**I N F S Y S**

**R** E S E A R C H

**R** E P O R T

INSTITUT FÜR INFORMATIONSSYSTEME

ABTEILUNG WISSENSBASIERTE SYSTEME

# ON COMPUTING SOLUTIONS TO BELIEF CHANGE SCENARIOS


James P. DELGRANDE Torsten SCHAUB
Hans TOMPITS Stefan WOLTRAN



INFSYS RESEARCH REPORT 1843-03-03

MÄRZ 2003



Institut für Informationssysteme
Abtg. Wissensbasierte Systeme
Technische Universität Wien
Favoritenstraße 9-11
A-1040 Wien, Austria
Tel: +43-1-58801-18405
Fax: +43-1-58801-18493
sek@kr.tuwien.ac.at
www.kr.tuwien.ac.at


TECHNISCHE UNIVERSITÄT WIEN

INFSYS RESEARCH REPORT

INFSYS RESEARCH REPORT 1843-03-03, MÄRZ 2003

# ON COMPUTING SOLUTIONS TO BELIEF CHANGE SCENARIOS

James P. Delgrande[1] Torsten Schaub[2],
Hans Tompits[3], and Stefan Woltran[3]

**Abstract.** *Belief change scenarios* have been introduced as a framework for expressing different forms of belief change. The essential idea is that for a belief change scenario $(K, R, C)$ the set of formulas $K$, representing the knowledge base, is modified so that the sets of formulas $R$ and $C$ are respectively true in, and consistent with, the result. By restricting the form of a belief change scenario one obtains specific belief change operators including belief revision, contraction, update, and merging. In this paper, we show how the belief revision and belief contraction operators can be axiomatised by means of quantified Boolean formulas. Basically for both the general approach and for specific operators, we give a quantified Boolean formula such that satisfying truth assignments to the free variables correspond to belief change *extensions* in the original approach. Hence, in this paper we reduce the problem of determining the results of a belief change operation to that of satisfiability. This approach has several benefits. First, it furnishes an axiomatic specification of belief change with respect to belief change scenarios. This alternative characterisation then leads to further insight into the belief change framework. Second, this axiomatisation allows us to identify strict complexity bounds for the considered reasoning tasks. Finally, we obtain an implementation of different forms of belief change by appeal to existing solvers for quantified Boolean formulas.

[1]School of Computing Science Simon Fraser University Burnaby, B.C., Canada V5A 1S6 E-mail: jim@cs.sfu.ca

[2]Institut für Informatik, Universität Potsdam Postfach 90 03 27, D–14439 Potsdam, Germany. E-mail: [sarsakov, torsten]@cs.uni-potsdam.de

[3]Institut für Informationssysteme, Abteilung Wissensbasierte Systeme 184/3, Technische Universität Wien, Favoritenstraße 9-11, A–1040 Wien, Austria. E-mail: [tompits, stefan]@kr.tuwien.ac.at

**Acknowledgements**: This work was partially supported by the Austrian Science Fund Project under grant P15068 as well as a Canadian NSERC Research Grant.

A preliminary version of this paper appeared in the *Proceedings of the Sixth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty* (*ECSQARU'2001*).

# Contents

# 1 Introduction

In [7] a consistency-based framework for expressing belief change operators is developed. The basic idea with respect to belief revision is that, given a knowledge base $K$ and a sentence $\alpha$ for revision, $K$ and $\alpha$ are expressed in disjoint languages; the languages are coerced (via a maximisation process) to agree on truth values of atoms wherever consistently possible; and the result is then re-expressed in the original language. Informally, in the maximisation step, models of $K$ are syntactically forced to correlate with those of $\alpha$ insofar as consistently possible. The inherent nondeterminism of the maximisation process gives rise to two notions of revision. In *choice revision* one such *belief change extension* is selected as the revised state. In general (*skeptical*) *revision*, the revised state consists of the intersection of all such extensions. Belief contraction is defined similarly.

In this paper, we discuss a method to implement this approach to belief change, based on reductions to quantified Boolean formulas. By a quantified Boolean formula (or QBF for short) one understands a formula which is constructed like an ordinary propositional formula, except that quantifiers ranging over propositional variables may also occur. Quantified Boolean formulas thus belong to the language of second-order logic. As well, they allow a compact representation of a large class of properties. The latter point is reflected by the fact that the evaluation problem of QBFs—i.e., the problem of determining the truth of a given QBF—is PSPACE-complete, whilst the evaluation problem of QBFs having prenex normal form with $i - 1$ alternating (groups of) quantifiers is complete for the $i$-th level of the polynomial hierarchy [34; 39].

The general mechanism of our approach is to translate (in polynomial time) a given reasoning task into the evaluation problem for QBFs and then use a QBF evaluator to compute the resultant instances. The existence of efficient QBF solvers, such as the systems developed by Cadoli *et al.* [4], Giunchiglia *et al.* [18], Rintanen [32], or Feldmann *et al.* [15], makes such a rapid prototyping approach practicably applicable.

A similar approach for solving various reasoning tasks belonging to the area of nonmonotonic reasoning has been realized in the system QUIP [12; 11; 27]. This prototype implementation currently handles the computation of the main reasoning tasks for logic-based abduction, default logic, several types of modal nonmonotonic logics, and equilibrium logic, a generalisation of the stable model semantics for logic programs. We have implemented the translations for belief change problems by incorporating them into the system QUIP.

Reduction methods to QBFs naturally generalise similar approaches for problems in NP; these latter problems can in turn be solved by translating them (in polynomial time) to SAT, the satisfiability problem of classical propositional logic (see e.g., [20] for such an application in Artificial Intelligence). Besides the implementation of different nonmonotonic reasoning tasks as realized by the system QUIP, successful applications based on reductions to QBFs have also been applied to conditional planning [31].

There are several reasons why we are interested in a reformulation of the approach of [7] using QBFs. First, it provides a straightforward implementation of the general framework by appeal to extant QBF solvers. Second, in the original approach several steps were expressed at the metalevel. In particular, there is a metatheoretic step in which pairs of atoms are asserted to be equivalent wherever consistent. Here in contrast, we obtain an object-level representation of the approach. In fact, we provide an axiomatisation of the original belief change method in terms of QBFs by constructing suitable translation schemas such that there is a one-to-one correspondence between the satisfying assignments to the free propositional atoms of the QBFs and the belief change extensions obtained in the original framework. This in turn leads to further insight into the original approach. Finally, the expression of belief change problems in terms of QBFs gives a direct way to estimate the computational complexity of the considered reasoning tasks. More specifically, by using the respective QBF encodings, we show that reasoning from choice revision is complete for $\Sigma_2^P$, and, dually, reasoning from skeptical revision is complete for $\Pi_2^P$. Additionally, we also discuss the complexity of other decision problems associated with belief change. In this regard we generalise and improve on the results in [7].

In the next section we briefly introduce notions of belief change as well as those aspects of belief change scenarios that interest us. In Section 3, we similarly introduce quantified Boolean formulas. Section 4 gives the polynomial-time constructible reductions of the relevant reasoning tasks into QBFs. Section 5 discusses complexity issues, while Section 6 briefly sketches an implementation of the reductions. Section 7 supplies some concluding remarks.

# 2 Belief Change and Belief Change Scenarios

## 2.1 Basic Notation

We deal with propositional languages and use the logical symbols $\top$, $\bot$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, and $\equiv$ to construct formulas in the standard way. We write $\mathcal{L}_\mathcal{P}$ to denote a language over an alphabet $\mathcal{P}$ of *propositional variables* or *atoms*. Formulas are denoted by Greek lower-case letters (possibly with subscripts). Disjunctions of form $\bigvee_{i \in I} \psi_i$ are assumed to stand for the logical constant $\bot$ whenever $I = \emptyset$. A *literal*, $L$, is either an atom $p$ (a *positive literal*) or a negated atom $\neg p$ (a *negative literal*). The set of all atoms occurring in a formula $\phi$ is denoted by $Var(\phi)$. Similarly, for a set $S$ of formulas, $Var(S)$ is the set of all atoms occurring in elements of $S$, i.e., $Var(S) = \bigcup_{\phi \in S} Var(\phi)$.

The (propositional) derivability operator, $\vdash$, is defined in the usual way, and likewise its semantic counterpart $\models$. The *deductive closure* of a set $S \subseteq \mathcal{L}_\mathcal{P}$ of formulas is given by $Cn_\mathcal{P}(S) = \{\phi \in \mathcal{L}_\mathcal{P} \mid S \vdash \phi\}$. We say that $S$ is *deductively closed* iff $S = Cn_\mathcal{P}(S)$. Furthermore, $S$ is *consistent* providing $\bot \notin Cn_\mathcal{P}(S)$. If the language is clear from the context, we usually drop the index "$\mathcal{P}$" from $Cn_\mathcal{P}(\cdot)$ and simply write $Cn(\cdot)$. *Knowledge bases*, or, equivalently, *belief sets*, are initially identified with deductively-closed sets of formulas; later we relax this restriction. We use $K, K_1, \ldots$ to denote knowledge bases.

Given an alphabet $\mathcal{P}$, we define a disjoint alphabet $\mathcal{P}'$ as $\mathcal{P}' = \{p' \mid p \in \mathcal{P}\}$. Then, for $\alpha \in \mathcal{L}_\mathcal{P}$, we define $\alpha'$ as the result of replacing in $\alpha$ each atom $p$ from $\mathcal{P}$ by the corresponding atom $p'$ in $\mathcal{P}'$ (so implicitly there is an isomorphism between $\mathcal{P}$ and $\mathcal{P}'$). This is defined analogously for sets of formulas.

## 2.2 General Approaches to Belief Change

A common approach in belief revision and other belief change functions is to provide a set of *rationality postulates* that constrain the results of any such function. The *AGM approach* of Alchourron, Gärdenfors, and Makinson [1; 17] provides the best-known set of such postulates. Belief states are modelled by deductively-closed sets of sentences, called *belief sets*, where the underlying logic includes classical propositional logic. $K + \alpha$, the *expansion* of $K$ by $\alpha$, is defined to be $Cn(K \cup \{\alpha\})$. $K_\bot$ is the inconsistent belief set (i.e. $K_\bot = \mathcal{L}_\mathcal{P}$).

A *revision function*, $\dot{+}$, is a mapping from $2^{\mathcal{L}_\mathcal{P}} \times \mathcal{L}_\mathcal{P}$ to $2^{\mathcal{L}_\mathcal{P}}$ satisfying the following postulates.

$(K\dot{+}1)$ $K\dot{+}\alpha$ is a belief set.

$(K\dot{+}2)$ $\alpha \in K\dot{+}\alpha$.

$(K\dot{+}3)$ $K\dot{+}\alpha \subseteq K + \alpha$.

$(K\dot{+}4)$ If $\neg\alpha \notin K$, then $K + \alpha \subseteq K\dot{+}\alpha$.

$(K\dot{+}5)$ $K\dot{+}\alpha = K_\perp$ iff $\models \neg\alpha$.

$(K\dot{+}6)$ If $\models \alpha \equiv \beta$, then $K\dot{+}\alpha = K\dot{+}\beta$.

$(K\dot{+}7)$ $K\dot{+}(\alpha \wedge \beta) \subseteq (K\dot{+}\alpha) + \beta$.

$(K\dot{+}8)$ If $\neg\beta \notin K\dot{+}\alpha$, then $(K\dot{+}\alpha) + \beta \subseteq K\dot{+}(\alpha \wedge \beta)$.

Informally, these postulates state that the result of revising $K$ by $\alpha$ is a belief set in which $\alpha$ is believed; whenever the result is consistent, revision consists of the expansion of $K$ by $\alpha$; the only time that $K_\perp$ is obtained is when $\alpha$ is inconsistent; and revision is independent of the syntactic form of $K$ and $\alpha$. The last two postulates assert that in revising by a conjunction, an expansion with a conjunct is employed where consistent.

*Contraction* is the dual notion of revision, in which beliefs are retracted but no new beliefs are added. Postulates governing a contraction function $\dot{-}$ are similarly given. The intuition underlying revision and contraction is that an agent receives new information concerning a static world or domain. [19] explores the distinct notions of belief *update* and *erasure* in which an agent changes its beliefs in response to changes in its external environment. As well, belief set *merging* (or *arbitration*), in which the contents of two belief sets are combined, is addressed in [23].

There has also been work on specific revision operators based on the *distance* between models of a knowledge base and a sentence to be incorporated in the knowledge base [3; 36; 5; 33; 38; 16]. For example, in [5] the revision operator measures the distance between interpretations by the number of propositional variables on which the interpretations differ. It is shown that this operator satisfies the AGM postulates.

Another direction in belief revision is to assume that revision is not carried out on a belief set itself, but rather on a finite subset of the theory. Belief change operations would take place with respect to this *belief base*, while the underlying belief set would correspond to the deductive closure of this base. The notion of belief base revision is proposed in [24], and independently, with respect to database systems in [14]. These approaches are fully explored in [25]. While conceptually simple, revision in these approaches frequently relies on arbitrary syntactic distinctions.

There has been some work with respect to implementations. For example, the aforecited distance-based approaches admit straightforward implementations. Otherwise, [37] provides an example of a computational model for belief base revision. (See also [35; 2] for other approaches.)

### 2.3 Belief Change via Belief Change Scenarios

In [7] a consistency-based framework for expressing a suite of belief change operators is developed. The intent was to specify an approach that has good formal properties, but that particularly lent itself to implementation. The approach is discussed formally in the next section; here we give an informal introduction to the approach to revision. As a starting point, it is clear that the syntactic form of a sentence does not give a firm indication as to which sentences should be included in a revision $K\dot{+}\alpha$. Alternately one can consider interpretations, and look at the models of $K$ and $\alpha$. Informally, if $K \cup \{\alpha\}$ is unsatisfiable, a model of $K\dot{+}\alpha$ should contain models of $\alpha$, but in a sense retaining aspects of models of $K$ that do not conflict with those of $\alpha$.

We accomplish this by first expressing $K$ and $\alpha$ in different languages, in essence replacing every occurrence of an atomic sentence $p$ in $K$ by a new atomic sentence $p'$ yielding knowledge base $K'$, and leaving $\alpha$ unchanged. Under this relabelling, the models of $K'$ and $\alpha$ are independent and $K' \cup \{\alpha\}$ is satisfiable (assuming that each of $K$, $\alpha$ are satisfiable). The models of $K'$ and $\alpha$ are linked by asserting that the languages are (with respect to truth conditions) the same wherever consistently possible. That is, for every $p \in \mathcal{P}$, we assert that $p \equiv p'$ wherever consistently possible. We obtain a set of such equivalences, call it $EQ$, such that $K' \cup \{\alpha\} \cup EQ$ is consistent. A model of $K' \cup \{\alpha\} \cup EQ$ then will be a model of $\alpha$ where the truth values of atomic sentences in $K'$ and $\alpha$ are linked wherever possible. A candidate "choice" revision of $K$ by $\alpha$ then consists of $K' \cup \{\alpha\} \cup EQ$ re-expressed in the original language. General revision corresponds to the intersection of all candidate choice revisions.

For example, consider where

$$K = Cn(\{(p \vee q) \wedge r\}) \qquad \text{and} \qquad \alpha = (\neg p \vee \neg q) \wedge \neg r.$$

Renaming the atoms in $K$ gives $K' = Cn(\{(p' \vee q') \wedge r'\})$. Clearly $K' \cup \{\alpha\}$ is consistent, even though $K \cup \{\alpha\}$ is not. In the step to link the respective interpretations of $K'$ and $\alpha$ we obtain that $Cn(K' \cup \{\alpha\} \cup \{p' \equiv p, q' \equiv q\})$ is consistent, but $Cn(K' \cup \{\alpha\} \cup \{p' \equiv p, q' \equiv q, r' \equiv r\})$ is not. Hence we take $EQ = \{p' \equiv p, q' \equiv q\}$. Intersecting $Cn(K' \cup \{\alpha\} \cup EQ)$ with the original language yields $Cn(\{(p \equiv \neg q) \wedge \neg r\})$ as the revised knowledge base.

The general framework allows the expression of contraction and integrity constraints, as well as update, erasure, and merging operations. Significantly, the approach is independent of how the knowledge base and formula for revision are represented. In particular, the original and revised knowledge base can be represented by a formula whose deductive closure gives the corresponding belief set. As well, the scope of a revision (for example) can be restricted to just those propositions common to the knowledge base and sentence for revision. The approach (essentially) satisfies the AGM postulates [1], with the exception of the revision postulate $(K\dot{+}8)$ and the contraction postulate $(K\dot{-}8)$. The approach to belief change is founded on the same intuitions as *consistency-based* reasoning methodologies in Artificial Intelligence. Examples of such systems include Theorist [29], diagnosis from first principles [30], and the assumption-based approach to truth maintenance [6].

## 2.4 Formal Elements of the Belief Change Framework

Following Delgrande and Schaub [7; 8], we define a *belief change scenario* in language $\mathcal{L}_\mathcal{P}$ as a triple $B = (K, R, C)$, where $K, R, C$ are sets of formulas in $\mathcal{L}_\mathcal{P}$. Informally, $K$ is a knowledge base that will be changed such that the set $R$ will be implied by the result, and the set $C$ will be consistent with the result. For a base approach to revision we take $C = \emptyset$ and for a base approach to contraction we take $R = \emptyset$.

We extend our notation $Var(\cdot)$ to belief change scenarios in the obvious way, i.e., for $B = (K, R, C)$, we define $Var(B) = Var(K \cup R \cup C)$. In the definition below, "maximal" is with respect to set containment (rather than set cardinality). The following definition is central [7; 8]:

**Definition 2.1** *Let $B = (K, R, C)$ be a belief change scenario in $\mathcal{L}_\mathcal{P}$. Define $EQ$ as a maximal set of equivalences $EQ \subseteq \{p \equiv p' \mid p \in \mathcal{P}\}$ such that*

$$K' \cup EQ \cup R \cup C \not\vdash \bot.$$

*Then,*

$$Cn(K' \cup EQ \cup R) \cap \mathcal{L}_\mathcal{P}$$

*is a* (consistent) belief change extension *of $B$.*

*If there is no such set $EQ$ then $B$ is* inconsistent *and $\mathcal{L}_\mathcal{P}$ is defined to be the sole* (inconsistent) belief change extension *of $B$.*

| $K'$ | $\alpha$ | $EQ$ | $K\dot{+}\alpha$ |
|---|---|---|---|
| $p' \wedge q'$ | $\neg q$ | $\{p \equiv p'\}$ | $p \wedge \neg q$ |
| $\neg p' \equiv q'$ | $\neg q$ | $\{ p \equiv p',\ q \equiv q' \}$ | $p \wedge \neg q$ |
| $p' \vee q'$ | $\neg p \vee \neg q$ | $\{ p \equiv p',\ q \equiv q' \}$ | $p \equiv \neg q$ |
| $p' \wedge q'$ | $\neg p \vee \neg q$ | $\{p \equiv p'\},\ \{q \equiv q'\}$ | $p \equiv \neg q$ |

Table 1: (Skeptical) revision examples.

So, a (consistent) belief change extension of $B$ is a modification of $K$ in which $R$ is true, and in which $C$ is consistent. We say that $EQ$ *determines* the belief change extension $Cn(K' \cup EQ \cup R) \cap \mathcal{L_P}$ of $B$. Clearly, for a given belief change scenario there may be more than one belief change extension.

Definition 2.1 provides a very general framework for specifying belief change. In what follows, we give specific definitions for the belief change operations *revision* and *contraction*. In these definitions we make use of the notion of a *selection function*, $c$, that for any set $I \neq \emptyset$ has as value some element of $I$. These primitive functions can be regarded as inducing selection functions $c'$ on belief change scenarios, such that $c'(B)$ has as value some belief change extension of $B = (K, R, C)$. This is a slight generalisation of selection functions as found in the AGM approach [17].

**Definition 2.2 (Revision)** *Let $K$ be a knowledge base and $\alpha$ a formula, and let $(E_i)_{i\in I}$ be the family of all belief change extensions of $(K, \{\alpha\}, \emptyset)$. Then,*

1. $K\dot{+}_c\alpha = E_i$ *is a* choice revision *of $K$ by $\alpha$ with respect to some selection function $c$ with $c(I) = i$; and*
2. $K\dot{+}\alpha = \bigcap_{i\in I} E_i$ *is the* (skeptical) revision *of $K$ by $\alpha$.*

Table 1 gives examples of (skeptical) revision. The first column gives the original knowledge base, but with atoms already renamed. The second column gives the revision formula, while the third gives the $EQ$ set(s) and the last column gives the results of the revision. For the first and last column, we give a formula whose deductive closure gives the corresponding belief set.

In detail, for the last example, we wish to determine

$$\{p \wedge q\}\dot{+}(\neg p \vee \neg q)\ .$$

We find maximal sets $EQ \subseteq \{p \equiv p', q \equiv q'\}$ such that

$$\{p' \wedge q'\} \cup EQ \cup \{\neg p \vee \neg q\} \cup \emptyset \not\vdash \bot.$$

We get two such sets of equivalences, viz. $EQ_1 = \{p \equiv p'\}$ and $EQ_2 = \{q \equiv q'\}$. Accordingly, we obtain

$$\{p \wedge q\}\dot{+}(\neg p \vee \neg q) = \bigcap_{i=1,2} Cn(\{p' \wedge q'\} \cup EQ_i \cup \{\neg p \vee \neg q\}) \cap \mathcal{L_P}.$$

In addition to $(\neg p \vee \neg q)$, we get $(p \vee q)$, jointly implying $(p \equiv \neg q)$.

Contraction is defined similarly to revision.

**Definition 2.3 (Contraction)** *Let $K$ be a knowledge base and $\alpha$ a formula, and let $(E_i)_{i\in I}$ be the family of all belief change extensions of $(K, \emptyset, \{\neg\alpha\})$. Then,*

| $K'$ | $\alpha$ | $EQ$ | $K \dot{-} \alpha$ |
|---|---|---|---|
| $p' \wedge q'$ | $q$ | $\{p \equiv p'\}$ | $p$ |
| $p' \wedge q' \wedge r'$ | $p \vee q$ | $\{r \equiv r'\}$ | $r$ |
| $p' \vee q'$ | $p \wedge q$ | $\{ p \equiv p',\ q \equiv q' \}$ | $p \vee q$ |
| $p' \wedge q'$ | $p \wedge q$ | $\{p \equiv p'\},\ \{q \equiv q'\}$ | $p \vee q$ |

Table 2: (Skeptical) contraction examples.

1. $K \dot{-}_c \alpha = E_i$ *is a* choice contraction *of* $K$ *by* $\alpha$ *with respect to some selection function* $c$ *with* $c(I) = i$*; and*
2. $K \dot{-} \alpha = \bigcap_{i \in I} E_i$ *is the* (skeptical) contraction *of* $K$ *by* $\alpha$.

Table 2 gives examples of (skeptical) contraction, using the same format and conventions as Table 1.

In detail, for the first example we wish to determine

$$\{p \wedge q\} \dot{-} q \, .$$

We compute the belief change extensions of $(\{p \wedge q\}, \emptyset, \{\neg q\})$. We rename the propositions in $\{p \wedge q\}$ and look for maximal subsets $EQ$ of $\{p \equiv p', q \equiv q'\}$ such that

$$\{p' \wedge q'\} \cup EQ \cup \emptyset \cup \{\neg q\} \not\vdash \bot.$$

We obtain $EQ = \{p \equiv p'\}$, yielding

$$\begin{aligned} \{p \wedge q\} \dot{-} q &= Cn(\{p' \wedge q'\} \cup \{p \equiv p'\} \cup \emptyset) \cap \mathcal{L}_{\mathcal{P}} \\ &= Cn(\{p\}). \end{aligned}$$

# 3 Quantified Boolean Formulas

Quantified Boolean formulas (QBFs) generalise ordinary propositional formulas by the addition of quantifiers over propositional variables. Informally, a QBF of form $\forall p \, \exists q \, \Phi$ means that for all truth assignments of $p$ there is a truth assignment of $q$ such that $\Phi$ is true. For instance, it is easily seen that the QBF $\exists p \, \exists q \, ((p \rightarrow q) \wedge \forall r (r \rightarrow q))$ evaluates to true. QBFs are denoted by Greek upper-case letters.

The precise semantical meaning of QBFs is defined as follows. First, some ancillary notation. An occurrence of a propositional variable $p$ in a QBF $\Phi$ is *free* iff it does not appear in the scope of a quantifier $\mathsf{Q}p$ ($\mathsf{Q} \in \{\forall, \exists\}$), otherwise the occurrence of $p$ is *bound*. If $\Phi$ contains no free variable occurrences, then $\Phi$ is *closed*, otherwise $\Phi$ is *open*. Furthermore, $\Phi[p_1/\phi_1, \ldots, p_n/\phi_n]$ denotes the result of uniformly substituting each free occurrence of a variable $p_i$ in $\Phi$ by a formula $\phi_i$, for $1 \leq i \leq n$.

By an *interpretation*, $M$, we understand a set of atoms. Informally, an atom $p$ is true under $M$ iff $p \in M$. In general, the truth value, $\nu_M(\Phi)$, of a QBF $\Phi$ under an interpretation $M$ is recursively defined as follows:

1. if $\Phi = \top$, then $\nu_M(\Phi) = 1$;
2. if $\Phi = p$ is an atom, then $\nu_M(\Phi) = 1$ if $p \in M$, and $\nu_M(\Phi) = 0$ otherwise;
3. if $\Phi = \neg\Psi$, then $\nu_M(\Phi) = 1 - \nu_M(\Psi)$;

4. if $\Phi = (\Phi_1 \wedge \Phi_2)$, then $\nu_M(\Phi) = min(\{\nu_M(\Phi_1), \nu_M(\Phi_2)\})$;

5. if $\Phi = \forall p\, \Psi$, then $\nu_M(\Phi) = \nu_M(\Psi[p/\top] \wedge \Psi[p/\bot])$;

The truth conditions for $\bot$, $\vee$, $\rightarrow$, $\equiv$, and $\exists$ follow from the above in the usual way. Note that $\exists$ is defined here similarly as in first-order logic, i.e., $\exists p\Psi = \neg\forall p\neg\Psi$, for each formula $\Psi$. Hence, the truth value for $\exists$ is given by

$$\nu_M(\exists p\Psi) = \nu_M(\Psi[p/\top] \vee \Psi[p/\bot]).$$

We say that $\Phi$ is *true under* $M$ iff $\nu_M(\Phi) = 1$, otherwise $\Phi$ is *false under* $M$. If $\nu_M(\Phi) = 1$, then $M$ is a *model* of $\Phi$. The set of all models of $\Phi$ is denoted by $Mod(\Phi)$. If $Mod(\Phi) \neq \emptyset$, then $\Phi$ is said to be *satisfiable*. If $\Phi$ is true under every interpretation, then $\Phi$ is *valid*. As usual, we write $\models \Phi$ to express that $\Phi$ is valid.

It is easily seen that the truth value of a QBF $\Phi$ under interpretation $M$ depends only on the free variables in $\Phi$. Hence, without loss of generality, for determining the truth value of QBFs, we may restrict our attention to interpretations which contain only atoms occurring free in the given QBF. In particular, closed QBFs are either true under every interpretation or false under every interpretation, i.e., they are either valid or unsatisfiable, so for closed QBFs there is no need to refer to particular interpretations. As well, if a closed QBF $\Phi$ is valid, we say that $\Phi$ *evaluates to true*, and, correspondingly, if $\Phi$ is unsatisfiable, we say that $\Phi$ *evaluates to false*. Two sets of formulas (i.e., ordinary propositional formulas or QBFs) are *logically equivalent* iff they possess the same models. Thus, formulas $\Phi$ and $\Psi$ are logically equivalent iff $\Phi \equiv \Psi$ is valid.

In the sequel, we use the following abbreviations in the context of QBFs: Let $S = \{\phi_1, \ldots, \phi_n\}$ and $T = \{\psi_1, \ldots, \psi_n\}$ be indexed sets of formulas. Then, $S \leq T$ abbreviates $\bigwedge_{i=1}^{n}(\phi_i \rightarrow \psi_i)$, and $S \equiv T$ is an abbreviation for $\bigwedge_{i=1}^{n}(\phi_i \equiv \psi_i)$. Furthermore, for an indexed set $V = \{p_1, \ldots, p_n\}$ of propositional variables and a quantifier $\mathsf{Q} \in \{\forall, \exists\}$, we let $\mathsf{Q}V\,\Phi$ stand for the formula $\mathsf{Q}p_1\mathsf{Q}p_2 \cdots \mathsf{Q}p_n\,\Phi$. Finally, a finite set $T = \{\phi_1, \ldots, \phi_n\}$ of formulas is usually identified with the conjunction $\bigwedge_{i=1}^{n} \phi_i$ of its elements.

The operator $\leq$ is a fundamental tool for expressing tests on sets of formulas in terms of QBFs. In particular, we use $\leq$ in conjunction with the following task:

Given finite sets $S$ and $T$ of formulas, compute all subsets $R \subseteq S$ such that $T \cup R$ is consistent.

This problem can be encoded by a QBF in the following way:

**Proposition 3.1** *Let $S = \{\phi_1, \ldots, \phi_n\}$ and $T$ be finite sets of formulas, let $V$ be the set of atoms occurring in $S \cup T$, and let $G = \{g_1, \ldots, g_n\}$ be a set of new variables not occurring in $S$ or $T$. Furthermore, consider some $R \subseteq S$ and some $M \subseteq G$ such that $\phi_i \in R$ iff $g_i \in M$, for $1 \leq i \leq n$.*

*Then, $T \cup R$ is consistent iff $M$ is a model of the QBF*

$$\mathcal{C}[T, S] = \exists V(T \wedge (G \leq S)).$$

Note that $\mathcal{C}[T, S]$ is an open QBF having $G$ as its set of free variables. These variables facilitate the selection of those elements of $S$ which determine the sets $R$ such that $T \cup R$ is consistent. Moreover, $\mathcal{C}[T, S]$ is designed to express *all* potential subsets $R \subseteq S$ such that $T \cup R$ is consistent. To express, for example, all *maximal* such subsets, some additional elements are required. The computation of maximal sets satisfying certain criteria, using QBFs, is discussed in Section 4.

Finally, we note some useful relations concerning the shifting and renaming of quantifiers, paralleling similar results from standard first-order logic.

**Proposition 3.2** *Let $p$, $q$ be atoms, $\mathsf{Q} \in \{\forall, \exists\}$, and let $\Phi$, $\Psi$ be QBFs such that $\Psi$ does not contain free occurrences of $p$. Then,*

1. $\models (\neg\exists p\, \Phi) \equiv \forall p(\neg\Phi)$;
2. $\models (\neg\forall p\, \Phi) \equiv \exists p(\neg\Phi)$;
3. $\models (\Psi \circ \mathsf{Q}p\, \Phi) \equiv \mathsf{Q}p(\Psi \circ \Phi)$ *for* $\circ \in \{\wedge, \vee, \rightarrow\}$; *and*
4. $\models (\mathsf{Q}q\, \Psi) \equiv (\mathsf{Q}p\, \Psi[q/p])$.

We say that QBF $\Phi$ is in *prenex form* if $\Phi = \mathsf{Q}_1 p_1 \ldots \mathsf{Q}_n p_n \phi$, where $\mathsf{Q}_i \in \{\forall, \exists\}$, for $1 \leq i \leq n$, and $\phi$ is some propositional formula. Proposition 3.2 implies that any QBF $\Psi$ can be effectively transformed into a logical equivalent QBF $\Psi^*$ in prenex form.

# 4 Reductions

In this section, we present efficient (polynomial-time constructible) reductions of the relevant reasoning tasks in the context of belief change scenarios into QBFs. More specifically, these reductions are constructed in such a way that there is a one-to-one correspondence between belief change extensions and models of the translated QBFs. Based on these reductions, in Section 5 we analyse the computational complexity of the considered reasoning tasks.

Concerning the specific tasks, we deal with the following decision problems and their corresponding search problems:

EXT : Decide whether a given belief change scenario $B$ has some consistent belief change extension.

CHOICE : Given a belief change scenario $B$ and some formula $\phi$, decide whether $\phi$ is contained in at least one consistent belief change extension of $B$.

SKEPTICAL : Given a belief change scenario $B$ and some formula $\phi$, decide whether $\phi$ is contained in all belief change extensions of $B$.

Note that EXT and CHOICE are specified with respect to consistent belief change extensions. Dropping the consistency condition in EXT would result in a trivial decision problem, because, according to Definition 2.1, any belief change scenario possesses always at least one belief change extension. Concerning CHOICE, although here, like for EXT, we are primarily interested in consistent belief change extensions, later we relax this condition and deal also with the inconsistent case. For SKEPTICAL, however, the consistency requirement is actually irrelevant, because it holds that a formula $\phi$ is contained in all belief change extensions of a given belief change scenario $B$ iff it is contained in all consistent belief change extensions of $B$. In general, EXT is arguably less interesting than CHOICE or SKEPTICAL, given that it depends only on the consistency of the constituents of the given belief change scenario—however, the relevance of this task lies in fact in the corresponding *search problem*, i.e., in the actual computation of the consistent belief change extensions.

## 4.1 Encodings of the Basic Tasks

From now on we assume that, for any belief change scenario $B = (K, R, C)$, its constituents $K$, $R$, and $C$ are finite; thus, these sets are also represented as the conjunction of their elements. Furthermore, for our subsequent encodings it is convenient to use the following alternative characterisation of belief change extensions, which is a straightforward consequence of results given in [8]. Basically, this characterisation shows that sets of equivalences can be restricted to subsets of $\{p \equiv p' \mid p \in Var(B)\}$.

**Proposition 4.1** *For any belief change scenario $B = (K, R, C)$ in $\mathcal{L}_\mathcal{P}$, there is a one-to-one correspondence between the determining sets $EQ \subseteq \{p \equiv p' \mid p \in \mathcal{P}\}$ of the belief change extensions of $B$ and sets $EQ^\sharp \subseteq \{p \equiv p' \mid p \in Var(B)\}$ satisfying the following conditions:*

*$(a)$ $K' \cup EQ^\sharp \cup R \cup C \nvdash \bot$; and*

*$(b)$ for each $p \in Var(B)$ with $(p \equiv p') \notin EQ^\sharp$, we have $K' \cup EQ^\sharp \cup \{p \equiv p'\} \cup R \cup C \vdash \bot$.*

*In particular, for belief change extension $E$ of $B$ with determining set $EQ \subseteq \{p \equiv p' \mid p \in \mathcal{P}\}$, it holds that $E = Cn(K' \cup EQ^\sharp \cup R) \cap \mathcal{L}_\mathcal{P}$, where $EQ^\sharp$ satisfies Conditions $(a)$ and $(b)$, and $EQ^\sharp = EQ \cap \{p \equiv p' \mid p \in Var(B)\}$.*

Thus, with a slight abuse of notation, we also refer to sets $EQ \subseteq \{p \equiv p' \mid p \in Var(B)\}$ satisfying the above Conditions (a) and (b) as a determining set of belief change extension $Cn(K' \cup EQ \cup R) \cap \mathcal{L}_\mathcal{P}$.

We proceed with the following basic QBF module:

**Definition 4.1** *Let $B = (K, R, C)$ be a belief change scenario over $\mathcal{L}_\mathcal{P}$, let $V = Var(B)$ be the set of variables occurring in $B$, and let $V_{eq} = \{p_{eq} \mid p \in V\}$ be a set of new variables. Then,*

$$\mathcal{M}[B] \quad = \quad K' \wedge (V_{eq} \leq (V \equiv V')) \wedge R.$$

The computation of belief change extensions can be expressed in terms of QBFs as follows:

**Theorem 4.2** *Let $B = (K, R, C)$ be a belief change scenario in $\mathcal{L}_\mathcal{P}$, let $V = Var(B)$ be the atoms occurring in $B$, and let $V_{eq} = \{p_{eq} \mid p \in V\}$ be a set of variables disjoint from $V$ and $V'$. Furthermore, let $EQ \subseteq \{p \equiv p' \mid p \in Var(B)\}$ be a set of equivalences and let $M \subseteq V_{eq}$ be defined such that $p_{eq} \in M$ iff $(p \equiv p') \in EQ$.*

*Then, $Cn(K' \cup EQ \cup R) \cap \mathcal{L}_\mathcal{P}$ is a belief change extension of $B$ iff $M$ is a model of the QBF*

$$\begin{aligned} \mathcal{T}_{ext}[B] \quad = \quad & \exists V \exists V' (\mathcal{M}[B] \wedge C) \wedge \\ & \bigwedge_{p \in V} \Big( \neg p_{eq} \rightarrow \neg \exists V \exists V' ((p \equiv p') \wedge \mathcal{M}[B] \wedge C) \Big). \end{aligned}$$

Note that $V_{eq}$ constitutes the set of free variables of $\mathcal{T}_{ext}[B]$. Intuitively, $V_{eq}$ guesses a set $EQ$ of equivalences determining a belief change extension of $B$. The first conjunct of $\mathcal{T}_{ext}[B]$ checks consistency, and the second conjunct checks whether $EQ$ is maximal with respect to set containment.

For an illustration of the translation $\mathcal{T}_{ext}[\cdot]$, consider the belief change scenario $B = (\{p \wedge q\}, \{\neg p \vee \neg q\}, \emptyset)$ from Section 2.4. The free variables of $\mathcal{T}_{ext}[B]$ are given by $\{p_{eq}, q_{eq}\}$, so we get the following four interpretations serving as potential models of $\mathcal{T}_{ext}[B]$:

$$M_1 = \{\}; \qquad M_3 = \{q_{eq}\};$$
$$M_2 = \{p_{eq}\}; \qquad M_4 = \{p_{eq}, q_{eq}\}.$$

Since $B$ has two belief change extensions, generated by $EQ_1 = \{p \equiv p'\}$ and $EQ_2 = \{q \equiv q'\}$ (cf. Table 1), we expect $M_2$ and $M_3$ to be models of $\mathcal{T}_{ext}[B]$. Let us first look at the left conjunct $\exists V \exists V'(\mathcal{M}[B] \wedge C)$ of $\mathcal{T}_{ext}[B]$. For $B$ as above, we obtain

$$\begin{aligned}&\exists V \exists V'(\mathcal{M}[B] \wedge C) = \\ &\qquad \exists pqp'q'\Big((p' \wedge q') \wedge (p_{eq} \rightarrow (p \equiv p')) \wedge (q_{eq} \rightarrow (q \equiv q')) \wedge (\neg p \vee \neg q)\Big).\end{aligned} \tag{1}$$

This QBF has three models, viz. $M_1$, $M_2$, and $M_3$. Interpretation $M_1$ is a model because both conjuncts $(p_{eq} \rightarrow (p \equiv p'))$ and $(q_{eq} \rightarrow (q \equiv q'))$ of (1) evaluate to true (given that $p_{eq}, q_{eq} \notin M_1$), and since the remaining formula $(p' \wedge q') \wedge (\neg p \vee \neg q)$ is consistent. For $M_2$, we similarly get that $(q_{eq} \rightarrow (q \equiv q'))$ is true and that $(p' \wedge q') \wedge (p_{eq} \rightarrow (p \equiv p')) \wedge (\neg p \vee \neg q)$ is consistent, since $\{p', q', p\}$ is a model of $(p' \wedge q') \wedge (p \equiv p') \wedge (\neg p \vee \neg q)$. $M_3$ is a model by analogous arguments. However, $M_4$ is not a model of (1). This is because, under $M_4$, formula (1) can be reduced to

$$(p' \wedge q') \wedge (p \equiv p') \wedge (q \equiv q') \wedge (\neg p \vee \neg q), \tag{2}$$

which is not satisfiable.

Hence, we are left with three possible models of $\mathcal{T}_{ext}[B]$, viz. $M_1$, $M_2$, and $M_3$.

Now we investigate the remaining conjuncts of $\mathcal{T}_{ext}[B]$, which are given by

$$\begin{aligned}\Phi_1 \;=\; \Big[&\neg p_{eq} \rightarrow \neg\exists pqp'q'\Big((p \equiv p') \wedge (p' \wedge q') \wedge (p_{eq} \rightarrow (p \equiv p')) \wedge \\ &\qquad \wedge (q_{eq} \rightarrow (q \equiv q')) \wedge (\neg p \vee \neg q)\Big)\Big]\end{aligned}$$

and

$$\begin{aligned}\Phi_2 \;=\; \Big[&\neg q_{eq} \rightarrow \neg\exists pqp'q'\Big((q \equiv q') \wedge (p' \wedge q') \wedge (p_{eq} \rightarrow (p \equiv p')) \wedge \\ &\qquad \wedge (q_{eq} \rightarrow (q \equiv q')) \wedge (\neg p \vee \neg q)\Big)\Big].\end{aligned}$$

First, consider interpretation $M_2$. Since $p_{eq} \in M_2$, conjunct $\Phi_1$ evaluates to true, and it remains to analyse $\Phi_2$. The latter formula evaluates to true if

$$\Big((q \equiv q') \wedge (p' \wedge q') \wedge (p_{eq} \rightarrow (p \equiv p')) \wedge (q_{eq} \rightarrow (q \equiv q')) \wedge (\neg p \vee \neg q)\Big) \tag{3}$$

is not satisfiable. However, given $M_2$, (3) reduces to (2), which is indeed unsatisfiable. Hence, $M_2$ is a model of $\Phi_2$ and thus also a model of $\mathcal{T}_{ext}[B]$. By a similar argument it follows that $M_3$ is a model of $\Phi_1 \wedge \Phi_2$. It remains to see that $M_1$ is not a model of $\Phi_1 \wedge \Phi_2$. In fact, it holds that $\nu_{M_1}(\Phi_1) = \nu_{M_1}(\Phi_2) = 0$. We show the case of $\Phi_1$ (the case of $\Phi_2$ follows analogously). Since $M_1 = \{\}$, $\Phi_1$ is false under $M_1$ iff

$$\exists pqp'q'\Big((p \equiv p') \wedge (p' \wedge q') \wedge (p_{eq} \rightarrow (p \equiv p')) \wedge (q_{eq} \rightarrow (q \equiv q')) \wedge (\neg p \vee \neg q)\Big)$$

is true under $M_1$. Given that both $p_{eq}$ and $q_{eq}$ are false under $M_1$, the previous condition holds iff

$$\Big((p \equiv p') \wedge (p' \wedge q') \wedge (\neg p \vee \neg q)\Big) \tag{4}$$

is satisfiable. Clearly, this is the case, since $\{p, p', q'\}$ is a satisfying truth assignment for (4). Thus, $M_1$ is not a model of $\Phi_1$. This shows that $M_1$ is not a model of $\mathcal{T}_{ext}[B]$.

Concerning a QBF encoding for EXT, from Theorem 4.2 it immediately follows that a belief change scenario $B = (K, R, C)$ with $V = Var(B)$ has a consistent belief change extension iff $\exists V_{eq}\mathcal{T}_{ext}[B]$ evaluates to true. However, this encoding is in some sense not optimal because it is possible to characterise EXT in terms of a simpler QBF, corresponding to an ordinary satisfiability problem, by observing that $B$ has a consistent belief change extension iff $K' \cup R \cup C$ is consistent. Consequently, we can state the following result:

**Theorem 4.3** *Let $B = (K, R, C)$ be a belief change scenario in $\mathcal{L}_\mathcal{P}$ and let $V = Var(B)$.*
*Then, $B$ has a consistent belief change extension iff $\exists V \exists V' (K' \wedge R \wedge C)$ evaluates to true.*

Next, we discuss the translations of the reasoning tasks CHOICE and SKEPTICAL. We begin with the encodings of the corresponding search problems.

**Theorem 4.4** *Let $B = (K, R, C)$ be a belief change scenario in $\mathcal{L}_\mathcal{P}$ and let $\phi$ be a formula. Furthermore, let $V = Var(B)$, let $W = Var(B) \cup Var(\phi)$, and let $V_{eq} = \{p_{eq} \mid p \in V\}$ be a set of globally new variables. Finally, consider the QBF*

$$\mathcal{M}[B] \quad = \quad K' \wedge (V_{eq} \leq (V \equiv V')) \wedge R.$$

*from Definition 4.1.*
*Then, for $EQ \subseteq \{p \equiv p' \mid p \in Var(B)\}$ and $M \subseteq V_{eq}$ such that $(p \equiv p') \in EQ$ iff $p_{eq} \in M$, the following properties hold:*

1. *$Cn(K' \cup EQ \cup R) \cap \mathcal{L}_\mathcal{P}$ is a belief change extension of $B$ containing $\phi$ iff $M$ is a model of the QBF*

$$\mathcal{T}_{choice}[B, \phi] \quad = \quad \mathcal{T}_{ext}[B] \wedge \forall W \Big( (\exists W' \mathcal{M}[B]) \rightarrow \phi \Big);$$

2. *$Cn(K' \cup EQ \cup R) \cap \mathcal{L}_\mathcal{P}$ is a belief change extension of $B$ not containing $\phi$ iff $M$ is a model of the QBF*

$$\mathcal{T}_{skept}[B, \phi] \quad = \quad \mathcal{T}_{ext}[B] \wedge \neg\forall W \Big( (\exists W' \mathcal{M}[B]) \rightarrow \phi \Big).$$

Intuitively, the two encodings $\mathcal{T}_{choice}[B, \phi]$ and $\mathcal{T}_{skept}[B, \phi]$ are realised by (i) checking whether a selected set of equivalences determines a consistent belief change extension $E$ of $B$, and (ii) checking whether $E$ contains a given formula $\phi$, or checking whether $E$ does not contain $\phi$. Task (i) is modeled using the basic encoding $\mathcal{T}_{ext}[B]$, and Task (ii) is captured by a suitable QBF expressing derivability (in case of $\mathcal{T}_{choice}[B, \phi]$) or non-derivability (in case of $\mathcal{T}_{skept}[B, \phi]$) of $\phi$ from the selected belief change extension. Observe that the selection process is facilitated in terms of the members from $V_{eq}$, which represent the free variables of $\mathcal{T}_{choice}[B, \phi]$ and $\mathcal{T}_{skept}[B, \phi]$.

Concerning the decision problems CHOICE and SKEPTICAL, QBF encodings for these tasks are obtained from $\mathcal{T}_{choice}[B, \phi]$ and $\mathcal{T}_{skept}[B, \phi]$ as follows. CHOICE is expressed by the closed QBF $\exists V_{eq}\mathcal{T}_{choice}[B, \phi]$, which states that there is some some set $M \subseteq V_{eq}$ corresponding to a set $EQ$ of equivalences such that $EQ$ determines a consistent belief change extension $E$ entailing $\phi$, and SKEPTICAL is realised by the closed QBF $\neg\exists V_{eq}\mathcal{T}_{skept}[B, \phi]$, which expresses that there is no set $M \subseteq V_{eq}$ corresponding to a set $EQ$ of equivalences such that $EQ$ determines a consistent belief change extension $E$ not entailing $\phi$. Thus, we obtain the following corollary:

**Corollary 4.5** *Let $B$ be a belief change scenario and $\phi$ a formula. Then,*

1. *$\phi$ is contained in at least one consistent belief change extension of $B$ iff $\exists V_{eq}\, \mathcal{T}_{choice}[B, \phi]$ evaluates to true; and*

2. *$\phi$ is contained in all belief change extensions of $B$ iff $\neg \exists V_{eq}\, \mathcal{T}_{skept}[B, \phi]$ evaluates to true.*

In contrast to task EXT, which can be expressed by a QBF containing only one sort of quantifier, here we obtain encodings *possessing both existential quantifiers as well as universal ones*. As we show in Section 5, this quantifier alternation is in some sense unavoidable and reflects the inherent complexity of CHOICE and SKEPTICAL.

We remark that discarding the consistency condition of CHOICE can be easily incorporated into the translation $\exists V_{eq}\, \mathcal{T}_{choice}[\cdot, \cdot]$. Indeed, it is a simple matter to check that there is a (possibly inconsistent) belief change extension of $B = (K, R, C)$ containing formula $\phi$ iff the QBF

$$(\exists V \exists V' (K' \wedge R \wedge C)) \rightarrow (\exists V_{eq}\, \mathcal{T}_{choice}[B, \phi])$$

evaluates to true.

Finally, observe that Theorems 4.2, 4.3, and 4.4, as well as Corollary 4.5, provide encodings of reasoning tasks for *arbitrary* belief change scenarios. In particular, they subsume the characterisation of the corresponding reasoning tasks associated with revision and contraction, as illustrated by the revision example discussed previously. For convenience, we list the tasks for revision:

REXT : Given a knowledge base $K$ and some formula $\alpha$, decide whether a consistent belief change extension of $B = (K, \{\alpha\}, \emptyset)$ exists.

RCHOICE : Given a knowledge base $K$ and formulas $\alpha$ and $\phi$, decide whether there is some consistent choice revision $K \dot{+}_c \alpha$ containing $\phi$.

RSKEPTICAL : Given a knowledge base $K$ and formulas $\alpha$ and $\phi$, decide whether $\phi$ is contained in the skeptical revision $K \dot{+} \alpha$.

The corresponding tasks for belief contraction, denoted by CEXT, CCHOICE, and CSKEPTICAL, are defined accordingly.

### 4.2 Expressing Changed Knowledge Bases

Another interesting issue in the context of belief change is to determine the actual form of a given knowledge base after a revision or contraction operation has been applied to it. This task has already been analysed in [7; 8], and, as we show in the following, it can also be described in terms of QBFs. Before going into details, we briefly summarize the relevant results from [7; 8], starting with some notation.

Given a belief change scenario $B$ in language $\mathcal{L}_{\mathcal{P}}$ along with a set of equivalences $EQ_i \subseteq \{p \equiv p' \mid p \in \mathcal{P}\}$, define

$$\begin{aligned} \mathcal{P}_{EQ_i} &= \{p \in \mathcal{P} \mid p \equiv p' \in EQ_i\}; \quad \text{and} \\ \mathcal{P}_{\overline{EQ_i}} &= \mathcal{P} \setminus \mathcal{P}_{EQ_i}. \end{aligned}$$

Then, for $\phi \in \mathcal{L}_{\mathcal{P}}$, let $\lceil \phi \rceil_i$ be the result of replacing in $\phi$ each $p \in \mathcal{P}_{\overline{EQ_i}}$ by $\neg p$. Furthermore, for a set of functions

$$\Pi_i \;=\; \{\pi_i^k \mid \pi_i^k : \mathcal{P}_{\overline{EQ_i}} \rightarrow \{\top, \bot\}\},$$

let $\lfloor\phi\rfloor_i^k$ be the result of replacing in $\phi$ each $p \in \mathcal{P}_{\overline{EQ}_i}$ by $\pi_i^k$.

Then, the result of applying revision or contraction to a given knowledge base is described by the following formulas [7; 8]:

**Proposition 4.6** *Let $K$ be a finite knowledge base and $\alpha$ some formula. Then,*

1. *$K\dot{+}\alpha$ is logically equivalent to $\bigvee_{i\in I}\lceil K\rceil_i \wedge \alpha$; and*
2. *$K\dot{-}\alpha$ is logically equivalent to $\bigvee_{i\in I,\pi_i^k\in\Pi_i}\lfloor K\rfloor_i^k$,*

*for $(EQ_i)_{i\in I}$ being the family of sets of equivalences determining the belief change extensions of $(K,\{\alpha\},\emptyset)$ and $(K,\emptyset,\{\neg\alpha\})$, respectively.*

We can express the models of revision and contraction by QBFs using the following construction.

**Definition 4.2** *Let $B$ be a belief change scenario in $\mathcal{L}_\mathcal{P}$ with $V = \mathit{Var}(B)$, and let $V_{eq} = \{p_{eq} \mid p \in \mathit{Var}(B)\}$ be a set of new variables. Then,*

$$\mathcal{T}_m[B] \quad = \quad \exists V_{eq}(\mathcal{T}_{ext}[B] \wedge \exists V'\mathcal{M}[B]).$$

Observe that $\mathcal{T}_m[B]$ is an open QBF having $V$ as its set of free variables.

We obtain the following result:

**Theorem 4.7** *Let $K$ be a finite knowledge base, $\alpha$ some formula, and $V = \mathit{Var}(K \cup \{\alpha\})$. Then,*

1. *$M \subseteq V$ is a model of $K\dot{+}\alpha$ iff $M$ is a model of $\mathcal{T}_m[(K,\{\alpha\},\emptyset)]$, and*
2. *$M \subseteq V$ is a model of $K\dot{-}\alpha$ iff $M$ is a model of $\mathcal{T}_m[(K,\emptyset,\{\neg\alpha\})]$.*

By collecting all the models of $K\dot{+}\alpha$ or $K\dot{-}\alpha$ into a single formula, we thus obtain a disjunctive normal form of $K$ after revision or contraction with $\alpha$.

To formally express this, given a belief change scenario $B$ with $V = \mathit{Var}(B)$, define

$$F[B] \quad = \quad \bigvee_{M\in Mod(\mathcal{T}_m[B])}\Big(M \wedge \bigwedge_{p\in V\setminus M}\neg p\Big).$$

Then, we get the following characterisation, representing an alternative to Proposition 4.6:

**Corollary 4.8** *Let $K$ and $\alpha$ be as in Theorem 4.7. Then,*

1. *$K\dot{+}\alpha$ is logically equivalent to $F[(K,\{\alpha\},\emptyset)]$, and*
2. *$K\dot{-}\alpha$ is logically equivalent to $F[(K,\emptyset,\{\neg\alpha\})]$.*

For illustration of transformation $\mathcal{T}_m[\cdot]$, consider again the belief change scenario $B = (\{p \wedge q\},\{\neg p \vee \neg q\},\emptyset)$. As already discussed in Section 2.4, $B$ possesses the skeptical revision $Cn(p \equiv \neg q)$, which can be rewritten as $Cn((p \wedge \neg q) \vee (\neg p \wedge q))$. Thus, in virtue of Theorem 4.7, we expect $\{p\}$ and $\{q\}$ to be the models of

$$\mathcal{T}_m[B] \quad = \quad \exists V_{eq}\Big(\mathcal{T}_{ext}[B] \wedge \exists V'\mathcal{M}[B]\Big).$$

We first compute the models of the QBF

$$\mathcal{T}_{ext}[B] \wedge \exists V' \mathcal{M}[B]. \quad (5)$$

The free variables of (5) are given by $V_{eq} \cup V = \{p_{eq}, q_{eq}\} \cup \{p, q\}$. As argued previously, $\mathcal{T}_{ext}[B]$ is true under interpretations $\{p_{eq}\}$ and $\{q_{eq}\}$. In fact, since $\mathcal{T}_{ext}[B]$ has no free variables from $V$, it holds that $\{p_{eq}\} \cup U$ and $\{q_{eq}\} \cup U$ are also models of $\mathcal{T}_{ext}[B]$, for any set $U \subseteq V$. Consider now the second conjunct of (5), which is given by

$$\exists p' q' \Big( (p' \wedge q') \wedge (p_{eq} \rightarrow (p \equiv p')) \wedge (q_{eq} \rightarrow (q \equiv q')) \Big).$$

This formula is obviously true under the following interpretations:

$$M_1 = \{p_{eq}, p\}; \qquad M_2 = \{q_{eq}, q\},$$

as well as under every interpretation in which both $p_{eq}$ and $q_{eq}$ are false. Thus, only $M_1$ and $M_2$ satisfy both conjuncts of (5). Taking the existential closure of QBF (5) with respect to the variables $p_{eq}$ and $q_{eq}$, we get that $\{p\}$ and $\{q\}$ are the only models of $\mathcal{T}_m[B]$.

## 5 Complexity Results

In this section, we analyse the computational complexity of the tasks considered so far. Additionally, we also deal with the complexity of checking whether a given set of equivalences determines some consistent belief change extension of a given belief change scenario.

A particular advantage of our reduction approach is that upper complexity bounds are derived directly from the respective QBF encodings. This is due to the fact that our QBF translations are polynomial in the size of a given belief change scenario, and that the complexity of evaluating a given QBF $\Phi$ is determined by the quantifier order of $\Phi$. For each of the upper bounds obtained in this fashion, we show also that they are *strict*, i.e., they possess a matching lower bound. The results presented here strengthen the complexity analysis given in [7].

In what follows, we assume that the reader is familiar with the basic concepts of complexity theory (see, e.g., [26]). For convenience, we briefly recapitulate the definitions and some elementary properties of the complexity classes considered in the following. As usual, for any complexity class $C$, by co-$C$ we understand the class of all problems which are complementary to the problems in $C$.

Four complexity classes are relevant here, viz. NP, $\mathrm{D}^P$, $\Sigma_2^P$, and $\Pi_2^P$. The class NP consists of all decision problems which can be solved with a nondeterministic Turing machine working in polynomial time; $\mathrm{D}^P$ is defined as the class of all problems which can be described as the conjunction of two (independent) problems from NP and co-NP; $\Sigma_2^P$ is the class of all problems solvable on a nondeterministic Turing machine in polynomial time having access to an oracle for problems in NP; and $\Pi_2^P = \text{co-}\Sigma_2^P$.

Observe that NP, $\Sigma_2^P$, and $\Pi_2^P$ are part of the polynomial hierarchy, which is given by the following sequence of objects: the initial elements are

$$\Delta_0^P = \Sigma_0^P = \Pi_0^P = \mathrm{P};$$

and, for $i > 0$,

$$\Delta_i^P = \mathrm{P}^{\Sigma_{i-1}^P}; \qquad \Sigma_i^P = \mathrm{NP}^{\Sigma_{i-1}^P}; \qquad \Pi_i^P = \text{co-NP}^{\Sigma_{i-1}^P}.$$

Here, P is the class of all problems solvable on a deterministic Turing machine in polynomial time, and, for complexity classes $C$ and $A$, the notation $C^A$ stands for the *relativized version* of $C$, consisting of all problems which can be decided by Turing machines of the same sort and time bound as in $C$, only that the machines have access to an oracle for problems in $A$. It holds that $\Sigma_1^P = \mathrm{NP}$, $\Sigma_2^P = \mathrm{NP}^{\mathrm{NP}}$, and $\Pi_2^P = \text{co-NP}^{\mathrm{NP}}$. A problem is said to be at the *k-th level* of the polynomial hierarchy iff it is in $\Delta_{k+1}^P$ and either $\Sigma_k^P$-hard or $\Pi_k^P$-hard.

The class $\mathrm{D}^P$ can be described as part of a family of complexity classes $\mathrm{D}_k^P$, $k \geq 1$, where $\mathrm{D}^P = \mathrm{D}_1^P$ and each $\mathrm{D}_k^P$ consists of all problems expressible as the conjunction of a problem in $\Sigma_k^P$ and a problem in $\Pi_k^P$. Notice that, for all $k \geq 1$, $\Sigma_k^P \subseteq \mathrm{D}_k^P \subseteq \Sigma_{k+1}^P$ holds; both inclusions are widely conjectured to be strict. Moreover, any problem in $\mathrm{D}_k^P$ can be solved with two $\Sigma_k^P$ oracle calls, and is thus intuitively easier than a problem complete for $\Delta_k^P$.

In the same way as the satisfiability problem of classical propositional logic is the "prototypical" problem of NP, i.e., being an NP-complete problem, the satisfiability problem of QBFs possessing $k-1$ quantifier alternations is the "prototypical" problem of the $k$-th level of the polynomial hierarchy. More specifically, the following property holds:

**Proposition 5.1 ([39])** *Given a propositional formula $\phi$ whose atoms are partitioned into $i \geq 1$ sets $V_1, \ldots, V_i$, deciding whether $\exists V_1 \forall V_2 \exists V_3 \ldots \mathsf{Q} V_i \phi$ evaluates to true, where $\mathsf{Q} = \exists$ if $i$ is odd and $\mathsf{Q} = \forall$ if $i$ is even, is $\Sigma_i^P$-complete. Moreover, the problem remains $\Sigma_i^P$-hard even if $\phi$ is in conjunctive normal form and $i$ is odd, or if $\phi$ is in disjunctive normal form and $i$ is even.*

From this result it follows that the evaluation problem of QBFs of form $\forall V_1 \exists V_2 \forall V_3 \ldots \mathsf{Q} V_i \phi$ is $\Pi_i^P$-complete, where $\mathsf{Q} = \forall$ if $i$ is odd and $\mathsf{Q} = \exists$ if $i$ is even. As well, the problem remains $\Pi_i^P$-hard even if $\phi$ is in disjunctive normal form and $i$ is odd, or if $\phi$ is in conjunctive normal form and $i$ is even.

Given the above characterisations, we can estimate upper complexity bounds for the decision problems discussed in Section 4 by simply inspecting the quantifier order of the respective QBF encodings. This can be argued as follows. First of all, by applying the transformation rules described in Proposition 3.2, each of the above QBF encodings can be transformed in polynomial time into a closed QBF in prenex form. Then, by invoking Proposition 5.1 and observing that completeness of a decision problem $D$ for a complexity class $C$ implies membership of $D$ in $C$, the quantifier order of the resultant QBFs determines in which class of the polynomial hierarchy the corresponding decision problem lies.

Applying this method to the decision problems EXT, CHOICE, and SKEPTICAL, we get the following results. To begin with, according to Theorem 4.3, we have that EXT lies in NP. Hence, REXT and CEXT are also in NP because they are just special cases of EXT. Furthermore, the encoding $\exists V_{eq}\, \mathcal{T}_{choice}[B, \phi]$ for CHOICE can be transformed into a QBF of prenex form $\exists W_1 \forall W_2 \psi$, and, dually, the encoding $\neg \exists V_{eq}\, \mathcal{T}_{skept}[B, \phi]$ for SKEPTICAL can be transformed into a QBF of prenex form $\forall Z_1 \exists Z_2 \varphi$. Thus, CHOICE is in $\Sigma_2^P$, and SKEPTICAL is in $\Pi_2^P$. Similar to the case of EXT, $\Sigma_2^P$ is also an upper bound for RCHOICE and CCHOICE, and $\Pi_2^P$ is an upper bound for RSKEPTICAL and CSKEPTICAL.

Concerning lower complexity bounds, it turns out that all of the above given estimations are *strict*, i.e., the considered decision problems are hard for the respective complexity classes. Summarizing, we can state the following results:

**Theorem 5.2** *The decision problems* EXT, CHOICE, *and* SKEPTICAL, *as well as its variants for revision and contraction, enjoy the following completeness properties:*

1. EXT, REXT, *and* CEXT *are* NP*-complete;*

2. CHOICE, RCHOICE, *and* CCHOICE *are* $\Sigma_2^P$*-complete; and*

3. SKEPTICAL, RSKEPTICAL, *and* CSKEPTICAL *are* $\Pi_2^P$*-complete.*

Thus, the completeness results for CHOICE and SKEPTICAL, as well as for their specialisations for revision and contraction, imply that, unless the polynomial hierarchy collapses, it is not possible to efficiently represent these tasks in terms QBFs having a prenex form with only one sort of quantifier, i.e., *these tasks cannot be polynomially reduced to standard propositional logic*. Hence, under the above proviso, the encodings described in Corollary 4.5 cannot be simplified further to avoid an inherent quantifier alternation.

Rounding off our complexity analysis, we deal with the problem of checking whether a given set of equivalences determines some consistent belief change extension of a given belief change scenario.

**Theorem 5.3** *Given a belief change scenario $B$ and a set $EQ \subseteq \{p \equiv p' \mid p \in Var(B)\}$, checking whether $EQ$ determines some consistent belief change extension of $B$ is* $\mathrm{D}^P$*-complete.*

# 6 Implementation

Our methodology for expressing reasoning tasks associated with belief change scenarios in terms of quantified Boolean formulas is motivated by the availability of several practicably efficient QBF-solvers. Among the different tools, there is a propositional theorem-prover, `boole`, based on *binary decision diagrams* (the system can be downloaded from the Web at http://www.cs.cmu.edu/∼modelcheck/bdd.html), a system using a generalised resolution principle [21], several provers implementing an extended Davis-Putnam procedure [4; 18; 22; 15; 32], as well as a distributed algorithm running on a PC-cluster [15].

The translations discussed in the previous section have been implemented as a special module of the reasoning system QUIP [12; 11; 10; 27], a prototype tool for solving various nonmonotonic reasoning tasks based on reductions to QBFs. Among others, QUIP handles tasks for logic-based abduction, default logic, several types of modal nonmonotonic logics, and the stable model semantics for logic programs.

The general architecture of QUIP is depicted in Figure 1. QUIP consists of three parts, namely the `filter` program, a QBF-evaluator, and the interpreter `int`. The input filter translates the given problem description (in our case, a belief change scenario and a specified reasoning task) into the corresponding quantified Boolean formula, which is then sent to the QBF-evaluator. The current version of QUIP provides interfaces to most of the sequential QBF-solvers mentioned above. For the solvers requiring prenex normal form, the QBFs are translated into structure preserving normal form [9; 28]. The result of the QBF-evaluator is interpreted by `int`. Depending on the capabilities of the employed QBF-evaluator, `int` provides an explanation in terms of the underlying problem instance (e.g., listing all consistent definitional extensions of a given belief change scenario). This task relies on a protocol mapping of internal variables of the generated QBF into concepts of the problem description which is provided by `filter`.

The system QUIP has been implemented in C using standard tools like LEX and YACC (comprising a total of 2000 lines of code, excluding the used QBF-solver); it runs currently in a Unix environment (Sun/Solaris and Linux), but is easily portable to other operating systems.

# 7 Conclusion

We have shown how belief revision and belief contraction, as defined using belief change scenarios, can be axiomatised by means of quantified Boolean formulas. The general mechanism of our approach is to

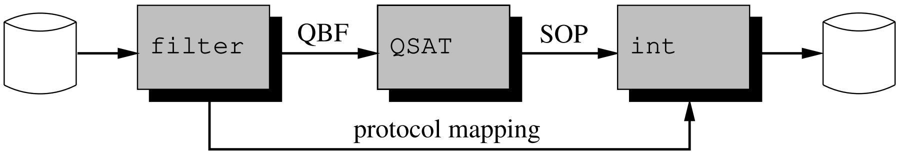


Figure 1: Architecture to use different QBF-solvers.

translate (in polynomial time) a reasoning problem, expressed in terms of belief change scenarios, into the evaluation problem for QBFs. Following this we use a QBF-evaluator to compute the resultant instances.

The approach has several benefits. First, the given axiomatics provides us with further insight about how belief revision and contraction work within belief change scenarios. As well, this axiomatisation allows us to furnish upper bounds for precise complexity results, going beyond those presented in [7]. Last but not least we obtain a straightforward implementation technique of belief change in belief change scenarios by appeal to the existing system QUIP [12; 11]. The implemented operators possess good formal properties, in that most (except for the last) AGM postulates obtain. In particular, the postulate of irrelevance of syntax is retained, and so the results of a belief change operation is independent of the syntactic expression of its arguments. While the interesting decision problems involving reasoning lie at the second level of the polynomial hierarchy, it remains to be seen whether the implementation may nonetheless prove practical for large-scale applications.

# A Proofs

**Proof**. According to the semantics of the existential quantifier $\exists$, and since for any set $U$ of formulas, $U$ is consistent iff $U$ is satisfiable, it obviously holds that $T \cup R$ is consistent iff $\exists V(T \wedge R)$ evaluates to true. By observing that $R$ and $M$ are chosen in such a way that $\phi_i \in R$ iff $g_i \in M$, for $1 \leq i \leq n$, it follows that $\exists V(T \wedge R)$ can be rewritten as

$$\exists V\Big(T \wedge \bigwedge_{g_i \in M} \phi_i\Big). \tag{6}$$

But (6) evaluates to true exactly if

$$\exists V\Big(T \wedge \bigwedge_{i=1}^{n} (g_i \rightarrow \phi_i)\Big) \tag{7}$$

is true under $M$, because each $g_i \rightarrow \phi_i$ is equivalent to $\phi_i$ if $g_i \in M$, and $g_i \rightarrow \phi_i$ is trivially true under $M$ if $g_i \notin M$. Finally, it remains to observe that QBF (7) coincides with $\mathcal{C}[T, S]$, by definition of $G \leq S$. $\Box$

**Proof**. Let us write $\mathcal{T}_{ext}[B]$ as $\Phi_1 \wedge \Phi_2$, where

$$\begin{aligned}
\Phi_1 &= \exists V \exists V'(\mathcal{M}[B] \wedge C); \quad \text{and} \\
\Phi_2 &= \bigwedge_{p \in V} \Big(\neg p_{eq} \rightarrow \neg \exists V \exists V'((p \equiv p') \wedge \mathcal{M}[B] \wedge C))\Big).
\end{aligned}$$

Consider Conditions (a) and (b) of Proposition 4.1. We show that Condition (a) holds iff $M$ is a model of $\Phi_1$, and that (b) holds iff $M$ is a model of $\Phi_2$.

To begin with, since, by hypothesis, $EQ$ and $M$ satisfy the condition that $(p \equiv p') \in EQ$ iff $p_{eq} \in M$, we can apply Proposition 3.1 and obtain that $K' \cup EQ \cup R \cup C$ is satisfiable iff $M$ is a model of

$$\exists V \exists V' \Big( K' \wedge R \wedge C \wedge (V_{eq} \leq (V \equiv V')) \Big),$$

which is obviously equivalent to $\Phi_1$. It remains to show that Condition (b) holds iff $M$ is a model of $\Phi_2$.

Consider some $p \in Var(B)$ such that $(p \equiv p') \notin EQ$ and $K' \cup EQ \cup \{p \equiv p'\} \cup R \cup C \vdash \perp$. Invoking Proposition 3.1 again, it follows that the last condition holds exactly if the QBF

$$\neg \exists V \exists V' \Big( K' \wedge (p \equiv p') \wedge R \wedge C \wedge (V_{eq} \leq (V \equiv V')) \Big)$$

is true under $M$. In general, if we perform this test for each $p \in Var(B)$ with $(p \equiv p') \notin EQ$, we get that Condition (b) is equivalent to the condition that the QBF

$$\bigwedge_{p \in V, (p \equiv p') \notin EQ} \neg \exists V \exists V' \Big( K' \wedge (p \equiv p') \wedge R \wedge C \wedge (V_{eq} \leq (V \equiv V')) \Big) \tag{8}$$

is true under $M$. Observing that, for any $p \in V$, $(p \equiv p') \in EQ$ iff $p_{eq} \in M$, it follows that $M$ is a model of (8) iff it is a model of

$$\bigwedge_{p \in V} \neg p_{eq} \rightarrow \neg \exists V \exists V' \Big( K' \wedge (p \equiv p') \wedge R \wedge C \wedge (V_{eq} \leq (V \equiv V')) \Big). \tag{9}$$

Since (9) is logically equivalent to $\Phi_2$, we conclude that Condition (b) holds iff $M$ is a model of $\Phi_2$. $\Box$

**Proof**. According to Theorem 4.2, $Cn(K' \cup EQ \cup R) \cap \mathcal{L}_{\mathcal{P}}$ is a consistent belief change extension of $B$ iff $M$ is a model of $\mathcal{T}_{ext}[B]$. Thus, for proving Part 1 of the theorem, it suffices to show that the following condition holds:

$(*)$ $\phi \in Cn(K' \cup EQ \cup R)$ iff $M$ is a model of $\forall W \Big( (\exists W' \mathcal{M}[B]) \rightarrow \phi \Big)$.

Furthermore, since $M$ is a model of $\forall W ((\exists W' \mathcal{M}[B]) \rightarrow \phi)$ precisely if $M$ is not a model of $\neg \forall W ((\exists W' \mathcal{M}[B]) \rightarrow \phi)$, we get that Condition $(*)$ implies that $\phi \notin Cn(K' \cup EQ \cup R)$ iff $M$ is a model of $\neg \forall W ((\exists W' \mathcal{M}[B]) \rightarrow \phi)$, which in turn proves Part 2 of the theorem. It remains to show that $(*)$ holds.

Since $\phi \in Cn(K' \cup EQ \cup R)$ iff $K' \cup EQ \cup R \cup \{\neg \phi\}$ is unsatisfiable, Proposition 3.1 implies that $\phi \in Cn(K' \cup EQ \cup R)$ iff $M$ is a model of

$$\neg \exists W \exists W' \Big( K' \wedge R \wedge \neg \phi \wedge (V_{eq} \leq (V \equiv V')) \Big). \tag{10}$$

Given that $\phi$ does not contain any primed atoms, we can rewrite (10) by moving $\neg \phi$ outside the scope of the quantifier $\exists W'$, thus obtaining

$$\neg \exists W \Big( \exists W' (K' \wedge R \wedge (V_{eq} \leq (V \equiv V'))) \wedge \neg \phi \Big),$$

which is in turn equivalent to

$$\neg \exists W \Big( (\exists W' \mathcal{M}[B]) \wedge \neg \phi \Big). \tag{11}$$

But (11) is clearly equivalent to

$$\forall W\Big((\exists W'\mathcal{M}[B]) \;\rightarrow\; \phi\Big), \tag{12}$$

and therefore we obtain that $\phi \in Cn(K' \cup EQ \cup R)$ iff $M$ is a model of (12). $\square$

**Proof**. Let $K$ be a finite knowledge base, $\alpha$ some formula, and $V = Var(K \cup \{\alpha\})$. Recall that, for any set $EQ$ of equivalences, $\mathcal{P}_{EQ} = \{p \in \mathcal{P} \mid p \equiv p' \in EQ\}$ and $\mathcal{P}_{\overline{EQ}} = \mathcal{P} \setminus \mathcal{P}_{EQ}$.

1. Consider $B = (K, \{\alpha\}, \emptyset)$ in $\mathcal{L}_\mathcal{P}$. Suppose that $M \subseteq V$ is a model of $K \dot{+} \alpha$. By Proposition 4.6, there is a consistent belief change extension $E_{i_0} = Cn(K' \cup EQ_{i_0} \cup \{\alpha\}) \cap \mathcal{L}_\mathcal{P}$ of $B$ (for some $i_0 \in I$) such that $\lceil K \rceil_{i_0} \wedge \alpha$ is true under $M$. Define $M_1 = M \cap \mathcal{P}_{EQ_{i_0}}$ and $M_2 = M \setminus M_1$. By construction of formula $\lceil K \rceil_{i_0}$, it follows that $M_1 \cup \overline{M}_2$ is a model of $K$, where $\overline{M}_2 = (V \cap \mathcal{P}_{\overline{EQ}_{i_0}}) \setminus M$. Hence, a simple renaming yields that $M_1' \cup \overline{M}_2'$ is a model of $K'$. It follows that $M \cup M_1' \cup \overline{M}_2'$ is a model of

   $$K' \;\wedge \bigwedge_{p \equiv p' \in EQ_{i_0}} (p \equiv p') \;\wedge\; \alpha.$$

   By setting $J = \{p_{eq} \mid p \equiv p' \in EQ_{i_0}\}$, we get that $M \cup J \cup M_1' \cup \overline{M}_2'$ is a model of

   $$\mathcal{M}[B] \;\;=\;\; K' \;\wedge\; \bigwedge_{p \in V} \Big(p_{eq} \;\rightarrow\; (p \equiv p')\Big) \;\wedge\; \alpha,$$

   which in turn implies that $M \cup J$ is a model of $\exists V'\mathcal{M}[B]$. On the other hand, since $E_{i_0}$ is a consistent belief change extension of $B$, Theorem 4.2 entails that $J$ is a model of $\mathcal{T}_{ext}[B]$. Consequently, $M \cup J$ is a model of $\mathcal{T}_{ext}[B] \;\wedge\; \exists V'\mathcal{M}[B]$, and therefore

   $$\mathcal{T}_m[B] \;\;=\;\; \exists V_{eq}\Big(\mathcal{T}_{ext}[B] \;\wedge\; \exists V'\mathcal{M}[B]\Big)$$

   is true under $M$.

   Conversely, assume that $M \subseteq V$ is a model of $\mathcal{T}_m[B]$. Then, there is an interpretation $J \subseteq V_{eq}$ such that $M \cup J$ is a model of

   $$\mathcal{T}_{ext}[B] \;\wedge\; \exists V'\mathcal{M}[B]. \tag{13}$$

   In particular, we have that $J$ is a model of $\mathcal{T}_{ext}[B]$, since the free variables of $\mathcal{T}_{ext}[B]$ are from $V_{eq}$. Hence, according to Theorem 4.2, we get that $E_{i_0} = Cn(K' \cup EQ_{i_0} \cup \{\alpha\}) \cap \mathcal{L}_\mathcal{P}$ is a consistent belief change extension of $B$, for $EQ_{i_0} = \{p \equiv p' \mid p_{eq} \in J\}$. Since $K \dot{+} \alpha \subseteq E_{i_0}$, for showing that $M$ is a model of $K \dot{+} \alpha$, it suffices to show that $M$ is a model of $E_{i_0}$. This can be seen as follows.

   Given that $M \cup J$ is a model of (13), we have that $M \cup J$ is *a fortiori* a model of $\exists V'\mathcal{M}[B]$. Hence, there is some interpretation $N' \subseteq V'$ such that $M \cup J \cup N'$ is a model of $\mathcal{M}[B]$. From this, we obtain that $M \cup N'$ is a model of

   $$K' \;\wedge \bigwedge_{(p \equiv p') \in EQ_{i_0}} (p \equiv p') \;\wedge\; \alpha,$$

which in turn implies that $M \cup N'$ is a model of $Cn(K' \cup EQ_{i_0} \cup \{\alpha\})$. In particular, $M$ must be a model of all those elements from $Cn(K' \cup EQ_{i_0} \cup \{\alpha\})$ which contain no atoms from $V'$. In other words, $M$ is a model of $E_{i_0} = Cn(K' \cup EQ_{i_0} \cup \{\alpha\}) \cap \mathcal{L}_{\mathcal{P}}$.

2. Consider $B = (K, \emptyset, \{\neg\alpha\})$ in $\mathcal{L}_{\mathcal{P}}$, and assume that $M \subseteq V$ is a model of $K \dot{-} \alpha$. From Proposition 4.6, we obtain that there is some consistent belief change extension $E_{i_0} = Cn(K' \cup EQ_{i_0}) \cap \mathcal{L}_{\mathcal{P}}$ of $B$ and some $\pi_{i_0}^{k_0} \in \Pi_{i_0}$ such that $\lfloor K \rfloor_{i_0}^{k_0}$ is true under $M$, for $\Pi_{i_0} = \{\pi_{i_0}^k \mid \pi_{i_0}^k : \mathcal{P}_{\overline{EQ}_{i_0}} \to \{\top, \bot\}\}$. Analogous to Part 1, define $M_1 = M \cap \mathcal{P}_{EQ_{i_0}}$ and $M_2 = M \setminus M_1$. Since $\lfloor K \rfloor_{i_0}^{k_0}$ is true under $M$, there is some $J \subseteq V \cap \mathcal{P}_{\overline{EQ}_{i_0}}$ such that $K$ is true under $M_1 \cup J$, and therefore $M_1' \cup J'$ is a model of $K'$. Hence, $M_1 \cup M_1' \cup J'$ is a model of

$$K' \wedge \bigwedge_{p \equiv p' \in EQ_{i_0}} (p \equiv p'). \tag{14}$$

Since no atoms from $V \cap \mathcal{P}_{\overline{EQ}_{i_0}}$ occur in (14), $M \cup M_1' \cup J'$ is also a model of (14). Applying similar arguments as in Part 1, it follows that $M$ is a model of $\mathcal{T}_m[B]$.

The proof of the converse direction proceeds analogously to Part 1.

**Proof**. Since the membership relations are already argued in the main body of the paper, it remains to show that the problems EXT, CHOICE, and SKEPTICAL, as well as its variants for revision and contraction, are hard for the respective classes.

1. REXT is NP-hard because a formula $\phi$ is satisfiable iff the belief change scenario $B_R = (\{\phi\}, \{\top\}, \emptyset)$ has a consistent belief change extension. Similarly, CEXT is NP-hard because $\phi$ is satisfiable iff $B_C = (\{\phi\}, \emptyset, \{\neg\bot\})$ possesses a consistent belief change extension. Either of these properties implies that EXT is NP-hard as well.

2. We show that RCHOICE and CCHOICE are $\Sigma_2^P$-hard; similar to the above, $\Sigma_2^P$-hardness of CHOICE is then an immediate consequence.

   We first deal with CCHOICE. According to Proposition 5.1, checking whether a closed QBF $\Phi$ of form $\exists P \forall Q \phi$, where $\phi$ is a propositional formula in disjunctive normal form and $P \cup Q$ is a partition of $Var(\phi)$, is $\Sigma_2^P$-hard. In order to show $\Sigma_2^P$-hardness of CCHOICE, we construct a polynomial-time transformation mapping each closed QBF $\Phi$ of the above form into a pair $(B_C, \phi^*)$, where $B_C = (K, \emptyset, \{\neg\alpha\})$ is a belief change scenario and $\phi^*$ is a formula, such that $\Phi$ is valid iff there is a consistent choice contraction $K \dot{-}_c \alpha$ containing $\phi^*$.

   The construction of $K$, $\alpha$, and $\phi^*$ is as follows. For $P = \{p_1, \ldots, p_n\}$, let $R = \{r_1, \ldots, r_n\}$ be a set of new atoms not occurring in $Var(\phi)$. Define

$$\begin{aligned} K &= \{p_i \wedge r_i \mid i = 1, \ldots, n\}; \quad \text{and} \\ \alpha &= \bigvee_{i=1}^{n} (p_i \wedge r_i), \end{aligned}$$

   and let $\phi^*$ be the result of replacing in $\phi$ each literal $\neg p_j$ with $p_j \in P$ by $r_j \in R$ ($1 \leq j \leq n$). We show that $\Phi = \exists P \forall Q \phi$ is valid iff there is a consistent belief change extension $Cn(K' \cup EQ_{i_0}) \cap \mathcal{L}_{\mathcal{P}}$ of $B_C = (K, \emptyset, \{\neg\alpha\})$ containing $\phi^*$.

To begin with, observe that, for any consistent belief change extension $Cn(K' \cup EQ) \cap \mathcal{L}_{\mathcal{P}}$ of $B_C$, it holds that

($*$) either $p_j \equiv p'_j \in EQ$ or $r_j \equiv r'_j \in EQ$, but not both, for each $1 \leq j \leq n$.

Assume that $\exists P \forall Q \phi$ is valid. Then, there is an interpretation $S_0 \subseteq P$ such that for each $U \subseteq Q$, $S_0 \cup U$ is a model of $\phi$. By construction of $\phi^*$, it follows that $S_0 \cup T_0 \cup U$ is a model of $\phi^*$, for $T_0 = \{r_j \in R \mid p_j \in P \setminus S_0\}$ and each $U \subseteq Q$. Now define

$$EQ_{i_0} \quad = \quad \{p_j \equiv p'_j \mid p_j \in S_0\} \cup \{r_j \equiv r'_j \mid p_j \in P \setminus S_0\}.$$

Clearly, $Cn(K' \cup EQ_{i_0}) \cap \mathcal{L}_{\mathcal{P}}$ is a consistent belief change extension of $B_C$. We claim that $\phi^* \in Cn(K' \cup EQ_{i_0}) \cap \mathcal{L}_{\mathcal{P}}$.

Since $\phi^* \in \mathcal{L}_{\mathcal{P}}$, it suffices to show that $K' \cup EQ_{i_0} \vdash \phi^*$. Let $M' \cup N$ be a model of $K' \cup EQ_{i_0}$, where $M' \subseteq P' \cup R'$ and $N \subseteq P \cup Q \cup R$. Define $N_1 = N \cap (P \cup R)$ and $N_2 = N \cap Q$. Hence, $N = N_1 \cup N_2$. Now, by definition of $K$ and $EQ_{i_0}$, it must hold that $N_1 = S_0 \cup T_0$. But $S_0 \cup T_0 \cup U$ is a model of $\phi^*$, for any $U \subseteq Q$; in particular, since $N_2 \subseteq Q$, $S_0 \cup T_0 \cup N_2$ is a model of $\phi^*$. Therefore, $N = N_1 \cup N_2$ is a model of $\phi^*$. Since $\phi^*$ contains no primed atoms, it follows that $M' \cup N$ is also a model of $\phi^*$. This proves the relation $K' \cup EQ_{i_0} \vdash \phi^*$. Hence, we showed that $\exists P \forall Q \phi$ is valid only if there is consistent belief change extension of $B_C$ containing $\phi^*$.

Conversely, assume that $\phi^* \in Cn(K' \cup EQ_{i_0}) \cap \mathcal{L}_{\mathcal{P}}$ for some consistent belief change extension $Cn(K' \cup EQ_{i_0}) \cap \mathcal{L}_{\mathcal{P}}$ of $B_C = (K, \emptyset, \{\neg\alpha\})$. We show that $\exists P \forall Q \phi$ is valid.

Observe that, according to Condition ($*$), we have either $p_j \equiv p'_j \in EQ_{i_0}$ or $r_j \equiv r'_j \in EQ_{i_0}$, but not both. Define

$$\begin{aligned} S_0 &= \{p_j \in P \mid p_j \equiv p'_j \in EQ_{i_0}\}; \quad \text{and} \\ T_0 &= \{r_j \in R \mid r_j \equiv r'_j \in EQ_{i_0}\}, \end{aligned}$$

and let $W_0 = S_0 \cup T_0$. Clearly, $p_j \in S_0$ iff $r_j \notin T_0$, for each $1 \leq j \leq n$. Furthermore, $W_0 \cup W'_0$ is a model of $EQ_{i_0}$. But, since $Var(EQ_{i_0}) = W_0 \cup W'_0$, and $W'_0 \subseteq P' \cup R'$, we have that $W_0 \cup P' \cup R'$ is a model of $EQ_{i_0}$, as well. Moreover, $W_0 \cup P' \cup R'$ is a model of $K' \cup EQ_{i_0}$. In fact, for any $U \subseteq Q$, $W_0 \cup P' \cup R' \cup U$ is a model of $K' \cup EQ_{i_0}$. Since, by hypothesis, $K' \cup EQ_{i_0} \vdash \phi^*$, it follows that $W_0 \cup P' \cup R' \cup U$ is a model of $\phi^*$, for any $U \subseteq Q$. But $\phi^*$ does not contain any primed atoms, so $W_0 \cup U = S_0 \cup T_0 \cup U$ must also be a model of $\phi^*$, for any $U \subseteq Q$. Hence, by definition of $\phi^*$, and since $p_j \in S_0$ iff $r_j \notin T_0$, we obtain that $S_0 \cup U$ is a model $\phi$, for each $U \subseteq Q$. We just proved that there is some $S_0 \subseteq P$ such that for each $U \subseteq Q$, $\phi$ is true under $S_0 \cup U$. This means that $\exists P \forall Q \phi$ is valid.

Now we turn our attention to RCHOICE. Consider $\Phi$, $K$, $\alpha$, and $\phi^*$ as above. We claim that $\Phi = \exists P \forall Q \phi$ is valid iff there is a consistent choice revision $K \dot{+}_c (\neg\alpha)$ containing $\phi^*$. From this, $\Sigma_2^P$-hardness of RCHOICE is an immediate consequence.

To prove the claim, we must show that $\Phi$ is valid iff there is a consistent belief change extension $Cn(K' \cup EQ_{i_0} \cup \{\neg\alpha\}) \cap \mathcal{L}_{\mathcal{P}}$ of $B_R = (K, \{\neg\alpha\}, \emptyset)$ containing $\phi^*$. Recall that we demonstrated above that $\Phi$ is valid iff there is a consistent belief change extension $Cn(K' \cup EQ_{i_0}) \cap \mathcal{L}_{\mathcal{P}}$ of $B_C = (K, \emptyset, \{\neg\alpha\})$ containing $\phi^*$. Furthermore, for any $EQ \subseteq \{v \equiv v' \mid v \in P \cup R\}$, it holds that $Cn(K' \cup EQ) \cap \mathcal{L}_{\mathcal{P}}$ is a consistent belief change extension of $B_C = (K, \emptyset, \{\neg\alpha\})$ iff $Cn(K' \cup EQ \cup \{\neg\alpha\}) \cap \mathcal{L}_{\mathcal{P}}$ is a consistent belief change extension of $B_R = (K, \{\neg\alpha\}, \emptyset)$ (cf. also Theorem 4.1 of [8]). Hence, it suffices to show that the following condition holds:

$(**)$ $\phi^* \in Cn(K' \cup EQ) \cap \mathcal{L}_{\mathcal{P}}$ iff $\phi^* \in Cn(K' \cup EQ \cup \{\neg\alpha\}) \cap \mathcal{L}_{\mathcal{P}}$, for any consistent belief change extension $Cn(K' \cup EQ) \cap \mathcal{L}_{\mathcal{P}}$ of $B_C$.

Consider some consistent belief change extension $Cn(K' \cup EQ) \cap \mathcal{L}_{\mathcal{P}}$ of $B_C$. If $\phi^*$ is contained in $Cn(K' \cup EQ) \cap \mathcal{L}_{\mathcal{P}}$, then $\phi^* \in Cn(K' \cup EQ \cup \{\neg\alpha\}) \cap \mathcal{L}_{\mathcal{P}}$, by monotonicity of $Cn(\cdot)$. So suppose $\phi^* \in Cn(K' \cup EQ \cup \{\alpha\}) \cap \mathcal{L}_{\mathcal{P}}$. We must show that $K' \cup EQ \vdash \phi^*$ holds.

Let $M' \cup N$ be some model of $K' \cup EQ$, where $M' \subseteq P' \cup R'$ and $N \subseteq P \cup R \cup Q$. Since $Cn(K' \cup EQ) \cap \mathcal{L}_{\mathcal{P}}$ is a consistent belief change extension of $B_C$, $EQ$ satisfies Condition $(*)$, i.e., for $P = \{p_1, \ldots, p_n\}$ and $R = \{r_1, \ldots, r_n\}$, we have that $p_j \equiv p'_j \in EQ$ or $r_j \equiv r'_j \in EQ$, but not both, for each $1 \leq j \leq n$. Define

$$\hat{N} \quad = \quad N \setminus (\{p_j \mid r_j \equiv r'_j \in EQ\} \cup \{r_j \mid p_j \equiv p'_j \in EQ\}).$$

Obviously, $M' \cup \hat{N}$ is a model of

$$K' \cup EQ \cup \{\neg\alpha\} \quad = \quad \{p'_i \wedge r'_i \mid i = 1, \ldots, n\} \cup EQ \cup \bigwedge_{i=1}^{n} (\neg p_i \vee \neg r_i).$$

Hence, since $\phi^* \in Cn(K' \cup EQ \cup \{\neg\alpha\})$, $M' \cup \hat{N}$ is a model of $\phi^*$. Moreover, since $\phi^*$ is a formula in disjunctive normal form, and no atom from $P$ or $R$ occurs negated in $\phi^*$, it follows that, for any $S \subseteq P$ and any $T \subseteq R$, $M' \cup \hat{N} \cup S \cup T$ is a model of $\phi^*$. In particular, $M' \cup N$ is a model of $\phi^*$. This proves $K' \cup EQ \vdash \phi^*$.

3. Again, we only show $\Pi_2^P$-hardness of RSKEPTICAL and CSKEPTICAL. To this end, we exploit some results from [13] and [7; 8].

   Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$ be two distinct sets of variables, and consider a closed QBF $\Phi$ of form $\forall P \exists Q \phi$, where $\phi$ is a propositional formula such that $P \cup Q = Var(\phi)$. Furthermore, let $R = \{r_1, \ldots, r_n\}$ be a set of variables distinct from $Var(\phi)$, and let $v$ be a further variable not occurring in $Var(\phi)$ or $R$. For $K$ and $\alpha$ as defined in the proof of CCHOICE above, define the following knowledge base $K_S$ and formulas $\beta$ and $\gamma$:

$$\begin{aligned} K_S &= K \cup Q \cup \{v\}; \\ \beta &= \neg\alpha \wedge (v \rightarrow \phi) \wedge ((q_1 \vee \ldots \vee q_m) \rightarrow v); \quad \text{and} \\ \gamma &= \bigwedge_{i=1}^{n} (p_i \vee r_i). \end{aligned}$$

   As shown in [13], $\Phi$ is valid iff $v \in K_S \dot{+}_s (\beta \wedge \gamma)$, where $\dot{+}_s$ is the Satoh revision operator. Since $K_S \dot{+}_s (\beta \wedge \gamma)$ is equivalent to $K_S \dot{+} (\beta \wedge \gamma)$ (cf. [7; 8]), we get that

$$\Phi \text{ is valid iff } v \in K_S \dot{+} (\beta \wedge \gamma). \tag{15}$$

Hence, RSKEPTICAL is $\Pi_2^P$-hard.

As for CSKEPTICAL, $\Pi_2^P$-hardness follows from (15) by showing that $v \in K_S \dot{+} (\beta \wedge \gamma)$ iff $v \in K_S \dot{-} (\neg\beta \vee \neg\gamma)$. To see this, analogous to the argumentation given above for RCHOICE, it suffices to show that

$$v \in Cn(K'_S \cup EQ \cup \{(\beta \wedge \gamma)\}) \cap \mathcal{L}_{\mathcal{P}} \text{ iff } v \in Cn(K'_S \cup EQ) \cap \mathcal{L}_{\mathcal{P}},$$

for any maximal set $EQ \subseteq \{w \equiv w' \mid w \in Var(K_S) \cup Var(\beta \wedge \gamma)\}$ such that $K'_S \cup EQ \cup \{(\beta \wedge \gamma)\} \not\vdash \bot$. But this property is obviously satisfied, because any such $EQ$ contains $v \equiv v'$, and $K'_S \cup \{v \equiv v'\} \vdash v$.

**Proof**. $\mathrm{D}^P$-membership can be seen as follows. According to Proposition 4.1, given a belief change scenario $B = (K, R, C)$ and some set $EQ \subseteq \{p \equiv p' \mid p \in Var(B)\}$ of equivalences, deciding whether $EQ$ determines a consistent belief change extension of $B$ is equivalent to

(i) deciding whether $K' \cup EQ \cup R \cup C$ is consistent, and

(ii) deciding whether $K' \cup EQ \cup (p \equiv p') \cup R \cup C$ is inconsistent, for each $p \in Var(B)$ such that $(p \equiv p') \notin EQ$.

Clearly, Task (i) is in NP and Task (ii) is in co-NP. Hence, the combined problem is in $\mathrm{D}^P$.
For showing $\mathrm{D}^P$-hardness, we consider the following well-known $\mathrm{D}^P$-complete problem (cf. [26]):

SAT-UNSAT : Given two propositional formulas $\phi$ and $\psi$, decide whether $\phi$ is satisfiable and $\psi$ is unsatisfiable.

We construct a polynomial transformation mapping each pair $(\phi, \psi)$ of propositional formulas into a belief change scenario $B$ and some set $EQ \subseteq \{p \equiv p' \mid p \in Var(B)\}$ of equivalences such that

$(*)$ if $\phi$ is satisfiable and $\psi$ is unsatisfiable, then $EQ$ determines a consistent belief change scenario of $B$, and vice versa.

The construction of $B$ and $EQ$ is as follows. Let $\phi$ and $\psi$ be propositional formulas. Without loss of generality, we can assume that $Var(\phi)$ and $Var(\psi)$ are disjoint. Furthermore, let $p$ and $q$ be distinct atoms not occurring in $Var(\phi) \cup Var(\psi)$. Then, define

$$\begin{aligned} B &= (\{p \rightarrow \phi, q \rightarrow \psi\}, \{p, q\}, \emptyset); \quad \text{and} \\ EQ &= \{p \equiv p'\} \cup \{v \equiv v' \mid v \in Var(\phi) \cup Var(\psi)\}. \end{aligned}$$

It is easy to see that this construction obeys Condition $(*)$. $\square$

## References

[1] C. Alchourrón, P. Gärdenfors, and D. Makinson. On the Logic of Theory Change: Partial Meet Contraction and Revision Functions. *Journal of Symbolic Logic*, 50:510–530, 1985.

[2] S. Benferhat, D. Dubois, and H. Prade. A Computational Model for Belief Change and Fusing Ordered Belief Bases. In M.-A. Williams and H. Rott, editors, *Frontiers in Belief Revision*, volume 22 of *Applied Logic Series*, pages 109–134. Kluwer Academic Publishers, 2001.

[3] A. Borgida. Language Features For Flexible Handling Of Exceptions In Information Systems. *ACM Transactions on Database Systems*, 10(4):565–603, 1985.

[4] M. Cadoli, A. Giovanardi, and M. Schaerf. An Algorithm to Evaluate Quantified Boolean Formulae. In *Proceedings of the AAAI National Conference on Artificial Intelligence*, pages 262–267, Madison, Wisconsin, 1998.

[5] M. Dalal. Investigations into Theory of Knowledge Base Revision. In *Proceedings of the AAAI National Conference on Artificial Intelligence*, pages 449–479, St. Paul, Minnesota, 1988.

[6] J. de Kleer. An Assumption-based TMS. *Artificial Intelligence*, 28:127–162, 1986.

[7] J. Delgrande and T. Schaub. A Consistency-Based Model for Belief Change: Preliminary Report. In *Proceedings of the AAAI National Conference on Artificial Intelligence*, pages 392–398, Austin, Texas, 2000.

[8] J. Delgrande and T. Schaub. A Consistency-Based Paradigm for Belief Change. *Artificial Intelligence*, 2003. Accepted for publication.

[9] E. Eder. *Relative Complexities of First-Order Calculi*. Artificial Intelligence. Vieweg Verlag, 1992.

[10] U. Egly, T. Eiter, R. Feldmann, V. Klotz, S. Schamberger, H. Tompits, and S. Woltran. On Mechanizing Modal Nonmonotonic Logics. In *Proceedings of the Fifth Dutch-German Workshop on Nonmonotonic Reasoning Techniques and their Applications*, pages 44–53, 2001.

[11] U. Egly, T. Eiter, V. Klotz, H. Tompits, and S. Woltran. Computing Stable Models with Quantified Boolean Formulas: Some Experimental Results. In *Proceedings of the 2001 AAAI Spring Symposium on Answer Set Programming*, pages 53–59, 2001.

[12] U. Egly, T. Eiter, H. Tompits, and S. Woltran. Solving Advanced Reasoning Tasks Using Quantified Boolean Formulas. In *Proceedings of the AAAI National Conference on Artificial Intelligence*, pages 417–422, Austin, Texas, 2000.

[13] T. Eiter and G. Gottlob. On the Complexity of Propositional Knowledge Base Revision, Updates, and Counterfactuals. *Artificial Intelligence*, 57:227–270, 1992.

[14] R. Fagin, J. Ullman, and M. Vardi. On the Semantics of Updates in Databases. Preliminary Report. In *Proceedings of the Second ACM Symposium on Principles of Database Systems SIGACT-SIGMOD*, pages 352–365, 1983.

[15] R. Feldmann, B. Monien, and S. Schamberger. A Distributed Algorithm to Evaluate Quantified Boolean Formulas. In *Proceedings of the AAAI National Conference on Artificial Intelligence*, pages 285–290, Austin, Texas, 2000.

[16] K. Forbus. Introducing Actions into Qualitative Simulation. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pages 1273–1278, 1989.

[17] P. Gärdenfors. *Knowledge in Flux: Modelling the Dynamics of Epistemic States*. The MIT Press, Cambridge, MA, 1988.

[18] E. Giunchiglia, M. Narizzano, and A. Tacchella. QUBE: A System for Deciding Quantified Boolean Formulas Satisfiability. In *Proceedings of the International Joint Conference on Automated Reasoning*, pages 364–369, 2001.

[19] H. Katsuno and A. Mendelzon. On the Difference between Updating a Knowledge Base and Revising it. In P. Gärdenfors, editor, *Belief Revision*, pages 183–203. Cambridge University Press, 1992.

[20] H. Kautz and B. Selman. Planning as Satisfiability. In *Proceedings of the European Conference on Artificial Intelligence*, pages 359–363, 1992.

[21] H. Kleine-Büning, M. Karpinski, and A. Flögel. Resolution for Quantified Boolean Formulas. *Information and Computation*, 117(1):12–18, 1995.

[22] R. Letz. Advances in Decision Procedures for Quantified Boolean Formulas. In *Proceedings of the IJCAR-01 Workshop on Theory and Applications of Quantified Boolean Formulas*, pages 55–64, 2001.

[23] P. Liberatore and M. Schaerf. Reducing Belief Revision to Circumscription (and Vice Versa). *Artificial Intelligence*, 93(1–2):261–296, 1997.

[24] D. Makinson. How to Give it Up: A Survey of Some Formal Aspects of the Logic of Theory Change. *Synthese*, 62:347–363, 1985.

[25] B. Nebel. Syntax Based Approaches to Belief Revision. In P. Gärdenfors, editor, *Belief Revision*, pages 52–88. Cambridge University Press, 1992.

[26] C. Papadimitriou. *Computational Complexity*. Addison-Wesley, Reading, Mass., 1994.

[27] D. Pearce, H. Tompits, and S. Woltran. Encodings for Equilibrium Logic and Logic Programs with Nested Expressions. In *Proceedings of the Tenth Portuguese Conference on Artificial Intelligence*, pages 306–320, 2001.

[28] D. A. Plaisted and S. Greenbaum. A Structure Preserving Clause Form Translation. *Journal of Symbolic Computation*, 2(3):293–304, 1986.

[29] D. Poole. A Logical Framework for Default Reasoning. *Artificial Intelligence*, 36(1):270–48, 1988.

[30] R. Reiter. A Theory of Diagnosis from First Principles. *Artificial Intelligence*, 32(1):57–96, 1987.

[31] J. Rintanen. Constructing Conditional Plans by a Theorem Prover. *Journal of Artificial Intelligence Research*, 10:323–352, 1999.

[32] J. Rintanen. Improvements to the Evaluation of Quantified Boolean Formulae. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pages 1192–1197, 1999.

[33] K. Satoh. Nonmonotonic Reasoning by Minimal Belief Revision. In *Proceedings of the International Conference on Fifth Generation Computer Systems*, pages 455–462, Tokyo, 1988.

[34] L. J. Stockmeyer. The Polynomial-Time Hierarchy. *Theoretical Computer Science*, 3(1):1–22, 1976.

[35] R. Wassermann. *Resource-Bounded Belief Revision*. PhD thesis, Institute for Logic, Language, and Computation, University of Amsterdam, 1999.

[36] A. Weber. Updating Propositional Formulas. *Proceedings of the First Conference on Expert Database Systems*, pages 487–500, 1986.

[37] M.-A. Williams. Theory Base Change: A Computational Model. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pages 1541–1547, Montréal, Canada, 1995.

[38] M. Winslett. Reasoning About Action Using a Possible Models Approach. In *Proceedings of the AAAI National Conference on Artificial Intelligence*, pages 89–93, St. Paul, Minnesota, 1988.

[39] C. Wrathall. Complete Sets and the Polynomial-Time Hierarchy. *Theoretical Computer Science*, 3(1):23–33, 1976.