsf{D}\wedge\mathsf{K})[\tfrac{5}{8}, 1]\}\,. \end{aligned}$$

By Theorem 5.1, KB is g-coherent iff both KB_1 and KB_2 are g-coherent. As both KB_1 and KB_2 are indeed g-coherent, also KB is g-coherent. □

The next theorem shows that g-coherent entailment of $(\beta|\alpha)[l, u]$ from a g-coherent probabilistic knowledge base KB coincides with g-coherent entailment of $(\beta|\alpha)[l, u]$ from the relevant subbase of KB w.r.t. $\beta \wedge \alpha$. This result can be proved in a similar way as Theorem 5.1.

Theorem 5.3 *Let KB be a g-coherent probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Then,* *(a)* $KB \mathrel{|\!\!\sim} (\beta|\alpha)[l,u]$ *iff* $rel_{\beta\wedge\alpha}(KB) \mathrel{|\!\!\sim} (\beta|\alpha)[l,u]$.

(b) $KB \mathrel{|\!\!\sim}_{tight} (\beta|\alpha)[l,u]$ *iff* $rel_{\beta\wedge\alpha}(KB) \mathrel{|\!\!\sim}_{tight} (\beta|\alpha)[l,u]$.

We illustrate the above result by an example.

Example 5.4 Consider again the g-coherent probabilistic knowledge base $KB=(L,P)$ given in Example 5.2. Then, $rel_{\neg\mathsf{D}\wedge\mathsf{K}\wedge\mathsf{H}}(KB)$ is the probabilistic knowledge base KB_2 of Example 5.2. By Theorem 5.3, as KB is g-coherent, $KB \mathrel{|\!\!\sim}_{tight} (\neg\mathsf{D}\wedge\mathsf{K}|\mathsf{H})[l,u]$ iff $KB_2 \mathrel{|\!\!\sim}_{tight} (\neg\mathsf{D}\wedge\mathsf{K}|\mathsf{H})[l,u]$. Since $KB_2 \mathrel{|\!\!\sim}_{tight} (\neg\mathsf{D} \wedge \mathsf{K}|\mathsf{H})[0,\frac{1}{3}]$, it thus holds $KB \mathrel{|\!\!\sim}_{tight} (\neg\mathsf{D}\wedge\mathsf{K}|\mathsf{H})[0,\frac{1}{3}]$. $\Box$

The following result shows that both the decomposition and the relevant subbase of a probabilistic knowledge base can be computed very efficiently. It follows from a reduction to the problem of computing the connected components of a hypergraph, which can be done in linear time.

Proposition 5.5 *Given a probabilistic knowledge base* KB *and an event* α*, computing* $dec_\alpha(KB)$ *and* $rel_\alpha(KB)$ *can be done in linear time.*

5.2 Removing inactive constraints

We next present a transformation where some logical and conditional constraints are characterized as inactive and then removed. This transformation applies only to the conjunctive case. It is inspired by ideas from [36, 18]. We start by giving some preparative definitions.

An event ϕ is *conjunctive* iff ϕ is either $\top$ or a conjunction of basic events. A conditional constraint $(\psi|\phi)[l,u]$ (resp., conditional event $\psi|\phi$) is *conjunctive* iff ψ is a conjunction of basic events, and ϕ is a conjunctive event. A logical constraint $\psi \Leftarrow \phi$ is *Horn* (resp., *definite-Horn*) iff ψ is either $\perp$ or a basic event (resp., ψ is a basic event), and ϕ is a conjunctive event. A probabilistic knowledge base $KB=(L,P)$ is *conjunctive* iff L is a finite set of Horn logical constraints, and P is a finite set of conjunctive conditional constraints. For sets of Horn logical constraints L, we use L^+ to denote the set of all definite members of L. For sets of conjunctive conditional constraints P, denote by P^+ the set of all definite-Horn logical constraints $b \Leftarrow \phi$ such that $b \in \Phi$ occurs in the consequent of some $(\psi|\phi)[l,u] \in P$ with $l>0$.

Let $KB=(L,P)$ be a conjunctive probabilistic knowledge base, and let α be a conjunctive event. A basic event $b \in \Phi$ is *active* w.r.t. KB and α, iff $L^+ \cup P^+ \cup \{\alpha\} \models b$. An event ϕ (resp., a logical or conditional constraint F) is *active* w.r.t. KB and α iff all basic events in ϕ (resp., F) are active w.r.t. KB and α. An event (resp., a logical or conditional constraint) is *inactive* w.r.t. KB and α iff it is not active w.r.t. KB and α. We define $act_\alpha(KB)$ as the probabilistic knowledge base (L_a,P_a), where L_a (resp., P_a) is the set of all members of L (resp., P) that are active w.r.t. KB and α.

The following result shows that a conjunctive $KB=(L,P)$ is g-coherent iff (i) the active part of KB, with $\alpha = \bigwedge_{\psi|\phi \in \mathcal{E}_{KB}} \phi$, is g-coherent and (ii) the premise of each inactive $F \in P$ is satisfiable under the active part of L. Here, the g-coherence of KB immediately implies (i) and (ii), while the converse holds, as inactive basic events can always be assigned the probability zero, and thus be ignored, while checking g-coherence.

Theorem 5.6 *Let* $KB=(L,P)=(L,\{(\psi_i|\phi_i)[l_i,u_i] \mid i\in\{1,\ldots,n\}\})$ *be a conjunctive probabilistic knowledge base. Let* $act_{\phi_1\wedge\cdots\wedge\phi_n}(KB)=(L_a,P_a)$*. Then,* KB *is g-coherent iff (i)* (L_a,P_a) *is g-coherent and (ii)* $L_a \cup \{\phi\}$ *is satisfiable for all* $(\psi|\phi)[l,u] \in P \setminus P_a$.

The following example illustrates the above result.

Example 5.7 Let $KB = (L, P)$ over $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$, where $L = \{\mathsf{A} \Leftarrow \mathsf{B}\}$ and $P = \{(\mathsf{B}|\mathsf{A})[0.1, 0.2], (\mathsf{C}|\mathsf{B})[0.1, 0.2], (\mathsf{D}|\mathsf{C})[0, 0.2]\}$. Then,

$$L^+ \cup P^+ \cup \{\mathsf{A}\wedge\mathsf{B}\wedge\mathsf{C}\} \quad = \quad \{\mathsf{A} \Leftarrow \mathsf{B}, \mathsf{B} \Leftarrow \mathsf{A}, \mathsf{C} \Leftarrow \mathsf{B}, \mathsf{A}\wedge\mathsf{B}\wedge\mathsf{C}\}\,.$$

Thus, A, B, and C are active w.r.t. KB and $\mathsf{A}\wedge\mathsf{B}\wedge\mathsf{C}$, and D is not active w.r.t. KB and $\mathsf{A}\wedge\mathsf{B}\wedge\mathsf{C}$. Hence, $act_{\mathsf{A}\wedge\mathsf{B}\wedge\mathsf{C}}(KB) = (L_a, P_a)$, where $L_a = \{\mathsf{A} \Leftarrow \mathsf{B}\}$ and $P_a = \{(\mathsf{B}|\mathsf{A})[0.1, 0.2], (\mathsf{C}|\mathsf{B})[0.1, 0.2]\}$. By Theorem 5.6, KB is g-coherent iff (L_a, P_a) is g-coherent and $\{\mathsf{A} \Leftarrow \mathsf{B}, \mathsf{C}\}$ is satisfiable. As indeed (L_a, P_a) is g-coherent and $\{\mathsf{A} \Leftarrow \mathsf{B}, \mathsf{C}\}$ is satisfiable, also KB is g-coherent. $\Box$

The next result shows that, in the conjunctive case, g-coherent entailment of $(\beta|\alpha)[l, u]$ from some g-coherent KB coincides with g-coherent entailment of $(\beta|\alpha)[l, u]$ from the active subbase of KB w.r.t. $\alpha \wedge \beta$. This result can be proved in a similar way as Theorem 5.6.

Theorem 5.8 *Let KB be a g-coherent conjunctive probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conjunctive conditional constraint. Then,*

(a) $KB \mathrel{|\!\sim} (\beta|\alpha)[l, u]$ *iff* $act_{\alpha\wedge\beta}(KB) \mathrel{|\!\sim} (\beta|\alpha)[l, u]$.

(b) $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$ *iff* $act_{\alpha\wedge\beta}(KB) \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$.

The above result is illustrated by the following example.

Example 5.9 Consider the probabilistic knowledge base $KB = (L, P)$ over $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{F}\}$, where $L = \{\mathsf{D} \Leftarrow \mathsf{A}\}$ and $P = \{(\mathsf{A}\wedge\mathsf{C}|\mathsf{D})[0.4, 0.5], (\mathsf{A}|\mathsf{D}\wedge\mathsf{F})[0.6, 0.8], (\mathsf{B}\wedge\mathsf{C}|\mathsf{F})[0.1, 0.8]\}$. It easy to verify that KB is g-coherent. To compute $l, u \in [0, 1]$ such that $KB \mathrel{|\!\sim}_{tight} (\mathsf{A}|\mathsf{D})[l, u]$, consider

$$L^+ \cup P^+ \cup \{\mathsf{A}\wedge\mathsf{D}\} \quad = \quad \{\mathsf{D} \Leftarrow \mathsf{A}, \mathsf{A} \Leftarrow \mathsf{D}, \mathsf{C} \Leftarrow \mathsf{D}, \mathsf{A} \Leftarrow \mathsf{D}\wedge\mathsf{F}, \mathsf{B} \Leftarrow \mathsf{F}, \mathsf{C} \Leftarrow \mathsf{F}, \mathsf{A}\wedge\mathsf{D}\}\,.$$

Observe that A, C, and D are active w.r.t. KB and $\mathsf{A}\wedge\mathsf{D}$, and that B and F are not active w.r.t. KB and $\mathsf{A}\wedge\mathsf{D}$. Thus, $act_{\mathsf{A}\wedge\mathsf{D}}(KB) = (L_a, P_a)$, where $L_a = \{\mathsf{D} \Leftarrow \mathsf{A}\}$ and $P_a = \{(\mathsf{A}\wedge\mathsf{C}|\mathsf{D})[0.4, 0.5]\}$. By Theorem 5.8, the bounds $l, u \in [0, 1]$ such that $KB \mathrel{|\!\sim}_{tight} (\mathsf{A}|\mathsf{D})[l, u]$ are given by $act_{\mathsf{A}\wedge\mathsf{D}}(KB) \mathrel{|\!\sim}_{tight} (\mathsf{A}|\mathsf{D})[l, u]$. Numerically, we obtain $[l, u] = [0.4, 1]$. $\Box$

The following result shows that the active subbase of a conjunctive probabilistic knowledge base can be computed very efficiently. It follows from the well-known result that the set of all basic events that are logically implied by a finite set of Horn logical constraints can be computed in linear time.

Proposition 5.10 *Given a conjunctive probabilistic knowledge base KB and a conjunctive event α, computing $act_{\alpha}(KB)$ can be done in linear time.*

6 New algorithms

Our results on the relationship between model-theoretic probabilistic reasoning and probabilistic reasoning under coherence in Section 3 open a new perspective on algorithms for deciding g-coherence, for deciding g-coherent and tight g-coherent consequence, and for computing tight intervals under g-coherent entailment.

They show how these problems can be reduced to standard tasks in model-theoretic probabilistic reasoning. That is, we obtain new algorithms for these problems, which are based on subprocedures for standard reasoning tasks in model-theoretic probabilistic logic. In the rest of this section, we first describe the new algorithms, and we then discuss the tasks in model-theoretic probabilistic reasoning.

The key features of these new algorithms can be briefly described as follows. First, they are surprisingly simple compared to existing algorithms. This helps to more deeply understand the semantic properties of g-coherence and of g-coherent entailment (see [5] for more details). Second, they show that efficient techniques for model-theoretic probabilistic reasoning can be applied for probabilistic reasoning under coherence. We will illustrate this aspect in Sections 7 and 8 by describing some such recent techniques.

6.1 G-coherence

As for g-coherence, a previous algorithm by Gilio [25] is reformulated using terminology from the area of default reasoning. Interestingly, the resulting algorithm for deciding g-coherence is closely related to an algorithm by Goldszmidt and Pearl [28] for checking ε-consistency in default reasoning.[1]

Let $KB = (L, P)$ be a probabilistic knowledge base, and let $\beta|\alpha$ be a conditional event. By Theorem 3.4 (a), deciding whether KB is g-coherent can be done by Algorithm g-coherence (see Fig. 1) as follows. We first compute the set Δ_0 of all elements in $P_0 = P$ tolerated by P_0 under L. We then compute the set Δ_1 of all elements in $P_1 = P_0 \setminus \Delta_0$ tolerated by P_1 under L. We iterate this procedure until no element in $P_k = P_{k-1} \setminus \Delta_{k-1}$ is tolerated by P_k under L. Then, KB is g-coherent iff P_k is empty.

Algorithm g-coherence

Input: Probabilistic knowledge base $KB = (L, P)$.
Output: "Yes", if KB is g-coherent, otherwise "No".

1. $Q := P$;
2. **repeat**
3. $\quad \Delta := \{F \in Q \mid F$ is tolerated under L by $Q\}$;
4. $\quad Q := Q \setminus \Delta$
5. **until** $\Delta = \emptyset$;
6. **if** $Q = \emptyset$ **then return** "Yes"
7. $\quad$ **else return** "No".

Figure 1: Algorithm g-coherence.

The following two examples illustrate Algorithm g-coherence.

Example 6.1 Let the probabilistic knowledge base $KB = (L, P)$ over $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$ be given by:

$$(L, P) \quad = \quad (\emptyset, \{(\mathsf{B}|\mathsf{A})[0, 0.2], (\mathsf{C}|\mathsf{A}\wedge\mathsf{B})[0, 0.2], (\mathsf{D}|\mathsf{C})[0, 0.2]\}) \, .$$

Is KB g-coherent? Using Algorithm g-coherence, we compute the set of all members of P that are tolerated by P under L. As this set already coincides with P, it then follows that KB is g-coherent. □

[1] Note that a relationship between the algorithms in [25] and [28] was suggested first by Didier Dubois (personal communication).

Example 6.2 Let the probabilistic knowledge base $KB = (L, P)$ over $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{I}, \mathsf{J}\}$ be given as follows:

$$\begin{aligned} L &= \{\mathsf{G} \Leftarrow \mathsf{J}, \mathsf{H} \Leftarrow \mathsf{J}, \mathsf{I} \Leftarrow \mathsf{J}, \mathsf{A} \Leftarrow \mathsf{G}, \mathsf{B} \Leftarrow \mathsf{G}, \mathsf{C} \Leftarrow \mathsf{G}, \mathsf{D} \Leftarrow \mathsf{I}, \mathsf{E} \Leftarrow \mathsf{I}, \mathsf{F} \Leftarrow \mathsf{I}\}\,, \\ P &= \{(\mathsf{J}|\mathsf{G})[0.2, 1], (\mathsf{G}|\mathsf{A})[0.2, 1], (\mathsf{G}|\mathsf{B})[0.2, 1], (\mathsf{G}|\mathsf{C})[0.2, 1], (\mathsf{J}|\mathsf{H})[0.2, 1], \\ &\quad (\mathsf{J}|\mathsf{I})[0.2, 1], (\mathsf{I}|\mathsf{D})[0.2, 1], (\mathsf{I}|\mathsf{E})[0.2, 1], (\mathsf{I}|\mathsf{F})[0.2, 1]\}\,. \end{aligned}$$

It is easy to verify that every member of P is tolerated by P under L. This shows that KB is g-coherent. $\Box$

6.2 G-coherent entailment

As for g-coherent entailment, we have the important result that deciding g-coherent (resp., tight g-coherent) consequence can be reduced to first performing a g-coherence check, and then deciding logical (resp., tight logical) consequence. Similarly, a tight interval under g-coherent entailment can be computed by first checking g-coherence, and then computing a tight interval under logical entailment. In this section, we describe only how to compute tight intervals under g-coherent entailment. The algorithms for deciding g-coherent (resp., tight g-coherent) consequence are similar.

Let $KB = (L, P)$ be a probabilistic knowledge base, and let $\beta|\alpha$ be a conditional event. By Theorems 3.7 and 3.8, the real numbers $l, u \in [0, 1]$ such that $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$ can be computed by Algorithm `tight-g-coherent-consequence` (see Fig. 2) as follows. We first compute the set Δ_0 of all elements in $P_0 = P$ that are tolerated by P_0 under $L \cup \{\bot \Leftarrow \alpha\}$. We then compute the set Δ_1 of all elements in $P_1 = P_0 \setminus \Delta_0$ that are tolerated by P_1 under $L \cup \{\bot \Leftarrow \alpha\}$. We iterate this procedure until no member of $P_k{=}P_{k-1}\setminus\Delta_{k-1}$ is tolerated by P_k under $L \cup \{\bot \Leftarrow \alpha\}$. Then, we compute the real numbers $l, u \in [0, 1]$ such that $L \cup P_k \models_{tight} (\beta|\alpha)[l, u]$. They are the requested bounds $l, u \in [0, 1]$ such that $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$.

Algorithm tight-g-coherent-consequence

Input: Probabilistic knowledge base $KB = (L, P)$ and two events α and β.
Output: Unique pair $(l, u) \in [0, 1]^2$ such that $KB \mathrel{|\!\sim}_{tight} (\beta|\alpha)[l, u]$.

1. $Q := P$;
2. **repeat**
3. $\quad\Delta := \{F \in Q \mid F$ is tolerated under $L \cup \{\bot \Leftarrow \alpha\}$ by $Q\}$;
4. $\quad Q := Q \setminus \Delta$
5. **until** $\Delta = \emptyset$;
6. compute $l, u \in [0, 1]$ such that $L \cup Q \models_{tight} (\beta|\alpha)[l, u]$;
7. **return** (l, u).

Figure 2: Algorithm `tight-g-coherent-consequence`.

The following two examples illustrate the above algorithm.

Example 6.3 Consider again the probabilistic knowledge base $KB = (L, P)$ given in Example 6.1. Suppose we want to know the real numbers $l, u \in [0, 1]$ such that $KB \mathrel{|\!\sim}_{tight} (\mathsf{B} \wedge \mathsf{C} \wedge \mathsf{D}|\mathsf{A})[l, u]$. Using Algorithm `tight-g-coherent- consequence`, we then first compute the set of all members of P that are tolerated by P under $L \cup \{\bot \Leftarrow \mathsf{A}\}$. That is, we ask the following questions:

(1.1) Has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr such that $Pr(\mathsf{A}) > 0$?

(1.2) Has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr such that $Pr(\mathsf{A}{\wedge}\mathsf{B}) > 0$?

(1.3) Has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr such that $Pr(\mathsf{C}) > 0$?

The answers to (1.1) and (1.2) are (trivially) "No", while the answer to (1.3) is "Yes". Thus, we continue by computing the set of all $F \in P' = P \setminus \{(\mathsf{D}|\mathsf{C})[0, 0.2]\}$ tolerated by P' under $L \cup \{\bot \Leftarrow \mathsf{A}\}$ by asking:

(2.1) Has $P' \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr such that $Pr(\mathsf{A}) > 0$?

(2.2) Has $P' \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr such that $Pr(\mathsf{A}{\wedge}\mathsf{B}) > 0$?

Since the answers to (2.1) and (2.2) are both (trivially) "No", the interval $[l, u] \subseteq [0, 1]$ such that $KB \;|\!\!\sim_{tight} (\mathsf{B}{\wedge}\mathsf{C}{\wedge}\mathsf{D}|\mathsf{A})[l, u]$ is given by $L \cup P' \models_{tight} (\mathsf{B}{\wedge}\mathsf{C}{\wedge}\mathsf{D}|\mathsf{A})[l, u]$. Numerically, $[l, u]$ is given by $[0, 0.04]$. $\Box$

We give an example where a tight g-coherent consequence coincides with a tight logical consequence.

Example 6.4 Consider again the probabilistic knowledge base $KB = (L, P)$ of Example 6.2. Suppose we want to know the real numbers $l, u \in [0, 1]$ such that $(\mathsf{J}|\mathsf{A})[l, u]$ is a tight g-coherent consequence of KB. Since no member of P is tolerated by P under $L \cup \{\bot \Leftarrow \mathsf{A}\}$, the interval $[l, u] \subseteq [0, 1]$ such that $KB \;|\!\!\sim_{tight} (\mathsf{J}|\mathsf{A})[l, u]$ is determined by $L \cup P \models_{tight} (\mathsf{J}|\mathsf{A})[l, u]$. Numerically, this interval $[l, u]$ is given by $[0.04, 1]$. $\Box$

6.3 Tasks in probabilistic logic

The new algorithms in Sections 6.1 and 6.2 reduce the problems of deciding g-coherence and of computing tight intervals under g-coherent entailment to the two standard tasks in model-theoretic probabilistic reasoning of deciding the existence of models that associate with a given event a positive probability (POS) and of computing tight intervals under logical entailment (TLC).

Task POS can be reduced to deciding if a system of linear constraints is solvable. This is more formally expressed by the following well-known result.

Theorem 6.5 *Let $KB = (L, P)$ be a probabilistic knowledge base, and let α be an event. Let $R = R_L(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$. Then, KB has a model Pr such that $Pr(\alpha) > 0$ iff the system of linear constraints* (3) *over the variables y_r $(r \in R)$ is solvable (see Fig. 3).*

Task TLC is reducible to computing the optimal values of two linear programs, as the next well-known result shows.

Theorem 6.6 *Let $KB = (L, P)$ be a probabilistic knowledge base, and let $\beta|\alpha$ be a conditional event. Let $R = R_L(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$. Suppose that KB has a model Pr such that $Pr(\alpha) > 0$. Then, l (resp., u) such that $KB \models_{tight} (\beta|\alpha)[l, u]$ is the optimal value of the following linear program over the variables y_r $(r \in R)$:*

$$\textit{minimize (resp., maximize)} \sum_{r \in R,\, r \models \beta \wedge \alpha} y_r \quad \textit{subject to } (3)\,.$$

The following example illustrates the above two results.

Example 6.7 Consider again the probabilistic knowledge base $KB = (L, P)$ given in Example 6.1. Assume first that we want to reply question (1.3). That is, has $P \cup \{\bot \Leftarrow \mathsf{A}\}$ a model Pr such that $Pr(\mathsf{C}) > 0$? We now apply Theorem 6.5. The set $\mathcal{E} = \mathcal{E}_{KB} \cup \{\mathsf{C}|\mathsf{C}\}$ is given as follows:

$$\mathcal{E} \;=\; \{\mathsf{B}|\mathsf{A}, \mathsf{C}|\mathsf{A}{\wedge}\mathsf{B}, \mathsf{D}|\mathsf{C}, \mathsf{C}|\mathsf{C}\}\,,$$

$$\begin{aligned}
\sum_{\substack{r \in R \\ r \models \neg\psi\wedge\phi}} -l\, y_r + \sum_{\substack{r \in R \\ r \models \psi\wedge\phi}} (1-l)\, y_r &\geq 0 \quad \text{(for all } (\psi|\phi)[l,u] \in P,\ l>0) \\
\sum_{\substack{r \in R \\ r \models \neg\psi\wedge\phi}} u\, y_r + \sum_{\substack{r \in R \\ r \models \psi\wedge\phi}} (u-1)\, y_r &\geq 0 \quad \text{(for all } (\psi|\phi)[l,u] \in P,\ u<1) \\
\sum_{\substack{r \in R \\ r \models \alpha}} y_r &= 1 \\
y_r &\geq 0 \quad \text{(for all } r \in R)
\end{aligned} \tag{3}$$

Figure 3: System of linear constraints for Theorems 6.5 and 6.6.

and the set $R = R_{L\cup\{\bot\Leftarrow\mathsf{A}\}}(\mathcal{E})$ is given as follows:

$$R \;\hat{=}\; \{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}, \bar{\mathsf{A}}\mathsf{C}\mathsf{D}\}\,.$$

Thus, the system of linear constraints (3) is given by:

$$\begin{aligned}
0.2\, y_{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}} - 0.8\, y_{\bar{\mathsf{A}}\mathsf{C}\mathsf{D}} &\geq 0 \\
y_{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}} + y_{\bar{\mathsf{A}}\mathsf{C}\mathsf{D}} &= 1 \\
y_{\bar{\mathsf{A}}\mathsf{C}\bar{\mathsf{D}}}, y_{\bar{\mathsf{A}}\mathsf{C}\mathsf{D}} &\geq 0\,.
\end{aligned} \tag{4}$$

As (4) is solvable, the reply to question (1.3) is “Yes”. Suppose next that we want to compute $l, u \in [0,1]$ such that $L \cup P' \models_{tight} (\mathsf{B}\wedge\mathsf{C}\wedge\mathsf{D}|\mathsf{A})[l,u]$, where $L = \emptyset$ and $P' = \{(\mathsf{B}|\mathsf{A})[0,0.2], (\mathsf{C}|\mathsf{A}\wedge\mathsf{B})[0,0.2]\}$. Then,

$$\mathcal{E} \;=\; \mathcal{E}_{(L,P')} \cup \{\mathsf{B}\wedge\mathsf{C}\wedge\mathsf{D}|\mathsf{A}\} \;=\; \{\mathsf{B}|\mathsf{A}, \mathsf{C}|\mathsf{A}\wedge\mathsf{B}, \mathsf{B}\wedge\mathsf{C}\wedge\mathsf{D}|\mathsf{A}\}\,,$$

and the set $R = R_L(\mathcal{E})$ is given as follows:

$$R \;\hat{=}\; \{\mathsf{A}\bar{\mathsf{B}}, \mathsf{A}\mathsf{B}\bar{\mathsf{C}}, \mathsf{ABCD}, \mathsf{ABC}\bar{\mathsf{D}}\}\,.$$

Thus, the system of linear constraints (3) is given by:

$$\begin{aligned}
0.2\, y_{\mathsf{A}\bar{\mathsf{B}}} - 0.8\, (y_{\mathsf{AB}\bar{\mathsf{C}}} + y_{\mathsf{ABCD}} + y_{\mathsf{ABC}\bar{\mathsf{D}}}) &\geq 0 \\
0.2\, y_{\mathsf{AB}\bar{\mathsf{C}}} - 0.8\, (y_{\mathsf{ABCD}} + y_{\mathsf{ABC}\bar{\mathsf{D}}}) &\geq 0 \\
y_{\mathsf{A}\bar{\mathsf{B}}} + y_{\mathsf{AB}\bar{\mathsf{C}}} + y_{\mathsf{ABCD}} + y_{\mathsf{ABC}\bar{\mathsf{D}}} &= 1 \\
y_{\mathsf{A}\bar{\mathsf{B}}}, y_{\mathsf{AB}\bar{\mathsf{C}}}, y_{\mathsf{ABCD}}, y_{\mathsf{ABC}\bar{\mathsf{D}}} &\geq 0\,.
\end{aligned} \tag{5}$$

Observe that (5) is solvable. By Theorem 6.5, it thus follows that $L \cup P'$ has a model Pr with $Pr(\mathsf{A}) > 0$. Thus, by Theorem 6.6, the requested l (resp., u) is the optimal value of the following linear program:

$$\text{minimize (resp., maximize) } y_{\mathsf{ABCD}} \text{ subject to } (5)\,. \;\Box$$

7 Efficient transformations in model-theoretic probabilistic logic

As we reduce deciding g-coherence and computing tight intervals under g-coherent entailment to tasks in probabilistic logic, we can now apply all the techniques that have been developed there for an increased efficiency.

In this section, we summarize some of such techniques. All these methods aim at reducing the exponential number of variables in the system of linear constraints (3). This can be achieved by making hidden logical knowledge explicit, by removing vacuous conditional constraints, by decomposing a given probabilistic knowledge base, and by removing inactive constraints.

7.1 Adding logical constraints

We now describe a technique taken from [36], which increases a probabilistic knowledge base by trivially entailed logical constraints. Any such newly derived logical constraint reduces the number of variables in (3).

In the sequel, let $KB = (L, P)$ be a probabilistic knowledge base. We now define the operator $S^{\star}_{P}$, which adds to L logical constraints that trivially follow from KB. The operator S_P adds to a set of logical constraints K every logical constraint $\bot \Leftarrow \phi$ such that either

(i) $K \models \bot \Leftarrow \psi \wedge \phi$, $(\psi|\phi)[l,u] \in P$, and $l > 0$, or

(ii) $K \models \psi \Leftarrow \phi$, $(\psi|\phi)[l,u] \in P$, and $u < 1$.

For sets of logical constraints K, we next define $S^0_P(K) = K$ and $S^{n+1}_P(K) = S_P(S^n_P(K))$ for all $n > 0$. We then define $S^{\star}_P(K) = S^n_P(K)$, where n is the least $n \geq 0$ such that $S^n_P(K) = S^{n+1}_P(K)$.

The following theorem shows that, as far as the task POS in Section 6.3 is concerned, the set of logical constraints L can be replaced by $S^{\star}_P(L)$. It follows from $L \cup P$ being logically equivalent to $S^{\star}_P(L) \cup P$.

Theorem 7.1 *Let $KB = (L, P)$ be a probabilistic knowledge base, and let α be an event. Suppose $KB^{\star} = (S^{\star}_P(L), P)$. Then, KB has a model Pr with $Pr(\alpha) > 0$ iff $KB^{\star}$ has a model Pr with $Pr(\alpha) > 0$.*

The following example illustrates the above result.

Example 7.2 Let $KB = (L, P)$ over $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, where $L = \{\bot \Leftarrow \mathsf{B} \wedge \mathsf{C}\}$ and $P = \{(\mathsf{A}|\mathsf{B})[0.4, 0.5], (\mathsf{B}|\mathsf{C})[0.6, 0.8]\}$. Then, we obtain $S^{\star}_P(L) = L \cup \{\bot \Leftarrow \mathsf{C}\}$, and thus $KB^{\star} = (\{\bot \Leftarrow \mathsf{B} \wedge \mathsf{C}, \bot \Leftarrow \mathsf{C}\}, P)$. □

The next theorem shows that a similar result holds for the problem TLC. It can be proved in a similar way as Theorem 7.1.

Theorem 7.3 *Let $KB = (L, P)$ be a probabilistic knowledge base, and let $(\beta|\alpha)[l,u]$ be a conditional constraint. Suppose $KB^{\star} = (S^{\star}_P(L), P)$. Then, $KB \models_{tight} (\beta|\alpha)[l,u]$ iff $KB^{\star} \models_{tight} (\beta|\alpha)[l,u]$.*

The following result shows that, in the conjunctive case, computing $S^{\star}_P(L)$ can be done in polynomial time. It follows from the well-known result that for finite sets of Horn logical constraints K and conjunctive $\psi|\phi$, deciding if $K \models \bot \Leftarrow \psi \wedge \phi$ (resp., $K \models \psi \Leftarrow \phi$) holds can be done in linear time.

Proposition 7.4 *Given a conjunctive probabilistic knowledge base $KB = (L, P)$, computing $S^{\star}_P(L)$ can be done in polynomial time.*

7.2 Removing vacuous conditional constraints

Another technique from [36] is to remove from a probabilistic knowledge base conditional constraints that are vacuous by our logical knowledge.

Let $KB = (L, P)$ be a probabilistic knowledge base. A conditional constraint $(\psi|\phi)[l, u]$ is *vacuous* under L iff either (i) $L \models \bot \Leftarrow \phi$, or (ii) $L \models \bot \Leftarrow \psi \wedge \phi$ and $l = 0$, or (iii) $L \models \psi \Leftarrow \phi$ and $u = 1$, or (iv) $l = 0$ and $u = 1$. We use $vac_L(P)$ to denote the set of all vacuous members of P under L.

The following theorem shows that KB has a model Pr with $Pr(\alpha) > 0$ iff its subbase without vacuous conditional constraints $KB^\star$ has such a model. It follows from KB and $KB^\star$ being logically equivalent.

Theorem 7.5 *Let $KB = (L, P)$ be a probabilistic knowledge base, and let α be an event. Suppose $KB^\star = (L, P \setminus vac_L(P))$. Then, KB has a model Pr with $Pr(\alpha) > 0$ iff $KB^\star$ has a model Pr with $Pr(\alpha) > 0$.*

We give an illustrative example of the above result.

Example 7.6 Consider $KB = (L, P)$ over $\Phi = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, where $L = \{\bot \Leftarrow \mathsf{C}\}$ and $P = \{(\mathsf{A}|\mathsf{B})[0.4, 0.5], (\mathsf{B}|\mathsf{C})[0.6, 0.8]\}$. Then, $vac_L(P) = \{(\mathsf{B}|\mathsf{C})[0.6, 0.8]\}$, and thus $KB^\star = (\{\bot \Leftarrow \mathsf{C}\}, \{(\mathsf{A}|\mathsf{B})[0.4, 0.5]\})$. $\Box$

As KB and $KB^\star$ are logically equivalent, it also follows that logical entailment from KB coincides with logical entailment from $KB^\star$.

Theorem 7.7 *Let $KB = (L, P)$ be a probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint. Suppose $KB^\star = (L, P \setminus vac_L(P))$. Then, $KB \models_{tight} (\beta|\alpha)[l, u]$ iff $KB^\star \models_{tight} (\beta|\alpha)[l, u]$.*

The following result shows that, in the conjunctive case, computing all conditional constraints in P that are vacuous under L can be done in polynomial time. It can be proved in a similar way as Proposition 7.4.

Proposition 7.8 *Given a conjunctive probabilistic knowledge base $KB = (L, P)$, computing $vac_L(P)$ can be done in polynomial time.*

7.3 Decomposition

We now describe how instances of the problems POS and TLC can be decomposed into smaller problem instances. This transformation is similar to the decomposition described in Section 5.1 for the notions of g-coherence and of g-coherent entailment. It is based on ideas from [39].

The following theorem shows that KB has a model Pr such that $Pr(\alpha) > 0$ iff the relevant subbase of KB w.r.t. α has such a model and all the other subbases in the decomposition of KB w.r.t. α are satisfiable.

Theorem 7.9 *Let KB be a probabilistic knowledge base, and let α be an event. Then, KB has a model Pr with $Pr(\alpha) > 0$ iff*

(i) $rel_\alpha(KB)$ has a model Pr with $Pr(\alpha) > 0$, and

(ii) every $KB_i \in dec_\alpha(KB) \setminus \{rel_\alpha(KB)\}$ is satisfiable.

Similarly, if KB has a model Pr such that $Pr(\alpha) > 0$, then logical entailment of $(\beta|\alpha)[l, u]$ from KB coincides with logical entailment of $(\beta|\alpha)[l, u]$ from the relevant subbase of KB w.r.t. $\beta \wedge \alpha$.

Theorem 7.10 *Let KB be a probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conditional constraint such that $KB \not\models (\alpha|\top)[0, 0]$. Then, $KB \models_{tight} (\beta|\alpha)[l, u]$ iff $rel_{\beta \wedge \alpha}(KB) \models_{tight} (\beta|\alpha)[l, u]$.*

7.4 Removing inactive constraints

As described in Section 5.1 for the notions of g-coherence and of g-coherent entailment, we may also remove inactive logical and conditional constraints when solving the problems POS and TLC. This transformation of instances of the problems POS and TLC is taken from [36].

The following theorem shows that KB has a model Pr with $Pr(\alpha) > 0$ iff the active subbase of KB w.r.t. α has such a model.

Theorem 7.11 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and let α be a conjunctive event. Then, KB has a model Pr with $Pr(\alpha) > 0$ iff $act_\alpha(KB)$ has a model Pr with $Pr(\alpha) > 0$.*

Similarly, logical entailment of $(\beta|\alpha)[l, u]$ from KB coincides with logical entailment of $(\beta|\alpha)[l, u]$ from the active subbase of KB w.r.t. $\beta \wedge \alpha$.

Theorem 7.12 *Suppose $KB = (L, P)$ is a conjunctive probabilistic knowledge base, and let $(\beta|\alpha)[l, u]$ be a conjunctive conditional constraint. Then, $KB \models_{tight} (\beta|\alpha)[l, u]$ iff $act_{\alpha\wedge\beta}(KB) \models_{tight} (\beta|\alpha)[l, u]$.*

8 Decomposable variables in model-theoretic probabilistic logic

In the conjunctive case, the number of variables in the systems of linear constraints for Theorems 6.5 and 6.6 can be reduced by a technique from [33]. The main idea behind this technique is that we actually do not need the fine grain structure of $R_L(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$ (resp., $R_L(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$), and thus some variables can be removed. We first describe the original formulation given in [33] and then present two new equivalent characterizations.

8.1 Closure operator on sets of events

We now recall from [33] the characterization of a reduced set of variables for Theorems 6.5 and 6.6 in the conjunctive case. We first give some preparative definitions. In the sequel, let L be a finite set of Horn logical constraints, and let $\mathcal{E}$ be a finite set of conjunctive conditional events.

We define $con(\mathcal{E}) = \bigcup\{\{\psi \wedge \phi, \phi\} \mid \psi|\phi \in \mathcal{E}\}$. The operator T_L associates with each $C \subseteq con(\mathcal{E})$ the set of all $\phi \in con(\mathcal{E})$ such that $L \cup C \models \phi$. We denote by $\mathcal{C}^\bullet_L(\mathcal{E})$ the least set of subsets of $con(\mathcal{E})$ such that:

(i) if $\phi \in con(\mathcal{E})$ and $L \cup \{\phi\}$ is satisfiable, then $T_L(\{\phi\}) \in \mathcal{C}^\bullet_L(\mathcal{E})$;

(ii) if $C_1, C_2 \in \mathcal{C}^\bullet_L(\mathcal{E})$, $C_1 \cap C_2 \neq \emptyset$, and $L \cup C_1 \cup C_2$ is satisfiable, then $T_L(C_1 \cup C_2) \in \mathcal{C}^\bullet_L(\mathcal{E})$.

Let $R^\bullet_L(\mathcal{E})$ denote the set of all r_C with $C \in \mathcal{C}^\bullet_L(\mathcal{E})$, where r_C associates with each $\psi|\phi \in \mathcal{E}$ an element of $\{\psi \wedge \phi, \neg\psi \wedge \phi, \neg\phi\}$ as follows:

$$r_C(\psi|\phi) \quad = \quad \begin{cases} \psi \wedge \phi & \text{if } \psi \in C \text{ and } \phi \in C \\ \neg\psi \wedge \phi & \text{if } \psi \notin C \text{ and } \phi \in C \\ \neg\phi & \text{if } \psi \notin C. \end{cases}$$

The following result from [33] shows that we can use $R^\bullet_L(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$ instead of $R_L(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$ as index set for the variables in Theorem 6.5.

Theorem 8.1 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and let α be a conjunctive event. Let $R = R^{\bullet}_{L}(\mathcal{E}_{KB} \cup \{\alpha|\alpha\})$. Then, KB has a model Pr with $Pr(\alpha) > 0$ iff the system of linear constraints (3) over the variables y_r $(r \in R)$ is solvable (see Fig. 3).*

The next result from [33] shows that we can use $R^{\bullet}_{L}(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$ instead of $R_L(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$ as index set for the variables in Theorem 6.6.

Theorem 8.2 *Let $KB = (L, P)$ be a conjunctive probabilistic knowledge base, and let $\beta|\alpha$ be a conjunctive conditional event. Let $R = R^{\bullet}_{L}(\mathcal{E}_{KB} \cup \{\beta|\alpha\})$. Suppose that KB has a model Pr with $Pr(\alpha) > 0$. Then, l (resp., u) such that $KB \models_{tight} (\beta|\alpha)[l, u]$ is given by the optimal value of the following linear program over the variables y_r $(r \in R)$:*

$$\textit{minimize (resp., maximize)} \sum_{r \in R,\, r \models \beta \wedge \alpha} y_r \quad \textit{subject to } (3)\,.$$

We give an example where the new index set $R^{\bullet}_{L}(\mathcal{E})$ is much smaller than the previous one $R_L(\mathcal{E})$.

Example 8.3 Consider again the probabilistic knowledge base $KB = (L, P)$ given in Example 6.2. Assume that we ask whether KB has a model Pr such that $Pr(\mathsf{A}) > 0$. Then, $\mathcal{E} = \mathcal{E}_{KB} \cup \{\mathsf{A}|\mathsf{A}\}$ is given as follows:

$$\mathcal{E} \;=\; \{\mathsf{J}|\mathsf{G}, \mathsf{G}|\mathsf{A}, \mathsf{G}|\mathsf{B}, \mathsf{G}|\mathsf{C}, \mathsf{J}|\mathsf{H}, \mathsf{J}|\mathsf{I}, \mathsf{I}|\mathsf{D}, \mathsf{I}|\mathsf{E}, \mathsf{I}|\mathsf{F}, \mathsf{A}|\mathsf{A}\}\,.$$

We observe that $R_L(\mathcal{E})$ contains 162 elements, while $\mathcal{C}^{\bullet}_{L}(\mathcal{E})$ contains only 10 elements and is given by:

$$\begin{aligned}\mathcal{C}^{\bullet}_{L}(\mathcal{E}) \;=\; \{&\{\mathsf{A}\}, \{\mathsf{B}\}, \{\mathsf{C}\}, \{\mathsf{D}\}, \{\mathsf{E}\}, \{\mathsf{F}\}, \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{G}, \mathsf{AG}, \mathsf{BG}, \mathsf{CG}\}, \{\mathsf{H}\},\\ &\{\mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{I}, \mathsf{DI}, \mathsf{EI}, \mathsf{FI}\}, \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{I}, \mathsf{AG}, \mathsf{BG}, \mathsf{CG},\\ &\mathsf{DI}, \mathsf{EI}, \mathsf{FI}, \mathsf{GJ}, \mathsf{HJ}, \mathsf{IJ}\}\}\,.\end{aligned}$$

Hence, also $R^{\bullet}_{L}(\mathcal{E})$ contains only 10 elements:

$$\begin{aligned}R^{\bullet}_{L}(\mathcal{E}) \;\hat{=}\; \{&\mathsf{A}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\mathsf{B}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\mathsf{C}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\mathsf{D}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}},\\ &\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\mathsf{E}\bar{\mathsf{F}}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\mathsf{F}\bar{\mathsf{G}}\bar{\mathsf{H}}\bar{\mathsf{I}}, \mathsf{A}\mathsf{B}\mathsf{C}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\mathsf{G}\bar{\mathsf{H}}\bar{\mathsf{I}}\,\bar{\mathsf{J}}, \bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}\bar{\mathsf{E}}\bar{\mathsf{F}}\bar{\mathsf{G}}\mathsf{H}\bar{\mathsf{I}}\,\bar{\mathsf{J}},\\ &\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\mathsf{D}\mathsf{E}\mathsf{F}\bar{\mathsf{G}}\bar{\mathsf{H}}\mathsf{I}\bar{\mathsf{J}}, \mathsf{ABCDEFGHIJ}\}\,.\end{aligned}$$

Thus, the number of variables reduces from 162 to 10. $\square$

8.2 Decomposable variables

We now present the new result that the reduced index set $R^{\bullet}_{L}(\mathcal{E})$ of the previous section can be characterized in terms of decomposable variables.

Let L be a finite set of Horn logical constraints, and let $\mathcal{E}$ be a finite set of conjunctive conditional events. For $r \in R_L(\mathcal{E})$, we define $r(\phi) = 1$ if $r \models \phi$, and $r(\phi) = 0$ if $r \not\models \phi$. We use $con(r)$ to denote the set of all $\phi \in con(\mathcal{E})$ such that $L \cup \{\wedge r\} \models \phi$. For $R \subseteq R_L(\mathcal{E})$, we say r is *decomposable* in R relative to an event ϕ (resp., a set of events $\mathcal{C}$) iff some $S \subseteq R$ exists such that $|S| \geq 2$ and $r(\phi) = \sum_{s \in S} s(\phi)$ (resp., $r(\phi) = \sum_{s \in S} s(\phi)$ for all $\phi \in \mathcal{C}$). A variable y_r with $r \in R_L(\mathcal{E})$ is *decomposable* in R relative to ϕ (resp., $\mathcal{C}$) iff r is decomposable in R relative to ϕ (resp., $\mathcal{C}$).

The following theorem shows that $R_L(\mathcal{E}) \setminus R^{\bullet}_{L}(\mathcal{E})$ is the set of all members of $R_L(\mathcal{E})$ that are decomposable in $R_L(\mathcal{E})$ relative to $con(\mathcal{E})$.

Theorem 8.4 *Let L be a finite set of Horn logical constraints, and let $\mathcal{E}$ be a finite set of conjunctive conditional events. Then:*

(a) Every $r \in R_L(\mathcal{E}) \setminus R^\bullet_L(\mathcal{E})$ is decomposable in $R^\bullet_L(\mathcal{E})$ relative to $con(\mathcal{E})$.

(b) No $r \in R^\bullet_L(\mathcal{E})$ is decomposable in $R^\bullet_L(\mathcal{E})$ relative to $con(\mathcal{E})$.

The next result shows that every member of $R_L(\mathcal{E}) \setminus R^\bullet_L(\mathcal{E})$ is also decomposable in $R^\bullet_L(\mathcal{E})$ relative to $\{\neg\psi \wedge \phi \mid \psi|\phi \in \mathcal{E}\}$.

Theorem 8.5 *Let L be a finite set of Horn logical constraints, and let $\mathcal{E}$ be a finite set of conjunctive conditional events. Then, every $r \in R_L(\mathcal{E}) \setminus R^\bullet_L(\mathcal{E})$ is decomposable in $R^\bullet_L(\mathcal{E})$ relative to $\{\neg\psi \wedge \phi \mid \psi|\phi \in \mathcal{E}\}$.*

Hence, Theorems 8.4 (a) and 8.5 show that each decomposable variable in the system (3) can be expressed through a finite number of non-decomposable variables. That is, each decomposable variable always occurs together with *exactly one* of its associated non-decomposable variables. Thus, the decomposable variables can simply be removed from the system (3).

8.3 Random gain

We finally describe the new result that the reduced index set $R^\bullet_L(\mathcal{E})$ of Section 8.1 can also be expressed in terms of random gain.

Let $KB = (L, P) = (L, \{(\psi_i|\phi_i)[l_i, u_i] \mid i \in \{1, \ldots, n\}\})$, where $n \geq 1$, be a probabilistic knowledge base. Let $R = R_L(\mathcal{E}_{KB})$. Then, for every $r \in R$, the corresponding random gain g_r can be expressed by:

$$g_r \;=\; \sum_{i=1}^{n} s_i(p_{ri} - l_i) - \sigma_i(q_{ri} - u_i)\,, \tag{6}$$

where $s_i \geq 0$ and $\sigma_i \geq 0$ are real numbers, and p_{ri} and q_{ri} are defined by (2), for all $r \in R$ and $i \in \{1, \ldots, n\}$.

The following theorem now shows that the reduced index set $R^\bullet_L(\mathcal{E}_{KB})$ can be characterized using the notion of random gain.

Theorem 8.6 *Let $KB{=}(L, P){=}(L, \{(\psi_i|\phi_i)[l_i, u_i] \mid i \in \{1, \ldots, n\}\})$, where $n \geq 1$ and $l_i < u_i$ for all $i \in \{1, \ldots, n\}$, be a conjunctive probabilistic knowledge base. Let $R = R_L(\mathcal{E}_{KB})$ and $R^\bullet = R^\bullet_L(\mathcal{E}_{KB})$. Then:*

(a) If $r \in R \setminus R^\bullet$, then $g_r = \sum_{s\in S} g_s$ for some $S \subseteq R^\bullet$ with $|S| \geq 2$.

(b) If $r \in R^\bullet$, then no $S \subseteq R^\bullet$ exists such that $|S| \geq 2$ and $g_r = \sum_{s\in S} g_s$.

9 Summary and outlook

We gave a precise picture of the computational complexity of probabilistic reasoning under coherence. Moreover, we presented linear-time transformations for reducing an instance of deciding g-coherence or of computing tight g-coherent intervals to a smaller problem instance. Furthermore, we presented new algorithms for deciding g-coherence and for computing tight g-coherent intervals, which reduce these tasks to standard problems in model-theoretic probabilistic reasoning, which can then be reduced to linear optimization problems. This shows that efficient methods for model-theoretic probabilistic reasoning can immediately be applied for probabilistic reasoning under coherence. We then described several such techniques, which transform problem instances in model-theoretic probabilistic reasoning into smaller problem instances. We also described a technique for producing a reduced set of variables for the associated linear optimization problems in the conjunctive case, and gave new characterizations of this reduced set as a set of non-decomposable variables, and using the concept of random gain.

The overall procedure for deciding g-coherence and for computing tight g-coherent intervals may now be formulated as follows. Given a problem instance of deciding g-coherence or of computing tight g-coherent intervals, we first remove all inactive constraints (Section 5.2; in the conjunctive case), and then perform a decomposition (Section 5.1). Thereafter, we perform a reduction to POS and TLC (Sections 6.1 and 6.2). For each instance of POS and TLC, we make hidden logical knowledge explicit (Section 7.1), we remove all vacuous conditional constraints (Section 7.2), we remove all inactive constraints (Section 7.4; in the conjunctive case), and we perform a decomposition (Section 7.3). Thereafter, we produce a reduced set of variables for the linear optimization problems (Section 8.1; in the conjunctive case).

We remark that the methods of Sections 5.1, 5.2, 6.1, 6.2, and 7.1–7.4 can all be applied in combination with column generation techniques. We expect that the same applies also to the reduced set of variables of Section 8.1. Exploring this aspect in depth is a subject of future research.

Another interesting topic of future research is to further explore the reduced set of variables of Section 8.1. In particular, it has been shown in [6, 4] that further variables can be removed in coherence-based probabilistic reasoning by a refined characterization based on random gain. We expect that this result carries over to model-theoretic probabilistic reasoning.

A Appendix: Proofs for Section 4

To prove Theorems 4.1–4.4, we first recall a *small-model theorem* from [36], which says that tight bounds under logical entailment have a polynomial size and are attained at some models of polynomial size.

The *size* of a nonnegative integer a, denoted $\|a\|$, is defined as $1 + \lfloor \log a \rfloor$, if $a{>}0$, otherwise 1. The *size* of a nonnegative rational number r, denoted $\|r\|$, is defined as $\max(\|p\|, \|q\|)$, where $p \geq 0$ and $q > 0$ are relatively prime integers such that $p/q = r$ (we assume $q = 1$ when $r = 0$).

Theorem A.1 *Let $\mathcal{F}$ be a finite set of conditional constraints and $(\beta|\alpha)[l,u]$ be a conditional constraint, where $\mathcal{F} \models_{tight} (\beta|\alpha)[l,u]$ and $\mathcal{F} \not\models (\alpha|\top)[0,0]$. Let f denote the cardinality of $\mathcal{F}$ and let r be the greatest size of a rational number in $\mathcal{F}$ (we assume $r = 1$ when $\mathcal{F} = \emptyset$). Let $m = 2f + 1$. Then:*

(a) *For each $p \in \{l, u\}$, there exists a model Pr of $\mathcal{F}$ such that:*

  (i) $Pr(\alpha) > 0$ and $Pr(\beta|\alpha) = p$,

  (ii) the number of all $I \in \mathcal{I}_\Phi$ with $Pr(I) > 0$ is at most m, and

  (iii) the size of each $Pr(I) > 0$ is at most $1 + rm^2 + m + \lfloor (m^2 + m + 1) \log m \rfloor$.

(b) *Each $p \in \{l, u\}$ has a size of at most $1 + rm^2 + \lfloor m^2 \log m \rfloor$.*

We are now ready to prove Theorems 4.1–4.4.

Proof of Theorem 4.1. We first prove NP-membership. By Theorem 3.4 (b), $KB = (L, P)$ is g-coherent iff an ordered partition $\mathcal{P} = (P_0, \ldots, P_k)$ of P exists such that for each $i \in \{0, \ldots, k\}$ and each $(\psi|\phi)[l,u] \in P_i$, the set $L \cup P_i \cup \cdots \cup P_k$ has a model Pr with $Pr(\phi) > 0$. By Theorem A.1, the latter is equivalent to

$L \cup P_i \cup \cdots \cup P_k$ having a model Pr of polynomial size with $Pr(\phi) > 0$. Hence, such $\mathcal{P}$ and Pr can be guessed and verified by a nondeterministic algorithm in polynomial time.

Hardness for NP is shown by a polynomial transformation from the NP-complete *graph 3-colorability* problem [22]: Given a finite undirected graph $G = (V, E)$, decide whether G is 3-colorable (i.e., whether there exists a mapping $c\colon V \to \{1, 2, 3\}$ such that $c(u) \neq c(v)$ for all edges $\{u, v\} \in E$).

Let $G = (V, E)$ be a finite undirected graph. We now construct an atomic probabilistic knowledge base $KB = (\emptyset, P)$ such that KB is g-coherent iff G is 3-colorable. The set of basic events Φ is defined as $\{p_{i,j} \mid i \in V,\ j \in \{1, 2, 3\}\}$. We then define $P = P_1 \cup P_2 \cup P_3$ as follows:

$$
\begin{aligned}
P_1 &= \{(p_{i,j} \,|\, p_{i,k})[0, 0] \mid i \in V,\ j, k \in \{1, 2, 3\},\ j < k\}\,,\\
P_2 &= \{(p_{i,j} \,|\, p_{k,j})[0, 0] \mid \{i, k\} \in E,\ j \in \{1, 2, 3\}\}\,,\\
P_3 &= \{(p_{i,j} \,|\top)[1/3, 1/3] \mid i \in V,\ j \in \{1, 2, 3\}\}\,.
\end{aligned}
$$

Clearly, KB is constructible in polynomial time from G. In [36], it has been shown that KB is satisfiable iff G is 3-colorable. We now show that KB is g-coherent iff KB is satisfiable. Suppose first KB is g-coherent. By Corollary 3.3, KB is also satisfiable. Conversely, assume KB is satisfiable, Thus, there exists a model Pr of KB such that $Pr(p_{i,j}) = 1/3$ for all $i \in V$ and $j \in \{1, 2, 3\}$, and thus $Pr(\phi) > 0$ for all $(\psi|\phi)[l, u] \in P$. That is, each member of P is tolerated under $L = \emptyset$ by P. By Theorem 3.4 (b), KB is g-coherent. In summary, KB is g-coherent iff G is 3-colorable. $\Box$

Proof of Theorem 4.2. We first show membership in co-NP. By Theorem 2.3, $KB \not\mathrel{|\!\!\sim} (\beta|\alpha)[l, u]$ iff $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is g-coherent for some $p \in [0, l) \cup (u, 1]$. By Theorem 3.7, it holds that $KB \mathrel{|\!\!\sim}_{\mathit{tight}} (\beta|\alpha)[r, s]$ iff $(L, P^\star) \models_{\mathit{tight}} (\beta|\alpha)[r, s]$, where $P^\star$ is the greatest element in $P_\alpha(KB)$. By Theorem A.1, the tight bounds r and s are of polynomial size. Hence, $KB \not\mathrel{|\!\!\sim} (\beta|\alpha)[l, u]$ iff $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is g-coherent for some $p \in [0, l) \cup (u, 1]$ of polynomial size. Thus, guessing such p and verifying that $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is g-coherent, by Theorem 4.1, can be done in nondeterministic polynomial time.

To show co-NP-hardness, we give a polynomial transformation from the complement of the NP-complete *graph 3-colorability* problem (see the proof of Theorem 4.1). Let $G = (V, E)$ be a finite undirected graph. We now construct a 1-conjunctive g-coherent probabilistic knowledge base $KB = (\emptyset, P)$ and an atomic $(b \,|\top)[l, 1]$ such that $KB \mathrel{|\!\!\sim} (b \,|\top)[l, 1]$ iff G is not 3-colorable. The set of basic events Φ is defined as $\{p\} \cup \{p_{i,j} \mid i \in V,\ j \in \{1, 2, 3\}\}$. We then define $P = P_1 \cup P_2 \cup P_3$ as follows:

$$
\begin{aligned}
P_1 &= \{(p \,|\, p_{i,j} \wedge p_{i,k})[1, 1] \mid i \in V,\ j, k \in \{1, 2, 3\},\ j < k\}\,,\\
P_2 &= \{(p \,|\, p_{i,j} \wedge p_{k,j})[1, 1] \mid \{i, k\} \in E,\ j \in \{1, 2, 3\}\}\,,\\
P_3 &= \{(p_{i,j} \,|\top)[1/3, 1/3] \mid i \in V,\ j \in \{1, 2, 3\}\}\,.
\end{aligned}
$$

Let $f = |P|$, and let r be the greatest size of a rational number in P. Let $a = 1 \,/\, 2^{1 + rm^2 + \lfloor m^2 \log m \rfloor}$, where $m = 2f + 1$. Let $(b \,|\top)[l, 1] = (p \,|\top)[a, 1]$.

Clearly, KB and $(b \,|\top)[l, 1]$ are constructible in polynomial time from G. We next show that KB is g-coherent. Let $I_1 \in \mathcal{I}_\Phi$ be defined by $I_1(b) = \mathbf{true}$ for all $b \in \Phi$. Let $I_2 \in \mathcal{I}_\Phi$ be defined by $I_2(p) = \mathbf{true}$ and $I_2(b) = \mathbf{false}$ for all other $b \in \Phi$. Define Pr by $Pr(I_1) = 1/3$, $Pr(I_2) = 2/3$, and $Pr(I) = 0$ for all other $I \in \mathcal{I}_\Phi$. Then, Pr is a model of P with $Pr(\phi) > 0$ for all $(\psi|\phi)[l, u] \in P$. That is, each member of P is tolerated under $L = \emptyset$ by P. By Theorem 3.4 (b), KB is g-coherent. Notice that the model Pr of P satisfies $Pr(p) = 1$. Thus, the tight upper bound for $p \,|\top$ under logical entailment from KB and also under g-coherent entailment from KB is given by 1.

By Theorem A.1, all models Pr of P satisfy $Pr(p) > 0$ iff they all satisfy $Pr(p) \geq a$. The former is equivalent to G not being 3-colorable (similarly as in the proof of Theorem 4.1), while the latter is equivalent to $KB \models (p \,|\, \top)[a, 1]$. By Theorem 3.7, $KB \models (p \,|\, \top)[a, 1]$ is equivalent to $KB \;|\!\!\sim\; (p \,|\, \top)[a, 1]$, as P is the unique greatest element in $P_\top(KB)$. In summary, G is not 3-colorable iff $KB \,|\!\!\sim (p \,|\, \top)[a, 1]$. □

Proof of Theorem 4.3. We first show membership in $\mathrm{D^P}$. By Theorem 2.4, $KB \,|\!\!\sim_{\mathit{tight}} (\beta|\alpha)[l, u]$ iff (i) $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is not g-coherent for all $p \in [0, l) \cup (u, 1]$, and (ii) $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is g-coherent for all $p \in [l, u]$. By Theorems 2.1 and 2.3, the latter is equivalent to (i) $KB \,|\!\!\sim (\beta|\alpha)[l, u]$, and (ii) $(L, P \cup \{(\beta|\alpha)[l, l]\})$ and $(L, P \cup \{(\beta|\alpha)[u, u]\})$ are g-coherent. By Theorem 4.2 (resp., 4.1), deciding whether (i) (resp., (ii)) holds is in co-NP (resp., NP). In summary, deciding if $KB \,|\!\!\sim_{\mathit{tight}} (\beta|\alpha)[l, u]$ is in $\mathrm{D^P}$.

Hardness for $\mathrm{D^P}$ is shown by a polynomial transformation from the $\mathrm{D^P}$-complete *exact traveling salesman* problem [46]: Given a set of $n \geq 1$ cities $V = \{1, 2, \ldots, n\}$, a nonnegative integer distance $d_{i,j} = d_{j,i}$ between any two cities i and j, and some integer d, decide whether d is the smallest length of a tour through all the cities, that is, the minimum of $\sum_{i=1}^{n} d_{\pi(i),\pi(\sigma(i))}$ subject to all permutations π on V, where $\sigma(n) = 1$, and $\sigma(i) = i + 1$ for all $i < n$. Without affecting the complexity of the problem, we can assume $n \geq 3$.

Let s be the sum of all $d_{i,j}$ with $i, j \in V$ and $i < j$. We now construct $KB = (L, P)$ and $(\beta|\alpha)[l, u]$ as in the statement of the theorem such that $KB \,|\!\!\sim_{\mathit{tight}} (\beta|\alpha)[l, u]$ iff d is the smallest length of a tour through all the cities.

Let $E = \{\{i, j\} \subseteq V \mid i \neq j\}$, $w = d/s$, and $w_{\{i,j\}} = d_{i,j}/s$ for all $\{i, j\} \in E$. The set of basic events Φ is defined as $\Phi_1 \cup \Phi_2$, where $\Phi_1 = \{p_{i,j} \mid i, j \in V\}$ and $\Phi_2 = \{p\} \cup \{p_e \mid e \in E\}$. We then define a set of 1-conjunctive conditional constraints $P_1 = P_{1,1} \cup P_{1,2} \cup P_{1,3}$ that describes the set of all permutations of the members in V as follows:

$$
\begin{aligned}
P_{1,1} &= \{(p_{i,j} \,|\, p_{i,k})[0, 0] \mid i, j, k \in V,\ j < k\}\,, \\
P_{1,2} &= \{(p_{i,j} \,|\, p_{k,j})[0, 0] \mid i, j, k \in V,\ i < k\}\,, \\
P_{1,3} &= \{(p_{i,j} \,|\, \top)[1/n, 1/n] \mid i, j \in V\}\,.
\end{aligned}
$$

Roughly speaking, each world I with $Pr_1(I) > 0$ for some model Pr_1 of P_1 corresponds to a permutation of the members in V, and vice versa. We next define a set of 1-conjunctive conditional constraints $P_2 = P_{2,1} \cup P_{2,2} \cup P_{2,3}$ that associates each such permutation with its tour length, and the predicate symbol p with the sum of all such tour lengths as follows:

$$
\begin{aligned}
P_{2,1} &= \{(p_{e_1} \,|\, p_{e_2})[0, 0] \mid e_1, e_2 \in E,\ e_1 \neq e_2\}\,, \\
P_{2,2} &= \{(p_{\{i,j\}} \,|\, p_{u,i} \wedge p_{\sigma(u),j})[w_{\{i,j\}}, w_{\{i,j\}}] \mid u \in V,\ \{i, j\} \in E\}\,, \\
P_{2,3} &= \{(p \,|\, p_e)[1, 1] \mid e \in E\}\,.
\end{aligned}
$$

We finally define $KB = (L, P) = (\emptyset, P_1 \cup P_2)$. We now show that KB is g-coherent. As shown in [36], for every permutation π on V, there exists a model Pr of P such that $Pr(p_{u,\pi(u)} \wedge p_{\sigma(u),\pi(\sigma(u))}) > 0$ for all $u \in V$. Let $u \in V$ and $\{i, j\} \in E$. Then, there exists a permutation π on V such that $\pi(u) = i$ and $\pi(\sigma(u)) = j$. Thus, there exists a model Pr of P such that $Pr(p_{u,i} \wedge p_{\sigma(u),j}) > 0$ and $Pr(\phi) > 0$ for all $(\psi|\phi)[l, u] \in P_1$. Observe then that for every $(\psi|\phi)[l, u] \in P_{2,1} \cup P_{2,3}$, a model Pr of $P_{2,1} \cup P_{2,3}$ exists such that $Pr(\phi) > 0$. By Theorem 3.4 (b), KB is g-coherent. By Theorem 3.7, $KB \,|\!\!\sim_{\mathit{tight}} (p \,|\, \top)[w, 1]$ iff $KB \models_{\mathit{tight}} (p \,|\, \top)[w, 1]$, as P is the unique greatest element in $P_\top(KB)$. As shown in [36], $KB \models_{\mathit{tight}} (p \,|\, \top)[w, 1]$ iff d is the smallest length of a tour through all the cities. In summary, $KB \,|\!\!\sim_{\mathit{tight}} (p \,|\, \top)[w, 1]$ iff d is the smallest length of a tour through all the cities. □

Proof of Theorem 4.4. We first prove membership in $\mathrm{FP}^{\mathrm{NP}}$. By Theorems 3.7 and 3.8, Algorithm `tight-g-coherent-consequence` (see Fig. 2) calculates the bounds $l, u \in [0,1]$ such that $KB \,|\!\!\sim_{tight} (\beta|\alpha)[l,u]$. It first performs a polynomial number of toleration checks (in step 3), where each check is in NP by the proof of Theorem 4.1. It then computes $l, u \in [0,1]$ such that $KB \models_{tight} (\beta|\alpha)[l,u]$ (in step 6), which is in $\mathrm{FP}^{\mathrm{NP}}$ [36]. In summary, computing $l, u \in [0,1]$ such that $KB \,|\!\!\sim_{tight} (\beta|\alpha)[l,u]$ is in $\mathrm{FP}^{\mathrm{NP}}$.

Hardness for $\mathrm{FP}^{\mathrm{NP}}$ is shown by a polynomial transformation from the $\mathrm{FP}^{\mathrm{NP}}$-complete *traveling salesman cost* problem [41]. Given a set of $n \geq 1$ cities $V = \{1, 2, \ldots, n\}$ and a nonnegative integer distance $d_{i,j} = d_{j,i}$ between any two cities i and j, we have to compute the smallest length d of a tour through all the cities, that is, the minimum of $\sum_{i=1}^{n} d_{\pi(i),\pi(\sigma(i))}$ subject to all permutations π, where $\sigma(n) = 1$, and $\sigma(i) = i+1$ for all $i < n$. Without affecting the complexity of the problem, we can assume $n \geq 3$.

Let $KB = (L, P)$ be defined as in the proof of Theorem 4.3. Recall that KB is g-coherent, $L = \emptyset$, and P is 1-conjunctive. Moreover, it holds that $d = l\, s$, where $KB \,|\!\!\sim_{tight} (p \,|\, \top)[l,1]$ and $s = \sum_{i,j \in V, i<j} d_{i,j}$. □

B Appendix: Proofs for Section 5

Proof of Theorem 5.1. Let $dec_{\top}(KB) = \{KB_i = (L_i, P_i) \mid i \in \{1, \ldots, k\}\}$.

($\Rightarrow$) Assume that KB is g-coherent. By Theorem 3.2, for every nonempty $P' \subseteq P$, there exists a model Pr of $L \cup P'$ such that $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$. In particular, for every $i \in \{1, \ldots, k\}$ and every nonempty $P_i{}' \subseteq P_i$, there exists a model Pr_i of $L \cup P_i{}'$ such that $Pr(\phi_i) > 0$ for some $(\psi_i|\phi_i)[l_i, u_i] \in P_i{}'$. Thus, for every $i \in \{1, \ldots, k\}$ and every nonempty $P_i{}' \subseteq P_i$, there exists a model Pr_i of $L_i \cup P_i{}'$ such that $Pr(\phi_i) > 0$ for some $(\psi_i|\phi_i)[l_i, u_i] \in P_i{}'$. By Theorem 3.2, this shows that every $KB_i = (L_i, P_i)$ with $P_i \neq \emptyset$ is g-coherent. As $P \neq \emptyset$, by Theorem 3.2, there exists a model of L. Hence, every $KB_i = (L_i, P_i)$ with $P_i = \emptyset$ is satisfiable.

($\Leftarrow$) Assume that every $KB_i = (L_i, P_i)$ with $i \in \{1, \ldots, k\}$ and $P_i \neq \emptyset$ (resp., $P_i = \emptyset$) is g-coherent (resp., satisfiable). By Theorem 3.2, for every $i \in \{1, \ldots, k\}$ with $P_i \neq \emptyset$ and every nonempty $P_i{}' \subseteq P_i$, there exists a model Pr_i of $L_i \cup P_i{}'$ such that $Pr(\phi_i) > 0$ for some $(\psi_i|\phi_i)[l_i, u_i] \in P_i{}'$. As $P_i \neq \emptyset$, it even follows that for every $i \in \{1, \ldots, k\}$ with $P_i \neq \emptyset$ and every $P_i{}' \subseteq P_i$, there exists a model Pr_i of $L_i \cup P_i{}'$ such that $Pr(\phi_i) > 0$ for some $(\psi_i|\phi_i)[l_i, u_i] \in P_i{}'$, if $P_i{}' \neq \emptyset$. Moreover, for every $i \in \{1, \ldots, k\}$ with $P_i = \emptyset$, there exists a model Pr_i of L_i. In summary, for every $i \in \{1, \ldots, k\}$ and every $P_i{}' \subseteq P_i$, there exists a model Pr_i of $L_i \cup P_i{}'$ such that $Pr(\phi_i) > 0$ for some $(\psi_i|\phi_i)[l_i, u_i] \in P_i{}'$, if $P_i{}' \neq \emptyset$. Thus, for every $i \in \{1, \ldots, k\}$ and every $P_i{}' \subseteq P_i$, there exists a model $Pr_i{}'$ on $\mathcal{I}_{\Phi_i}$ of $L_i \cup P_i{}'$ with $Pr(\phi_i) > 0$ for some $(\psi_i|\phi_i)[l_i, u_i] \in P_i{}'$. Hence, the probabilistic interpretation Pr' on $\mathcal{I}_{bas(KB) \cup bas(\alpha)}$ that is defined by $Pr'(I) = Pr_1{}'(I_1) \cdots Pr_k{}'(I_k)$ for all $I \in \mathcal{I}_{bas(KB) \cup bas(\alpha)}$, where I_i is the restriction of I to $\mathcal{I}_{\Phi_i}$, for all $i \in \{1, \ldots, k\}$, is a model of $L \cup P'$ such that $Pr'(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$, if $P' \neq \emptyset$, where $P' = P_1{}' \cup \cdots \cup P_k{}'$. Hence, for every $P' \subseteq P$, there exists a model Pr on $\mathcal{I}_{\Phi}$ of $L \cup P'$ with $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$, if $P' \neq \emptyset$. In particular, for every nonempty $P' \subseteq P$, there exists a model Pr of $L \cup P'$ with $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$. By Theorem 3.2, KB is g-coherent. □

Proof of Theorem 5.3. (a) Immediate by (b).

(b) By Theorem 2.4, $KB = (L, P) \,|\!\!\sim_{tight} (\beta|\alpha)[l,u]$ iff (i) KB_p is not g-coherent for all $p \in [0,l) \cup (u,1]$, and (ii) KB_p is g-coherent for all $p \in [l,u]$, where $KB_p = (L, P \cup \{(\beta|\alpha)[p,p]\})$. By Theorem 5.1, this is equivalent to (i) some $KB_i = (L_i, P_i) \in dec_{\top}(KB_p)$ with $P_i \neq \emptyset$ (resp., $P_i = \emptyset$) is not g-coherent (resp., satisfiable) for all $p \in [0,l) \cup (u,1]$, and (ii) every $KB_i = (L_i, P_i) \in dec_{\top}(KB_p)$ with $P_i \neq \emptyset$ (resp., $P_i = \emptyset$) is g-coherent (resp., satisfiable) for all $p \in [l,u]$. As KB is g-coherent, this is in turn equivalent to (i) $rel_{\beta\wedge\alpha}(KB_p)$ is not g-coherent for all $p \in [0,l) \cup (u,1]$, and (ii) $rel_{\beta\wedge\alpha}(KB_p)$ is g-coherent for all $p \in [l,u]$. Equivalently, (i) $KB_p{}'$ is not g-coherent for all $p \in [0,l) \cup (u,1]$, and (ii) $KB_p{}'$ is g-coherent for all $p \in [l,u]$, where $KB_p{}' = (L', P' \cup \{(\beta|\alpha)[p,p]\})$ and $(L',P') = rel_{\beta\wedge\alpha}(KB)$. By Theorem 2.4, this is in turn equivalent to $rel_{\beta\wedge\alpha}(KB) \mid\!\sim_{tight} (\beta|\alpha)[l,u]$. In summary, $KB \mid\!\sim_{tight} (\beta|\alpha)[l,u]$ iff $rel_{\beta\wedge\alpha}(KB) \mid\!\sim_{tight} (\beta|\alpha)[l,u]$. □

Proof of Proposition 5.5. Let $KB = (L,P)$. We first compute the set of all connected components of the hypergraph $G = (bas(KB) \cup bas(\alpha), \{bas(F) \mid F \in L \cup P\} \cup \{bas(\alpha)\})$, where $bas(F)$ is the set of all $b \in \Phi$ that occur in F. This can be done in linear time using standard methods and data structures. Once S is given, $dec_\alpha(KB)$ and $rel_\alpha(KB)$ can be computed in linear time. In summary, $dec_\alpha(KB)$ and $rel_\alpha(KB)$ can be computed in linear time. □

The following lemma prepares the proofs of Theorems 5.6 and 5.8.

Lemma B.1 *Let $KB = (L,P)$ be a conjunctive probabilistic knowledge base, and let α be a conjunctive event. Let $act_\alpha(KB) = (L_a, P_a)$. Then, for every probabilistic interpretation Pr, there is a model $Pr^\star$ of $(L \cup P) \setminus (L_a \cup P_a)$ such that $Pr(\phi) = Pr^\star(\phi)$ for all events ϕ that are active w.r.t. KB and α.*

Proof. Let Pr be a probabilistic interpretation. We define $Pr^\star$ as follows. For every $I \in \mathcal{I}_\Phi$ with $I(b) = \mathbf{true}$ for some inactive $b \in \Phi$, we define $Pr^\star(I) = 0$, and for every $I \in \mathcal{I}_\Phi$ with $I(b) = \mathbf{false}$ for all inactive $b \in \Phi$, we define $Pr^\star(I)$ as the sum of all $Pr(J)$ such that $J \in \mathcal{I}_\Phi$ and $I(b) = J(b)$ for all active $b \in \Phi$.

Notice that $Pr^\star$ is a probabilistic interpretation such that $Pr(\phi) = Pr^\star(\phi)$ for all active events ϕ. Notice then that $Pr^\star(b) = 0$ for all inactive $b \in \Phi$. We now show that $Pr^\star$ satisfies all inactive members of $L \cup P$. Towards a contradiction, assume first that $Pr^\star$ does not satisfy some inactive $\psi \Leftarrow \phi \in L$. Thus, ϕ must be active, as otherwise $Pr^\star(\phi) = 0$, and thus $Pr^\star \models \psi \Leftarrow \phi$. Hence, ψ must be an an inactive basic event, as otherwise $\psi \Leftarrow \phi$ would be active. Hence, $\psi \Leftarrow \phi \in L^+$, and thus ψ is active. This is a contradiction. Assume next that $Pr^\star$ does not satisfy some inactive $(\psi|\phi)[l,u] \in P$. Hence, ϕ must be active, as otherwise $Pr^\star(\phi) = 0$, and thus $Pr^\star \models (\psi|\phi)[l,u]$. Thus, ψ must contain an inactive basic event, as otherwise $(\psi|\phi)[l,u]$ would be active. Hence, $Pr^\star(\psi \wedge \phi) = 0$, and thus $l > 0$, as otherwise $Pr^\star \models (\psi|\phi)[l,u]$. Hence, $\psi \Leftarrow \phi \in P^+$. But this contradicts ψ being inactive. Hence, $Pr^\star$ satisfies all inactive members of $L \cup P$. □

We are now ready to prove Theorems 5.6 and 5.8.

Proof of Theorem 5.6. ($\Rightarrow$) Assume that KB is g-coherent. By Theorem 3.3, for every nonempty $P' \subseteq P$, there exists a model Pr of $L \cup P'$ such that $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$. Hence, for every nonempty $P_a{}' \subseteq P_a$, there exists a model Pr of $L_a \cup P_a{}'$ with $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P_a{}'$, and thus (L_a, P_a) is g-coherent, by Theorem 3.3. Furthermore, for every $(\psi|\phi)[l,u] \in P \setminus P_a$, there exists a model Pr of $L_a \cup \{(\psi|\phi)[l,u]\}$ such that $Pr(\phi) > 0$, and thus $L_a \cup \{\phi\}$ is satisfiable for all $(\psi|\phi)[l,u] \in P \setminus P_a$.

($\Leftarrow$) Assume that (L_a, P_a) is g-coherent and that $L_a \cup \{\phi\}$ is satisfiable for all $(\psi|\phi)[l,u] \in P \setminus P_a$. By Theorem 3.3, for every nonempty $P_a{}' \subseteq P_a$, there exists a model Pr of $L_a \cup P_a{}'$ with $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P_a{}'$. By Lemma B.1, for every nonempty $P' \subseteq P$ with $P' \cap P_a \neq \emptyset$, there exists a model Pr of $L \cup P'$ with $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$. As $L_a \cup \{\phi\}$ is satisfiable for all $(\psi|\phi)[l,u] \in P \setminus P_a$, there exists a model Pr of L_a with $Pr(\phi) > 0$. By Lemma B.1, as ϕ is active, for every nonempty $P' \subseteq P$ with $P' \cap P_a = \emptyset$, there exists a model Pr of $L \cup P'$ with $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$. In summary, for every nonempty $P' \subseteq P$, there exists a model Pr of $L \cup P'$ such that $Pr(\phi) > 0$ for some $(\psi|\phi)[l,u] \in P'$. By Theorem 3.3, KB is g-coherent. □

Proof of Theorem 5.8. Let $KB = (L,P)$ and $act_{\alpha\wedge\beta}(KB) = (L_a, P_a)$.

(a) By Theorem 2.3, it is sufficient to show that $(L, P \cup \{(\beta|\alpha)[p,p]\})$ is not g-coherent for all $p \in [0,l) \cup (u,1]$ iff $(L_a, P_a \cup \{(\beta|\alpha)[p,p]\})$ is not g-coherent for all $p \in [0,l) \cup (u,1]$.

($\Rightarrow$) Let $p \in [0, l) \cup (u, 1]$. Assume $(L_a, P_a \cup \{(\beta|\alpha)[p, p]\})$ is g-coherent. By Theorem 3.3, for each nonempty ${P_a}' \subseteq P_a \cup \{(\beta|\alpha)[p, p]\}$, a model Pr of $L_a \cup {P_a}'$ exists with $Pr(\phi)>0$ for some $(\psi|\phi)[l, u] \in {P_a}'$. By Lemma B.1, for every nonempty $P' \subseteq P \cup \{(\beta|\alpha)[p, p]\}$ with $P' \cap (P_a \cup \{(\beta|\alpha)[p, p]\}) \neq \emptyset$, a model Pr of $L \cup P'$ exists with $Pr(\phi) > 0$ for some $(\psi|\phi)[l, u] \in P'$. As KB is g-coherent, by Theorem 3.3, for every nonempty $P' \subseteq P \cup \{(\beta|\alpha)[p, p]\}$ with $P' \cap (P_a \cup \{(\beta|\alpha)[p, p]\}) = \emptyset$, there exists a model Pr of $L \cup P'$ such that $Pr(\phi) > 0$ for some $(\psi|\phi)[l, u] \in P'$. In summary, for every nonempty $P' \subseteq P \cup \{(\beta|\alpha)[p, p]\}$, a model Pr of $L \cup P'$ exists with $Pr(\phi) > 0$ for some $(\psi|\phi)[l, u] \in P'$. By Theorem 3.3, $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is g-coherent.

($\Leftarrow$) By Theorem 3.2, for all $p \in [0, l) \cup (u, 1]$, if $(L_a, P_a \cup \{(\beta|\alpha)[p, p]\})$ is not g-coherent, then $(L, P \cup \{(\beta|\alpha)[p, p]\})$ is not g-coherent.

(b) Immediate by (a). $\Box$

Proof of Proposition 5.10. Let $KB = (L, P)$. We first compute the sets L^+ and P^+. We then compute the set of all basic events $b \in \Phi$ that are active w.r.t. KB and α. We finally compute the set of all members of L and P that are active w.r.t. KB and α. As each of these three computations can be done in linear time, $act_\alpha(KB)$ can be computed in linear time. $\Box$

C Appendix: Proofs for Section 7

Proof of Theorem 7.1. Observe first that for all $i \in \{0, \ldots, n-1\}$, where $n \geq 0$ such that $S^\star_P(L) = S^n_P(L)$, it holds that every model of $S^i_P(L) \cup P$ is also a model of $S^{i+1}_P(L) \cup P$. Thus, every model of $L \cup P$ is also a model of $S^\star_P(L) \cup P$. Conversely, as $S^i_P(L) \subseteq S^{i+1}_P(L)$ for all $i \in \{0, \ldots, n-1\}$, every model of $S^\star_P(L) \cup P$ is trivially also a model of $L \cup P$. In summary, $L \cup P$ and $S^\star_P(L) \cup P$ have exactly the same models. Hence, KB has a model Pr with $Pr(\alpha) > 0$ iff $KB^\star$ has a model Pr with $Pr(\alpha) > 0$. $\Box$

Proof of Theorem 7.3. Immediate by the proof of Theorem 7.1. $\Box$

Proof of Proposition 7.4. Observe that $n \leq |P|$ for the number $n \geq 0$ such that $S^\star_P(L) = S^n_P(L)$. Thus, $S^\star_P(L)$ is computable by an iteration of at most $|P| + 1$ steps. In iteration step i, we first compute $S^i_P(L)$, which is the set of all $\bot \Leftarrow \phi$ such that either (i) or (ii) holds relative to $S^{i-1}_P(L)$, and then check whether $S^i_P(L)$ is logically equivalent to $S^{i-1}_P(L)$. In the conjunctive case, the former can be done in time $O(\|KB\||P|)$, while the latter is possible in time $O(\|KB\|)$, where $\|KB\|$ and $|P|$ denote the input size of KB and the cardinality of P, respectively. If $S^i_P(L)$ is logically equivalent to $S^{i-1}_P(L)$, then the iteration stops and $S^\star_P(L) = S^i_P(L)$. In summary, $S^\star_P(L)$ can be computed in time $O(\|KB\||P|^2)$, that is, in polynomial time. $\Box$

Proof of Theorem 7.5. As $L \cup (P \setminus vac_L(P)) \subseteq L \cup P$, every model of KB is also a model of $KB^\star$. Conversely, as L logically entails every member of $vac_L(P)$, every model of $KB^\star$ is also a model of KB. In summary, KB and $KB^\star$ have exactly the same models. Hence, KB has a model Pr with $Pr(\alpha) > 0$ iff $KB^\star$ has a model Pr with $Pr(\alpha) > 0$. $\Box$

Proof of Theorem 7.7. Immediate by the proof of Theorem 7.5. $\Box$

Proof of Proposition 7.8. The set $vac_L(P)$ can be computed by checking for every $(\psi|\psi)[l, u] \in P$, whether a condition among (i)–(iv) is satisfied, where each such check can be done in linear time in the conjunctive case. In summary, $vac_L(P)$ can be computed in polynomial time. $\Box$

Proof of Theorem 7.9. ($\Rightarrow$) Trivially, every model Pr of KB with $Pr(\alpha) > 0$ is also a model of every $KB_i \in dec_\alpha(KB)$.

($\Leftarrow$) Let $dec_\alpha(KB) = \{KB_1, \ldots, KB_k\}$, where $KB_1 = rel_\alpha(KB)$. Assume that (i) and (ii) hold. Thus, every KB_i has a model Pr_i on $\mathcal{I}_{\Phi_i}$, where $Pr_1(\alpha) > 0$. Hence, the probabilistic interpretation Pr on $\mathcal{I}_{bas(KB)\cup bas(\alpha)}$ that is defined by $Pr(I) = Pr_1(I_1) \cdots Pr_k(I_k)$ for all $I \in \mathcal{I}_{bas(KB)\cup bas(\alpha)}$, where every I_i is the restriction of I to $\mathcal{I}_{\Phi_i}$, is a model of KB with $Pr(\alpha) > 0$. $\Box$

Proof of Theorem 7.10. Let $dec_{\beta\wedge\alpha}(KB) = \{KB_1, \ldots, KB_k\}$, where $KB_1 = rel_{\beta\wedge\alpha}(KB)$. We define $R = \{Pr(\beta|\alpha) \mid Pr \models KB,\ Pr(\alpha) > 0\}$ and $R_1 = \{Pr(\beta|\alpha) \mid Pr \models KB_1,\ Pr(\alpha) > 0\}$. It is now sufficient to show that $R = R_1$. Since every model Pr of KB with $Pr(\alpha) > 0$ is also a model of KB_1 with $Pr(\alpha) > 0$, it follows that $R \subseteq R_1$. To show the converse, assume Pr is a model of KB_1 with $Pr(\alpha) > 0$. Hence, there exists a model Pr_1 of KB_1 on $\mathcal{I}_{\Phi_1}$ such that $Pr_1(\alpha) > 0$ and $Pr_1(\beta|\alpha) = Pr(\beta|\alpha)$. As $KB \not\models (\alpha|\top)[0,0]$, every KB_i with $i \in \{2, \ldots, k\}$ is satisfiable. It follows that every KB_i with $i \in \{2, \ldots, k\}$ has a model Pr_i on $\mathcal{I}_{\Phi_i}$. Thus, the probabilistic interpretation Pr' on $\mathcal{I}_{bas(KB)\cup bas(\beta\wedge\alpha)}$ that is defined by $Pr'(I) = Pr_1(I_1) \cdots Pr_k(I_k)$ for all $I \in \mathcal{I}_{bas(KB)\cup bas(\beta\wedge\alpha)}$, where every I_i is the restriction of I to $\mathcal{I}_{\Phi_i}$, is a model of KB such that $Pr'(\beta|\alpha) = Pr_1(\beta|\alpha) = Pr(\beta|\alpha)$ and $Pr'(\alpha) = Pr_1(\alpha) > 0$. This shows that $R \supseteq R_1$. In summary, it holds that $R = R_1$. $\Box$

Proof of Theorem 7.11. ($\Rightarrow$) Trivially, if KB has a model Pr with $Pr(\alpha) > 0$, then $act_\alpha(KB)$ has a model Pr with $Pr(\alpha) > 0$.

($\Leftarrow$) Let $(L_a, P_a) = act_\alpha(KB)$ and $(L_i, P_i) = (L \setminus L_a, P \setminus P_a)$. Let Pr be a model of $act_\alpha(KB)$ with $Pr(\alpha) > 0$. By Lemma B.1, there exists a model $Pr^\star$ of $L_i \cup P_i$ such that $Pr^\star(\phi) = Pr(\phi)$ for all active events ϕ. Hence, $Pr^\star$ is a model of $L \cup P$ such that $Pr^\star(\alpha) = Pr(\alpha) > 0$, as α is active. $\Box$

Proof of Theorem 7.12. Let $(L_a, P_a) = act_{\alpha\wedge\beta}(KB)$ and $(L_i, P_i) = (L \setminus L_a,\ P \setminus P_a)$. Let $l, u, r, s \in [0,1]$ such that $L \cup P \models_{tight} (\beta|\alpha)[l,u]$ and $L_a \cup P_a \models_{tight} (\beta|\alpha)[r,s]$. We now prove that $[l,u] = [r,s]$. Trivially, every model Pr of $L \cup P$ with $Pr(\alpha) > 0$ is also a model of $L_a \cup P_a$ with $Pr(\alpha) > 0$. This shows that $[l,u] \subseteq [r,s]$. To prove the converse inclusion, let Pr be a model of $L_a \cup P_a$ with $Pr(\alpha) > 0$. By Lemma B.1, there exists a model $Pr^\star$ of $L_i \cup P_i$ such that $Pr^\star(\phi) = Pr(\phi)$ for all active events ϕ. Thus, $Pr^\star$ is a model of $L \cup P$, and as both α and β are active, it holds that $Pr^\star(\alpha) = Pr(\alpha) > 0$ and $Pr^\star(\beta|\alpha) = Pr(\beta|\alpha)$. This shows that $[l,u] \supseteq [r,s]$. $\Box$

D Appendix: Proofs for Section 8

For the proof of Theorem 8.4, we need the following preliminary result.

Lemma D.1 *Let L be a finite set of Horn logical constraints and $\mathcal{E}$ be a finite set of conjunctive conditional events. Let $D, E \in \mathcal{C}^\bullet_L(\mathcal{E})$ with $D \subseteq E$, and let $\mathcal{C} \subseteq \mathcal{C}^\bullet_L(\mathcal{E})$ be a partition of E. Then, $D \subseteq C$ for some $C \in \mathcal{C}$.*

Proof. We prove the statement by structural induction on $D \in \mathcal{C}^\bullet_L(\mathcal{E})$. *Basis:* If $D = T_L(\{\phi\})$ for some $\phi \in con(\mathcal{E})$ such that $L \cup \{\phi\}$ is satisfiable, then $\phi \in C$ for some $C \in \mathcal{C}$, and thus $D = T_L(\{\phi\}) \subseteq C$. *Induction:* Assume that $D = T_L(D_1 \cup D_2)$ for two distinct $D_1, D_2 \in \mathcal{C}^\bullet_L(\mathcal{E})$ such that $D_1 \cap D_2 \neq \emptyset$ and $L \cup D_1 \cup D_2$ is satisfiable. As $D_1, D_2 \subseteq D \subseteq E$, by the induction hypothesis, $D_1 \subseteq C_1$ and $D_2 \subseteq C_2$ for some $C_1, C_2 \in \mathcal{C}$. As $D_1 \cap D_2 \neq \emptyset$, it follows that $C_1 = C_2$. Thus, $D = T_L(D_1 \cup D_2) \subseteq C_1$ for some $C_1 \in \mathcal{C}$. $\Box$

We are now ready to prove Theorem 8.4.

Proof of Theorem 8.4. (a) Let $r \in R_L(\mathcal{E}) \setminus R^\bullet_L(\mathcal{E})$. Let $\mathcal{C}_r$ be the set of all $C \in \mathcal{C}^\bullet_L(\mathcal{E})$ such that (i) $C \subseteq con(r)$ and (ii) $C \not\subseteq C'$ for all $C' \in \mathcal{C}^\bullet_L(\mathcal{E})$ with $C' \neq C$ and $C' \subseteq con(r)$. We now prove that any two distinct $C_1, C_2 \in \mathcal{C}_r$ are disjoint. Towards a contradiction, assume that $C_1 \cap C_2 \neq \emptyset$ for two distinct $C_1, C_2 \in \mathcal{C}_r$. As

$C_1, C_2 \subseteq con(r)$ and $L \cup \{\wedge r\}$ is satisfiable, it follows that $L \cup C_1 \cup C_2$ is satisfiable, and thus $T_L(C_1 \cup C_2) \in \mathcal{C}^\bullet_L(\mathcal{E})$. As $C_1, C_2 \subseteq T_L(C_1 \cup C_2)$ and $C_1 \neq C_2$, it follows that $C_i \subset T_L(C_1 \cup C_2) \subseteq con(r)$ for some $i \in \{1, 2\}$. But this contradicts (ii). Thus, any two distinct $C_1, C_2 \in \mathcal{C}_r$ are disjoint. Observe then that $T_L(\{\phi\}) \in \mathcal{C}^\bullet_L(\mathcal{E})$ for every $\phi \in con(r)$. It thus follows that $\bigcup\{C \mid C \in \mathcal{C}_r\} = con(r)$. In summary, $\mathcal{C}_r$ is a partition of $con(r)$. That is, $r(\phi) = \sum_{C \in \mathcal{C}_r} r_C(\phi)$ for all $\phi \in con(\mathcal{E})$. This shows that r is decomposable in $R^\bullet_L(\mathcal{E})$ relative to $con(\mathcal{E})$.

(b) Let $r \in R^\bullet_L(\mathcal{E})$. We now prove that r is not decomposable in $R^\bullet_L(\mathcal{E})$ relative to $con(\mathcal{E})$. Towards a contradiction, assume that $r(\phi) = \sum_{C \in \mathcal{C}_r} r_C(\phi)$ for all $\phi \in con(\mathcal{E})$, where $\mathcal{C}_r \subseteq \mathcal{C}^\bullet_L(\mathcal{E})$ and $|\mathcal{C}_r| \geq 2$. Thus, $\mathcal{C}_r$ is a partition of $con(r)$. Assume first that $con(r) = T_L(\{\phi\})$ for some $\phi \in con(\mathcal{E})$ where $L \cup \{\phi\}$ is satisfiable. Hence, $\phi \in C$ for some $C \in \mathcal{C}_r$, and thus $con(r) = T_L(\{\phi\}) \subseteq C \subseteq con(r)$. Hence, $C = con(r)$, and thus $\mathcal{C}_r = \{C\}$, but this contradicts $|\mathcal{C}_r| \geq 2$. Assume next that $con(r) = T_L(D_1 \cup D_2)$ for two distinct $D_1, D_2 \in \mathcal{C}^\bullet_L(\mathcal{E})$ such that $D_1 \cap D_2 \neq \emptyset$ and $L \cup D_1 \cup D_2$ is satisfiable. As $D_1, D_2 \subseteq con(r)$, by Lemma D.1, it follows that $D_1 \subseteq C_1$ and $D_2 \subseteq C_2$ for some $C_1, C_2 \in \mathcal{C}_r$. As $D_1 \cap D_2 \neq \emptyset$, we obtain $C_1 = C_2$. Hence, $con(r) = T_L(D_1 \cup D_2) \subseteq C_1 \subseteq con(r)$. Hence, $C_1 = con(r)$, and thus $\mathcal{C}_r = \{C_1\}$, but this contradicts $|\mathcal{C}_r| \geq 2$. In summary, it thus follows that r is not decomposable in $R^\bullet_L(\mathcal{E})$ relative to $con(\mathcal{E})$. □

We finally prove Theorems 8.5 and 8.6.

Proof of Theorem 8.5. Let $r \in R_L(\mathcal{E}) \setminus R^\bullet_L(\mathcal{E})$ and $\psi|\phi \in \mathcal{E}$. It is easy to verify that $r(\neg\psi \wedge \phi) = r(\phi) - r(\psi \wedge \phi)$. By Theorem 8.4 (a), some $S \subseteq R^\bullet_L(\mathcal{E})$ exists such that $|S| \geq 2$, $r(\phi) = \sum_{s \in S} s(\phi)$, and $r(\psi \wedge \phi) = \sum_{s \in S} s(\psi \wedge \phi)$. It thus follows that $r(\neg\psi \wedge \phi) = \sum_{s \in S}(s(\phi) - s(\psi \wedge \phi)) = \sum_{s \in S} s(\neg\psi \wedge \phi)$. That is, r is decomposable in $R^\bullet_L(\mathcal{E})$ relative to $\neg\psi \wedge \phi$. □

Proof of Theorem 8.6. (a) Let $r \in R \setminus R^\bullet$. By Theorem 8.4 (a), some $S \subseteq R^\bullet$ exists such that $|S| \geq 2$ and $r(\phi) = \sum_{s \in S} s(\phi)$ for all $\phi \in con(\mathcal{E}_{KB})$. Thus,

$$\begin{aligned}
g_r &= \sum_{i=1}^{n} s_i(p_{ri} - l_i) - \sigma_i(q_{ri} - u_i) \\
&= \sum_{i=1}^{n} s_i(r(\psi_i \wedge \phi_i) - l_i\, r(\phi_i)) - \sigma_i(r(\psi_i \wedge \phi_i) - u_i\, r(\phi_i)) \\
&= \sum_{i=1}^{n} \sum_{s \in S} s_i(s(\psi_i \wedge \phi_i) - l_i\, s(\phi_i)) - \sigma_i(s(\psi_i \wedge \phi_i) - u_i\, s(\phi_i)) \\
&= \sum_{s \in S} \sum_{i=1}^{n} s_i(s(\psi_i \wedge \phi_i) - l_i\, s(\phi_i)) - \sigma_i(s(\psi_i \wedge \phi_i) - u_i\, s(\phi_i)) \\
&= \sum_{s \in S} g_s\,.
\end{aligned}$$

(b) Let $r \in R^\bullet$. We now show that no $S \subseteq R^\bullet$ exists such that $|S| \geq 2$ and $g_r = \sum_{s \in S} g_s$. Towards a contradiction, assume that $g_r = \sum_{s \in S} g_s$ for some $S \subseteq R^\bullet$ with $|S| \geq 2$. In particular, for every $j \in \{1, \ldots, n\}$, if $s_j = \sigma_j > 0$ and $s_i = \sigma_i = 0$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$, then $(u_j - l_j)\, s_j\, r(\phi_j) = g_r = \sum_{s \in S} g_s = \sum_{s \in S}(u_j - l_j)\, s_j\, s(\phi_j)$, and thus $r(\phi_j) = \sum_{s \in S} s(\phi_j)$. Moreover, for every $j \in \{1, \ldots, n\}$, if $s_j = u_j$, $\sigma_j = l_j$, and $s_i = \sigma_i = 0$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$, then $(u_j - l_j)\, r(\psi_j \wedge \phi_j) = g_r = \sum_{s \in S} g_s = \sum_{s \in S}(u_j - l_j) s(\psi_j \wedge \phi_j)$, and so $r(\psi_j \wedge \phi_j) = \sum_{s \in S} s(\psi_j \wedge \phi_j)$. Thus, r is decomposable in $R^\bullet$ relative to $con(\mathcal{E}_{KB})$. But this contradicts Theorem 8.4 (b). Hence, no $S \subseteq R^\bullet$ exists with $|S| \geq 2$ and $g_r = \sum_{s \in S} g_s$. □

References

[1] Amarger, S., D. Dubois, and H. Prade: 1991, 'Constraint propagation with imprecise conditional probabilities'. In: *Proceedings UAI-91*. pp. 26–34.

[2] Benferhat, S., D. Dubois, and H. Prade: 1997, 'Nonmonotonic reasoning, conditional objects and possibility theory'. *Artif. Intell.* **92**(1–2), 259–276.

[3] Biazzo, V. and A. Gilio: 2000, 'A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments'. *Int. J. Approx. Reasoning* **24**(2–3), 251–272.

[4] Biazzo, V. and A. Gilio: 2002, 'On the linear structure of betting criterion and the checking of coherence'. *Ann. Math. Artif. Intell.* **35**, 83–106.

[5] Biazzo, V., A. Gilio, T. Lukasiewicz, and G. Sanfilippo, 'Probabilistic logic under coherence, model-theoretic probabilistic logic, and default reasoning in System P'. Accepted for publication in *Journal of Applied Non-Classical Logics*. Available at `http://www.kr.tuwien.ac.at/staff/lukasiew/jancl02.pdf`.

[6] Biazzo, V., A. Gilio, and G. Sanfilippo: 2000a, 'Computational aspects in checking of coherence and propagation of conditional probability bounds'. In: *Proceedings WUPES-2000*. Jindřichův Hradec, Czech Republic, pp. 1–13.

[7] Biazzo, V., A. Gilio, and G. Sanfilippo: 2000b, 'Efficient checking of coherence and propagation of imprecise probability assessments'. In: *Proceedings IPMU-2000*. Madrid, Spain, pp. 1973–1976.

[8] Boole, G.: 1854, *An Investigation of the Laws of Thought, on which are Founded the Mathematical Theories of Logic and Probabilities*. London: Walton and Maberley. (reprint: Dover Publications, New York, 1958).

[9] Capotorti, A., L. Galli, and B. Vantaggi: 2000, 'How to use strong local coherence in an inferential process based on upper-lower probabilities'. In: *Proceedings WUPES-2000*. Jindřichův Hradec, Czech Republic, pp. 14–28.

[10] Capotorti, A. and B. Vantaggi: 1998, 'An algorithm for coherent conditional probability assessments'. In: *Proceedings IV Congresso Nazionale SIMAI*, Vol. 2. Giardini Naxos (Messina), Italy, pp. 144–148.

[11] Coletti, G.: 1994, 'Coherent numerical and ordinal probabilistic assessments'. *IEEE Trans. Syst. Man Cybern.* **24**(12), 1747–1754.

[12] Coletti, G. and R. Scozzafava: 1996, 'Characterization of coherent conditional probabilities as a tool for their assessment and extension'. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **4**(2), 103–127.

[13] Coletti, G. and R. Scozzafava: 1997, 'Exploiting zero probabilities'. In: *Proceedings EUFIT-97*. Aachen, Germany, pp. 1499–1503.

[14] Coletti, G. and R. Scozzafava: 1999a, 'Coherent upper and lower Bayesian updating'. In: *Proceedings ISIPTA-99*. Ghent, Belgium, pp. 101–110.

[15] Coletti, G. and R. Scozzafava: 1999b, 'Conditioning and inference in intelligent systems'. *Soft Computing* **3**(3), 118–130.

[16] Du, D., J. Gu, and P. M. Pardalos (eds.): 1997, *Satisfiability Problem: Theory and Applications*, Vol. 35 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*. American Mathematical Society.

[17] Dubois, D., H. Prade, and J.-M. Touscas: 1990, 'Inference with imprecise numerical quantifiers'. In: Z. W. Ras and M. Zemankova (eds.): *Intelligent Systems*. Ellis Horwood, Chapt. 3, pp. 53–72.

[18] Eiter, T. and T. Lukasiewicz: 2000, 'Default reasoning from conditional knowledge bases: Complexity and tractable cases'. *Artif. Intell.* **124**(2), 169–241.

[19] Fagin, R., J. Y. Halpern, and N. Megiddo: 1990, 'A logic for reasoning about probabilities'. *Inf. Comput.* **87**, 78–128.

[20] Frisch, A. M. and P. Haddawy: 1994, 'Anytime deduction for probabilistic logic'. *Artif. Intell.* **69**, 93–122.

[21] Gabbay, D. M. and P. Smets (eds.): 1998, *Handbook on Defeasible Reasoning and Uncertainty Management Systems*. Dordrecht, Netherlands: Kluwer Academic.

[22] Garey, M. R. and D. S. Johnson: 1979, *Computers and Intractability: A Guide to the Theory of NP-Completeness*. New York: Freeman.

[23] Georgakopoulos, G., D. Kavvadias, and C. Papadimitriou: 1988, 'Probabilistic satisfiability'. *J. Complexity* **4**(1), 1–11.

[24] Gilio, A.: 1995a, 'Algorithms for precise and imprecise conditional probability assessments'. In: G. Coletti, D. Dubois, and R. Scozzafava (eds.): *Mathematical Models for Handling Partial Knowledge in Artificial Intelligence*. New York: Plenum Press, pp. 231–254.

[25] Gilio, A.: 1995b, 'Probabilistic consistency of conditional probability bounds'. In: *Advances in Intelligent Computing*, Vol. 945 of *LNCS*. pp. 200–209.

[26] Gilio, A.: 2002, 'Probabilistic reasoning under coherence in System P'. *Ann. Math. Artif. Intell.* **34**, 5–34.

[27] Gilio, A. and R. Scozzafava: 1994, 'Conditional events in probability assessment and revision'. *IEEE Trans. Syst. Man Cybern.* **24**(12), 1741–1746.

[28] Goldszmidt, M. and J. Pearl: 1991, 'On the consistency of defeasible databases'. *Artif. Intell.* **52**(2), 121–149.

[29] Hansen, P., B. Jaumard, G.-B. D. Nguetsé, and M. P. de Aragão: 1995, 'Models and algorithms for probabilistic and Bayesian logic'. In: *Proceedings IJCAI-95*. pp. 1862–1868.

[30] Jaumard, B., P. Hansen, and M. P. de Aragão: 1991, 'Column generation methods for probabilistic logic'. *ORSA J. Comput.* **3**, 135–147.

[31] Jenner, B. and J. Toran: 1995, 'Computing functions with parallel queries to NP'. *Theor. Comput. Sci.* **141**, 175–193.

[32] Johnson, D. S.: 1990, 'A catalog of complexity classes'. In: J. van Leeuwen (ed.): *Handbook of Theoretical Computer Science*, Vol. A. MIT Press, Cambridge, MA, Chapt. 2, pp. 67–161.

[33] Lukasiewicz, T.: 1997, 'Efficient global probabilistic deduction from taxonomic and probabilistic knowledge-bases over conjunctive events'. In: *Proceedings of the 6th International Conference on Information and Knowledge Management*. pp. 75–82.

[34] Lukasiewicz, T.: 1999a, 'Local probabilistic deduction from taxonomic and probabilistic knowledge-bases over conjunctive events'. *Int. J. Approx. Reasoning* **21**(1), 23–61.

[35] Lukasiewicz, T.: 1999b, 'Probabilistic deduction with conditional constraints over basic events'. *J. Artif. Intell. Res.* **10**, 199–241.

[36] Lukasiewicz, T.: 2001, 'Probabilistic logic programming with conditional constraints'. *ACM Transactions on Computational Logic* **2**(3), 289–339.

[37] Lukasiewicz, T.: 2002a, 'Nonmonotonic probabilistic logics between model-theoretic probabilistic logic and probabilistic logic under coherence'. In: *Proceedings of the 9th International Workshop on Non-Monotonic Reasoning*. Toulouse, France, pp. 265–274. Available at `http://www.kr.tuwien.ac.at/staff/lukasiew/nmr02.pdf`.

[38] Lukasiewicz, T.: 2002b, 'Probabilistic default reasoning with conditional constraints'. *Ann. Math. Artif. Intell.* **34**(1–3), 35–88.

[39] Luo, C., C. Yu, J. Lobo, G. Wang, and T. Pham: 1996, 'Computation of best bounds of probabilities from uncertain data'. *Computational Intelligence* **12**(4), 541–566.

[40] Nilsson, N. J.: 1986, 'Probabilistic logic'. *Artif. Intell.* **28**, 71–88.

[41] Papadimitriou, C. H.: 1994, *Computational Complexity*. Reading, MA: Addison-Wesley.

[42] Pelessoni, R. and P. Vicig: 1998, 'A consistency problem for imprecise conditional probability assessments'. In: *Proceedings IPMU-98*. Paris, France, pp. 1478–1485.

[43] Scozzafava, R.: 1996, 'Subjective conditional probability and coherence principles for handling partial information'. *Mathware Soft Comput.* **3**(1), 183–192.

[44] Selman, A.: 1994, 'A taxonomy of complexity classes of functions'. *J. Comput. Syst. Sci.* **48**, 357–381.

[45] Vicig, P.: 1996, 'An algorithm for imprecise conditional probability assessments in expert systems'. In: *Proceedings IPMU-96*. Granada, Spain, pp. 61–66.

[46] Wagner, K. W.: 1987, 'More complicated questions about maxima and minima, and some closures of NP'. *Theor. Comput. Sci.* **51**, 53–80.

[47] Walley, P.: 1991, *Statistical Reasoning with Imprecise Probabilities*. London: Chapman and Hall.