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DISJUNCTIONS OF HORN THEORIES AND THEIR CORES

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DISJUNCTIONS OF HORN THEORIES AND THEIR CORES

Thomas Eiter[1] Toshihide Ibaraki[2] Kazuhisa Makino[3]

Abstract. In this paper, we study issues on disjunctions of propositional Horn theories. In particular, we consider deciding whether a disjunction of Horn theories is Horn, and, if not, computing a Horn core (i.e., a maximal Horn theory included in this disjunction) and Horn envelope (i.e., the minimum Horn theory including the disjunction). The problems are investigated for different representations of Horn theories, namely for Horn CNFs and characteristic models. While the problems are shown to be intractable in general, in the case of bounded disjunctions, we present polynomial time algorithms for testing the Horn property in both representations, and for computing a core in the CNF representation. Even in the case of bounded disjunction, no polynomial algorithm exists (unless P=NP) for computing a core in the characteristic model representation.

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1 Introduction

Since deduction from a set of propositional clauses is a well-known co-NP-complete problem, different approximation methods for reasoning from a clausal theory Σ have been investigated, e.g., [14, 18, 3, 4, 1]. One of these approaches [14] uses a *greatest Horn lower bound* (also called *Horn core* [15]), i.e., if we view theories as sets of models, a maximal Horn theory $\Sigma_c \subseteq \Sigma$, and the *least Horn upper bound* (*Horn envelope* [15]), which is the minimal Horn theory $\Sigma_e \supseteq \Sigma$ such that Σ logically implies Σ_e. Note that in general, different Horn cores may exist, while it is known that the Horn envelope is always unique.

Computing Horn envelopes and Horn cores has been investigated in [14, 15, 2, 4, 5, 9, 7, 1]. It has been shown that a Horn core of a given CNF (conjunctive normal form) φ is computable in polynomial time with an oracle for NP [2], and that all Horn cores can be generated with polynomial delay, if the theory Σ is given by the set of its models. However, in the latter setting, computing a maximum (in terms of the numbers of models) Horn core is co-NP-hard [15].

In this paper, we consider the issue of computing Horn cores of the disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ of Horn theories Σ_i, represented either by Horn CNFs, or by their *characteristic models* [12]. Characteristic models have been proposed as a model-based alternative to the formula-based theory representation. These two approaches are orthogonal with respect to space requirements, in the sense that one approach sometimes allows for an exponentially smaller representation than the other; see [13, 17]. Observe that a disjunction Σ of Horn theories is in general not Horn. Hence, in particular, it is of interest to know whether Σ is Horn, since in this case the Horn core and the Horn envelope coincide to Σ.

Disjunctions of Horn theories may be encountered in different applications. For example, suppose that two groups have respectively formed logical hypotheses about an application domain (the "world"), e.g., relationships between medical tests and diseases, and they believe that the relationships amount to a Horn theory. The hypotheses Σ_1 and Σ_2 of two groups (amounting to sets of models) may be obtained from actual and conjectured cases, respectively; i.e., concrete measurement data (obtained by experiments) and data which are believed to be true. Suppose that the hypotheses Σ_1 and Σ_2 are merged. Then, at the logic level, the disjunction, i.e., union $\Sigma = \Sigma_1 \cup \Sigma_2$ describes the merged hypothesis. It is of particular interest to know whether Σ is Horn; if so, then the original hypotheses are compatible in the the sense that no further cases have to be adopted in order to preserve the Horn property, which may indicate that the individual hypotheses are sound. However, if Σ is not Horn, then either further cases have to be added to maintain the Horn property (which corresponds to derivation of new case knowledge), or some of the adopted hypothetical cases have to be abandoned. Applying Occam's razor, it is natural to add or to abandon a minimal set of cases. In the former case, this amounts to finding the Horn envelope of Σ, and in the latter case, to finding a Horn core Π of Σ. In finding a Horn core Π, it may be also asked to add such constraint as that Π includes all actual cases Σ_1. That is, computing a Horn core Π satisfying $\Sigma_1 \subseteq \Pi \subseteq \Sigma$ can be seen as one of the important problems.

Another application concerns inference from knowledge bases. Suppose that Horn theories $\Sigma_1, \Sigma_2, \ldots, \Sigma_l$ represent knowledge bases which are located at different sites $s_1, s_2, \ldots, s_l$, respectively. A formula φ is a logical consequence of all Σ_i, for $i = 1, 2, \ldots, l$, just if φ is a logical consequence of the disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$. Thus, in order to test whether φ is a consequence in all Σ_i, it is equivalent to test whether φ is a consequence of Σ. This may be profitably used if the on-line access to the theories Σ_i at the individual sites is costly or unreliable, for instance. If Σ is stored at a distinguished site s^*, then by accessing this single site the query φ to all knowledge bases can be answered. In the case where Σ is Horn, we can store a Horn CNF (resp., the characteristic set) of Σ, which can be more compact then simply mirroring the disjunction of Horn CNFs (resp., the characteristic sets) of the individual Horn theories Σ_i at s^*, since redundancies can be avoided. For example, if each Σ_i is represented by a Horn CNF $\varphi_i = \psi \wedge (x \vee \overline{y}_i)$, for $i = 1, 2, \ldots, l$, then the Horn CNF $\varphi = \psi \wedge (x \vee \overline{y}_1 \vee \overline{y}_2 \vee \ldots \vee \overline{y}_l)$ represents the disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$. Note that, in this case, storing the individual φ_i requires more than l times space compared to φ.

In this paper, we first address the problem of checking if a disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ of Horn theories Σ_i, represented either by Horn CNFs or by their characteristic models, becomes a Horn theory. We show that the problem is in general co-NP-complete for both representations, but is polynomially solvable if l is bounded by a constant. These results indicate that computing a Horn core is difficult in general. The polynomiality follows from syntactical and semantical characterizations of a disjunction of Horn theories.

We next deal with the problem of computing a Horn core Π of Σ satisfying $\Sigma_1 \subseteq \Pi \subseteq \Sigma$. We show that, if l is bounded by some constant, then the problem is polynomially solvable from Horn CNFs, while it is co-NP-hard from characteristic models, even if $l = 2$. For the formula-based representations, we first present a polynomial time algorithm CORE to compute a Horn core of the disjunction of two Horn theories Σ_1 and Σ_2. Since the algorithm CORE cannot be generalized directly to the general case $l\,(\geq 3)$, we develop an algorithm CORE* which computes a Horn core of the disjunction of $l\,(\geq 3)$ Horn theories by making uses of algorithm CORE repeatedly. It turns out that algorithm CORE* runs in polynomial time if l is bounded by some constant. On the other hand, we show that computing a Horn core is not possible in polynomial time for characteristic models, by exhibiting a family of theories Σ_1 and Σ_2 having small characteristic sets, while every Horn core of $\Sigma_1 \cup \Sigma_2$ is exponentially large. We also give structural characterizations of Horn cores of a disjunction of Horn theories.

As for the Horn envelope, we show that it can be computed from characteristic models in polynomial time, but cannot be efficiently computed from Horn CNFs, even if $l = 2$. The negative result follows from the fact that there exist Horn theories Σ_1 and Σ_2 such that their Horn CNFs are small, but the CNF representing the Horn envelope of $\Sigma = \Sigma_1 \cup \Sigma_2$ is exponentially large.

The rest of this paper is organized as follows. In the next section, we recall some basic concepts and introduce notations. In Sections 3 and 4, we consider Horn cores in both representations of Horn CNFs and characteristic sets. Section 5 considers the Horn envelope. Finally, Section 6 discusses related work and concludes the paper. The complexity results of all the above problems are summarized in Table 1 of Section 6.

2 Preliminaries

We assume a supply of propositional variables (atoms) $x_1, x_2, \ldots, x_n$, where each x_i evaluates to either 1 (true) or 0 (false). Negated variables are denoted by $\overline{x}_i$. These x_i and $\overline{x}_i$ are called literals. A clause is a disjunction $c = L_1 \vee \cdots \vee L_k$ of literals, while a term is a conjunction $t = L_1 \wedge \cdots \wedge L_h$ of literals. By $P(c)$ and $N(c)$ (resp., $P(t)$ and $N(t)$), we denote the sets of variables occurring positively and negatively in c (resp., t). By $\perp$ (resp.,$\top$) we denote the empty clause (resp., empty term) representing falsity (resp., truth). A formula is composed of literals and operators: or ($\vee$), and ($\wedge$), negation ($\overline{}$).

In particular, a conjunction of clauses $\varphi = \bigwedge_i c_i$ (resp., a disjunction of terms $\varphi = \bigvee_i t_i$) is called *conjunctive normal form* (CNF) (resp., *disjunctive normal form* (DNF)).

A *model* is a vector $v \in \{0,1\}^n$, whose i-th component is denoted by v_i, and a *theory* is any set $\Sigma \subseteq \{0,1\}^n$ of models. We denote by $v \leq w$ the usual componentwise ordering of models, i.e., $v_i \leq w_i$ for all $i = 1, 2, \ldots, n$, where $0 < 1$. By $\min(\Sigma)$ and $\max(\Sigma)$ we denote the sets of minimal and maximal models in Σ under $<$, respectively, where $v \in \Sigma$ is a *maximal* (resp., *minimal*) model in Σ, if there is no $w \in \Sigma$ such that $w > v$ (resp., $w < v$). For $B \subseteq \{1, 2, \ldots, n\}$, we denote by x^B the model v such that $v_i = 1$ for

$i \in B$ and $v_i = 0$ for $i \notin B$.

For any formula φ, let $T(\varphi) = \{v \in \{0,1\}^n \mid \varphi(v) = 1\}$ denote the set of models of φ. We call that a formula φ represents a theory Σ if $T(\varphi) = \Sigma$. We sometimes do not distinguish a formula from the theory it represents, if no confusion arises. For any formulas φ and ψ, we write $\varphi \leq \psi$ (also $\varphi \models \psi$) if $T(\varphi) \subseteq T(\psi)$ holds. A clause c (resp., term t) is an *implicate* (resp., *implicant*) of a theory Σ if $c(v) = 1$ for all $v \in \Sigma$ (i.e., $T(c) \supseteq \Sigma$) (resp., $t(v)$ = 0 for all $v \notin \Sigma$ (i.e., $T(t) \subseteq \Sigma$)); it is *prime* if no proper sub-clause (resp., sub-term) is an implicate (resp., implicant) of Σ.

A theory Σ is *Horn* if $\Sigma = Cl_\wedge(\Sigma)$ holds, where $Cl_\wedge(S)$ is the closure of a theory $S \subseteq \{0,1\}^n$ under component-wise AND (i.e., intersection) of models v, w, denoted by $v \bigwedge w$. Observe that any Horn theory Σ has the least (unique smallest) model, denoted $lm(\Sigma)$, which is given by $\bigwedge_{v\in\Sigma} v$.

A clause c is *Horn* if $|P(c)| \leq 1$, and a CNF is *Horn* if it contains only Horn clauses. It is well-known that a theory Σ is Horn if and only if it is represented by some Horn CNF, and that all prime implicates of Horn theory are Horn.

A Horn theory Σ_c is a *Horn core* of a theory Σ if $\Sigma_c \subseteq \Sigma$ holds and no Horn theory Σ' exists such that $\Sigma_c \subset \Sigma' \subseteq \Sigma$. Observe that, in general, Σ has more than one Horn core; e.g., $\Sigma = \{(110), (101)\}$ has two Horn cores $\Sigma_c^1 = \{(110)\}$ and $\Sigma_c^2 = \{(101)\}$. The *Horn envelope* of Σ is the Horn theory $\Sigma_e \supseteq \Sigma$ for which no Horn theory Σ' satisfies $\Sigma_e \supset \Sigma' \supseteq \Sigma$. For the above Σ, we have $\Sigma_e = \{(110), (101), (100)\}$. As easily seen, the Horn envelope is always unique. Let φ be a formula representing a theory Σ, and let φ_c and φ_e be formulas representing a Horn core and the Horn envelope of Σ, respectively. Then φ_c and φ_e are also called a *Horn core* and the *Horn envelope* of φ, respectively.

3 CNF Representations

In this section, we deal with the case, in which each Horn theory is represented by a Horn CNF.

3.1 Horn Property of a disjunction of Horn theories

Our first result deals with the Horn property for a disjunction of Horn CNFs.

Theorem 3.1 *Given Horn CNFs $\varphi_1, \varphi_2, \ldots, \varphi_l$, deciding whether $\varphi = \bigvee_{i=1}^{l} \varphi_i$ is Horn is* co-NP-*complete.*

Proof. The problem is in co-NP, since a guess for models v and w of φ whether $v \bigwedge w$ is not a model of φ can be verified in polynomial time.

For the hardness part, we reduce the problem of checking whether a DNF $\psi = \bigvee_{i=1}^{m} t_i$ is a tautology (which is known to be co-NP-complete) to our problem. Note that each t_i can be seen as a Horn CNF φ_i. Let x_{n+1} and x_{n+2} be two fresh variables, and let $t_{m+1} = x_{n+1}$ and $t_{m+2} = x_{n+2}$. Then we claim that $\varphi = \psi \vee t_{m+1} \vee t_{m+2}$ $(= \bigvee_{i=1}^{m+2} t_i)$ is Horn if and only if $\psi \equiv \top$ (and hence $\varphi \equiv \top$). The if-direction is obvious.

For the only if direction, suppose $\psi \not\equiv \top$. Then there exists a (non-tautological) prime implicate c of ψ. It is then easy to see that $c' = c \vee x_{n+1} \vee x_{n+2}$ is a prime implicate of φ. Since c' is not Horn and all prime implicates of a Horn theory are Horn, it follows that no Horn CNF is equivalent to φ. This proves the result. □

Observe that the reduction proof of the above theorem makes use of a number of Horn theories. For the case in which l is bounded by some constant k, we have a positive result.

Let

$$\varphi_i = \bigwedge_{j=1}^{m_i} c_{i,j}, \quad i = 1, 2, \ldots, l,$$

where $c_{i,j}$ are Horn clauses. We introduce the following set.

> $HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$: the set of all Horn clauses c defined by $N(c) = \bigcup_{i=1}^{l} N(c_{i,j_i})$ and $P(c) = P(c_{i,j_i})$ for some i, where all the choices of $j_i \in \{1, 2, \ldots, m_i\}$ for $i = 1, 2, \ldots, l$ are considered under the constraint that the disjunction of the selected clauses $c_{1,j_1} \vee c_{2,j_2} \vee \ldots \vee c_{l,j_l}$ is not tautology.

Example 3.1 *Let*

$$\begin{aligned}\varphi_1 &= (\overline{x}_1 \vee \overline{x}_3 \vee x_4)(\overline{x}_2 \vee \overline{x}_5 \vee x_6)\\ \varphi_2 &= (\overline{x}_1 \vee \overline{x}_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_3 \vee x_6).\end{aligned}$$

Then the set HC is given as follows.

$$\begin{aligned}HC(\varphi_1, \varphi_2) = \{&(\overline{x}_1 \vee \overline{x}_3 \vee x_4), (\overline{x}_1 \vee \overline{x}_3 \vee x_6), (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_5 \vee x_6),\\ &(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5), (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee x_6)\}.\end{aligned}$$

□

The following lemma relates a disjunction of Horn CNFs with clauses from $HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$.

Lemma 3.2 *Let $\varphi_1, \varphi_2, \ldots, \varphi_l$ be Horn CNFs, and let $\varphi = \bigvee_{i=1}^{l} \varphi_i$. Then φ represents a Horn theory if and only if $\varphi \equiv \bigwedge_{c \in S} c$ (i.e., φ is equivalent to the right hand side) holds for some subset $S \subseteq HC(\varphi_1, \varphi_2, \ldots . \varphi_l)$.*

Proof. The if-part is obvious because all clauses $c \in HC(\varphi_1, \varphi_2, \ldots . \varphi_l)$ are Horn. For the only-if-part, let φ represent a Horn theory Σ. Then observe that φ is equivalent to the CNF

$$\varphi' = \bigwedge_{i_1=1}^{m_1} \bigwedge_{i_2=1}^{m_2} \cdots \bigwedge_{i_l=1}^{m_l} (c_{1,i_1} \vee c_{2,i_2} \vee \cdots \vee c_{l,i_l}). \tag{3.1}$$

Since φ' also represents a Horn theory Σ and all prime implicates of a Horn theory are Horn, for each non-tautological clause $c' = (c_{1,i_1} \vee c_{2,i_2} \vee \cdots \vee c_{l,i_l})$ in φ', there exists a Horn prime implicate c of Σ satisfying $\varphi' \leq c \leq c'$. Note that $HC(\varphi_1, \varphi_2, \ldots . \varphi_l)$ contains all maximal Horn clauses c^* with $c^* \leq c'$. Therefore, some clause c^* in $HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$ satisfies

$$\varphi' \, (\equiv \varphi) \;\leq\; c \;\leq c^* \;\leq\; c'.$$

Thus, replacing each c' in φ' by such a c^* again produces Σ, which implies the only-if-part. □

Theorem 3.3 *For an integer l bounded by a constant k, the problem in Theorem* 3.1 *can be solved in polynomial time.*

Proof. By Lemma 3.2, for each c' in φ' of (3.1), we only find a clause c^* in $HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$ such that $\varphi \leq c^* \leq c'$. If every c' has such a c^*, we can conclude that φ is Horn; otherwise, we can conclude that φ is not Horn.

As for the time complexity, since $\varphi \leq c^*$ is equivalent to the condition that $\varphi_i \leq c^*$ holds for all i, and since each φ_i is Horn, $\varphi \leq c^*$ can be checked in polynomial time. Furthermore, if l is bounded by a constant, we have at most a polynomial number of c' and c^*. Thus, the overall time is polynomial. □

Let us now consider the problem of computing a Horn core. The following proposition, together with Theorem 3.1, implies that this is a difficult problem in general. In particular, a polynomial-time computation of some Horn core is infeasible, as well as the enumeration of Horn cores. As for enumeration problems, we measure their complexity in the combined size of input and output. An algorithm is called of *polynomial total time* [11] if its running time is polynomial in the length of input and output.

Proposition 3.4 *A theory Σ has a unique Horn core if and only if Σ is Horn.*

Proof. The if direction is trivial. For the only-if direction, let Π be an arbitrary Horn core of Σ, and let $v \in \Sigma \setminus \Pi$. Since theory $\{v\}$ is Horn, there exists a Horn core Π' such that $v \in \Pi'$, for which $\Pi' \neq \Pi$ holds. □

Corollary 3.5 *Unless* P=NP*, there is no polynomial total time algorithm which, given Horn CNFs φ_1, φ_2, $\ldots, \varphi_l$, computes any number k (≥ 1) of Horn cores of $\varphi = \bigvee_{i=1}^{l} \varphi_i$.*

Proof. Let us assume that there is a polynomial total time algorithm A for generating k Horn cores of φ with polynomial running time $p(I, O)$, where I is the input length and O is the output length. We then apply Algorithm A to the instance which is used to prove Theorem 3.1. Recall that, in this case, φ is Horn if and only if $\psi \equiv \varphi \equiv \top$. Let us execute A until either (i) it halts or (ii) time $p(I, 1)$ is reached. In the case of (i), if A outputs exactly one Horn CNF $\top$, then output "Yes"; otherwise, "No". In the case of (ii), output "No", since it implies that the output length is more than 1. Therefore, tautology problem can be solved in polynomial time, implying P=NP. □

Corollary 3.6 *Given Horn CNFs φ_1, φ_2, $\ldots$, φ_l, deciding whether $\varphi = \bigvee_{i=1}^{l} \varphi_i$ has a unique Horn core is* co-NP*-complete.*

These are rather negative results. However, the proofs do not apply for the disjunction of a small (bounded by a constant) number of Horn theories, as will be seen in the next subsection.

3.2 Horn cores of the disjunction of two Horn theories

In this subsection, we describe a polynomial time algorithm which computes a Horn core of the disjunction of two Horn theories. We start with the following lemma showing that any Horn core of a disjunction of Horn CNFs can be represented by a Horn CNF consisting of only clauses from $HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$. This is a generalization of the only-if-part of Lemma 3.2.

Lemma 3.7 *Let $\varphi_1, \varphi_2, \ldots, \varphi_l$ be Horn CNFs, and let ψ be any Horn core of $\varphi = \bigvee_{i=1}^{l} \varphi_i$. Then $\psi \equiv \bigwedge_{c \in S} c$ holds for some subset $S \subseteq HC(\varphi_1, \varphi_2, \ldots .\varphi_l)$.*

Proof. Recall that φ is equivalent to the CNF φ' of (3.1) (which may not be Horn). Since ψ is a Horn core of φ, each nontautological clause c' in φ' is subsumed by some prime Horn implicate c of ψ (i.e., $c \leq c'$). Recall that $HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$ is the set of all the maximal Horn clauses subsuming at least one of the clauses in φ'. Thus we have a clause $c^* \in HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$ such that $c \leq c^* \leq c'$. Let ψ^* be the CNF obtained from φ' by replacing each c' by such a c^*. Then $\psi \leq \psi^* \leq \varphi'$ $(\equiv \varphi)$ holds. Since ψ is a Horn core of φ, it then follows $\psi \equiv \psi^*$. $\square$

Note that the converse is not true in general, even if ψ is a CNF obtained from φ' by replacing each non-tautological clause c' in φ' by a clause $c^* \in HC(\varphi_1, \varphi_2, \ldots, \varphi_l)$ with $c^* \leq c'$. The following example gives such an instance.

Example 3.2 *Let φ_1 and φ_2 be given in Example* 3.1. *Then φ' of* (3.1) *is written as*

$$\varphi' = (\overline{x}_1 \vee \overline{x}_3 \vee x_4 \vee x_6)(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5 \vee x_6)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee x_6),$$

where we exclude the tautological clauses from φ'. Let us consider a Horn CNF

$$\psi = (\overline{x}_1 \vee \overline{x}_3 \vee x_4)(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee x_6).$$

Note that all clauses in ψ are contained in $HC(\varphi_1, \varphi_2)$ (see Example 3.1), *and that this ψ is obtained from φ' in the desired way, because $(\overline{x}_1 \vee \overline{x}_3 \vee x_4) \leq (\overline{x}_1 \vee \overline{x}_3 \vee x_4 \vee x_6)$, $(\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5) \leq (\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5 \vee x_6)$ and $(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee x_6) \leq (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee x_6)$. However, this ψ is not Horn core of $\varphi = \varphi_1 \vee \varphi_2$, since*

$$\psi' = (\overline{x}_1 \vee \overline{x}_3 \vee x_4)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_5 \vee x_6)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee x_6) \tag{3.2}$$

satisfies $\psi < \psi' \leq \varphi'$ $(\equiv \varphi)$. In fact, $\psi \leq \psi'$ follows from $\psi \leq (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_5 \vee x_6)$, and this combined with $\psi(110111) = 0$ and $\psi'(110111) = 1$ implies $\psi < \psi'$. As will be shown later, ψ' of (3.2) *is a Horn core of φ.* $\square$

Now we give an algorithm that obtains a core of the disjunction of two Horn theories.

Algorithm CORE

Input: Horn CNFs $\varphi_1 = \bigwedge_{i=1}^{m_1} c_{1,i}$ and $\varphi_2 = \bigwedge_{j=1}^{m_2} c_{2,j}$.

Output: A Horn core ψ of $\varphi = \varphi_1 \vee \varphi_2$.

Step 1. Set $S := \{c^*_{i,j} = c_{1,i} \cup c_{2,j} \mid c^*_{i,j} \not\equiv \top,\, i = 1, 2, \ldots, m_1,\, j = 1, 2, \ldots, m_2\}$ and
$S_2 := \{c \in S \mid |P(c)| = 2\}$.

For each $c^*_{i,j} \in S$, let $c^1_{i,j}$ (resp., $c^2_{i,j}$) denote the Horn clause c such that $N(c) = N(c^*_{i,j})$ and $P(c) = P(c_{1,i})$ (resp., $P(c) = P(c_{2,j})$). (Observe that $HC(\varphi_1, \varphi_2)$ is the set of all non-tautological clauses $c^1_{i,j}$ and $c^2_{i,j}$.)

Step 2. $S_a := S \setminus S_2$; $S_b := S_2$;
for each $c^*_{i,j} \in S_2$ **do**
  if $\varphi_1 \leq c^1_{i,j}$ and $\varphi_2 \leq c^1_{i,j}$ **then**
  begin $S_a := S_a \cup \{c^1_{i,j}\}$; $S_b := S_b \setminus \{c^*_{i,j}\}$ **end**
  elseif $\varphi_1 \leq c^2_{i,j}$ and $\varphi_2 \leq c^2_{i,j}$ **then**
  begin $S_a := S_a \cup \{c^2_{i,j}\}$; $S_b := S_b \setminus \{c^*_{i,j}\}$ **end**;

Step 3. Output $\psi := \bigwedge_{c \in S_a} c \wedge \bigwedge_{c^*_{i,j} \in S_b} c^1_{i,j}$. $\square$

Example 3.3 *Let us apply algorithm* CORE *to Horn CNFs φ_1 and φ_2 given in Example* 3.1. *In Step 1, we have $S = \{(\overline{x}_1 \vee \overline{x}_3 \vee x_4 \vee x_6), (\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5 \vee x_6), (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee x_6)\}$ and $S_2 = \{(\overline{x}_1 \vee \overline{x}_3 \vee x_4 \vee x_6), (\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5 \vee x_6)\}$. A clause $c^*_{2,1} = (\overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_5 \vee x_6)$ satisfies the if-statement of Step* 2 *(i.e., $\varphi_1 \leq c^1_{2,1}$ and $\varphi_2 \leq c^1_{2,1}$ holds for $c^1_{2,1} = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_5 \vee x_6)$); any other clause in S_2 satisfies neither the if- nor elseif-statement. Thus, Step 3 outputs the Horn CNF of* (3.2). $\square$

This algorithm runs in polynomial time. Indeed, both implication tests in Step 2 are done in linear time since φ_1 and φ_2 are Horn (cf. [6]), and all other steps are clearly polynomial.

Theorem 3.8 *Let φ_1 and φ_2 be Horn CNFs. Then, algorithm* CORE *computes a Horn core ψ of $\varphi = \varphi_1 \vee \varphi_2$ in polynomial time. Moreover, $\varphi_1 \leq \psi$ holds.*

Proof. Observe that

$$\varphi_1 \leq \psi \leq \varphi_1 \vee \varphi_2 \tag{3.3}$$

obviously holds. Indeed, each clause c in S_a of Step 3 is an implicate of φ_1, and each clause $c^1_{i,j} \in S_b$ is subsumed by some clause of φ_1; hence, the first implication holds. The second holds since $\varphi_1 \vee \varphi_2 \equiv \bigwedge_{c^*_{i,j} \in S} c^*_{ij}$ holds by definition, and each clause $c = c^*_{i,j}$ in $\varphi_1 \vee \varphi_2$ is subsumed by some clause in ψ.

We claim that this ψ is a Horn core. Towards a contradiction, suppose that there exists a Horn core ψ^* such that $\psi^* > \psi$. Then, $\psi^* \equiv \bigwedge_{c \in S_a} c \wedge \bigwedge_{c^*_{i,j} \in S_b} c^{h(i,j)}_{i,j}$ must hold for some $h(i,j) \in \{1,2\}$ (the proof is similar to that of Lemma 3.7). Since $\psi \neq \psi^*$, it follows that $\psi \leq c^2_{i,j}$ holds for some $c^*_{i,j} \in S_b$, and hence $\varphi_1 \leq c^2_{i,j}$ by (3.3). On the other hand, since $c^2_{i,j}$ is subsumed by some clause in φ_2, $\varphi_2 \leq c^2_{i,j}$ also holds. However, this means that the clause $c^*_{i,j}$ is removed from S_b in Step 2 and hence $c^*_{i,j} \notin S_b$ in Step 3, which is a contradiction. Consequently, ψ is maximal. $\square$

An analysis of the algorithm CORE reveals that it bears no nondeterminism in computing a Horn core ψ that satisfies $\varphi_1 \leq \psi \leq \varphi_1 \vee \varphi_2$. This may indicate that a Horn core including φ_1 is unique. This is in fact the case.

Proposition 3.9 *Let Σ_1 and Σ_2 be Horn theories, and let $\Sigma = \Sigma_1 \cup \Sigma_2$. Then, there exists a unique Horn core Π that satisfies $\Sigma_1 \subseteq \Pi \subseteq \Sigma$.*

Proof. We show that for any Horn theories Π_1 and Π_2 such that $\Sigma_1 \subseteq \Pi_1 \subseteq \Sigma$ and $\Sigma_1 \subseteq \Pi_2 \subseteq \Sigma$, it holds that $\Pi' = Cl_\wedge(\Pi_1 \cup \Pi_2)$ is Horn and satisfies $\Pi_1, \Pi_2 \subseteq \Pi' \subseteq \Sigma$; the uniqueness follows from this.

Let $w \in \Pi' \setminus (\Pi_1 \cup \Pi_2)$, and we show that $w \in \Sigma$. Note that w is of the form $w = \bigwedge_{u \in S_1} u \wedge \bigwedge_{v \in S_2} v$, where $S_1 \subseteq \Pi_1$ and $S_2 \subseteq \Pi_2$. Since Π_1 and Π_2 are Horn, $u' = \bigwedge_{u \in S_1} u \in \Pi_1$, $v' = \bigwedge_{v \in S_2} v \in \Pi_2$ and $w = u' \bigwedge v'$. Since $\Pi_1, \Pi_2 \subseteq \Sigma_1 \cup \Sigma_2$, the following four cases may occur: (1) $u', v' \in \Sigma_1$, (2) $u', v' \in \Sigma_2$, (3) $u' \in \Sigma_1$, $v' \in \Sigma_2$, and (4) $u' \in \Sigma_2$, $v' \in \Sigma_1$. Clearly, $w = u' \wedge v' \in \Sigma$ holds for the case (1) or (2). As for the case (3), $u' \in \Pi_2$ holds by $\Sigma_1 \subseteq \Pi_2$. This implies that $u', v' \in \Pi_2$, and hence $w \in \Pi_2 \subseteq \Sigma$ holds. The case (4) is similar to (3). $\square$

A a consequence, we call the two Horn cores Π_1 and Π_2 satisfying $\Sigma_1 \subseteq \Pi_1 \subseteq \Sigma_1 \cup \Sigma_2$ and $\Sigma_2 \subseteq \Pi_2 \subseteq \Sigma_1 \cup \Sigma_2$, respectively, the *canonical Horn cores* of Σ_1 and Σ_2 with respect to $\Sigma = \Sigma_1 \cup \Sigma_2$, and denote them by $can(\Sigma_1, \Sigma)$ and $can(\Sigma_2, \Sigma)$, respectively.

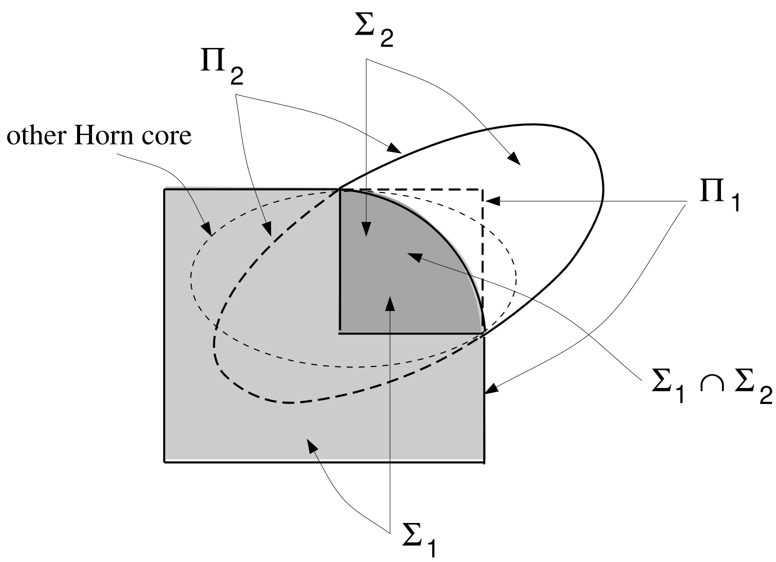

Figure 1: Structure of the family of Horn cores.

Observe that the generalization of Proposition 3.9 to a disjunction of $k(\geq 3)$ Horn theories does not hold. To see this, consider $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and assume that $\Sigma_1 = \emptyset$ holds. Then, if Σ_1 had the unique Horn core Π such that $\Sigma_1 \subseteq \Pi \subseteq \Sigma$ $(= \Sigma_2 \cup \Sigma_3)$, then this Π would be the unique Horn core of $\Sigma_2 \cup \Sigma_3$. Clearly, this conflicts Proposition 3.4.

Finally, we have the following structural result.

Proposition 3.10 *Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_l$ be Horn theories. Then $\Delta = \bigcap_{i=1}^{l} \Sigma_i$ is contained in every Horn core of $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$.*

Proof. Let Π be any Horn core of Σ. We show that $\Omega = Cl_\wedge(\Pi \cup \Delta)$ is contained in Σ; this completes the proof, since the maximality of Π then implies $\Delta \subseteq \Pi = \Omega$.

Take any model $w \in \Omega$. Since Π and Δ are Horn, we only need to consider the case of $w = u \bigwedge v$ for some $u \in \Pi$ and $v \in \Delta$. (The same reasoning is used in the proof of Proposition 3.9.) Now, $u \in \Sigma_i$ holds for some i. Since $v \in \Delta$ is also contained in this Σ_i and Σ_i is Horn, it holds that $w \in \Sigma_i$. Hence, $w \in \Sigma$ holds, implying $\Omega \subseteq \Sigma$. $\Box$

Figure 1 shows the structure of the family of Horn cores of the disjunction of two Horn theories.

3.3 Horn cores of the disjunction of more than two Horn theories

In this subsection we develop an algorithm to compute a Horn core of a general disjunction of $\varphi = \varphi_0 \vee \varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_l$ of Horn CNFs φ_i. Let us first show that the direct generalization of algorithm CORE does not produce a Horn core for the case $l \geq 3$.

Example 3.4 *Let*

$$\begin{aligned}\varphi_0 &= (\overline{x}_1 \vee \overline{x}_2 \vee x_4)(\overline{x}_2 \vee x_7)(\overline{x}_3 \vee x_5 \vee \overline{x}_7)\\ \varphi_1 &= (\overline{x}_3 \vee x_5)\\ \varphi_2 &= (\overline{x}_1 \vee \overline{x}_3 \vee x_6),\end{aligned}$$

and let $\varphi = \varphi_0 \vee \varphi_1 \vee \varphi_2$. *Then* φ *is equivalent to the CNF*

$$\varphi' = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4 \vee x_5 \vee x_6)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_5 \vee x_6 \vee x_7)(\overline{x}_1 \vee \overline{x}_3 \vee x_5 \vee x_6 \vee \overline{x}_7). \tag{3.4}$$

Since no clause c^* *in* $HC(\varphi_0, \varphi_1, \varphi_2)$ *satisfies* $|P(c^*)| = 1$ *in this case, the* S_b *in the beginning of Step* 2 *of the generalized algorithm* CORE *becomes the set of all clauses in* φ' *of* (3.4)*. Thus, in Step* 3*, we have*

$$\psi = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_7)(\overline{x}_1 \vee \overline{x}_3 \vee x_5 \vee \overline{x}_7).$$

However, this ψ *is not Horn core of* φ*, since*

$$\psi' = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_5)(\overline{x}_1 \vee \overline{x}_3 \vee x_5 \vee \overline{x}_7)$$

satisfies $\psi < \psi' \leq \varphi'$ $(\equiv \varphi)$*. We will show in Example* 3.5 *that this* ψ' *is in fact a Horn core of* φ*.* □

However, by using algorithm CORE repeatedly, we can get an algorithm to compute a Horn core of $\varphi = \vee_{i=0}^{l} \varphi_i$. This algorithm is polynomial if l is bounded by a constant k. Informally, it constructs a sequence of (not necessarily strictly) increasing Horn theories $\psi_0 \leq \psi_1 \leq \psi_2 \leq \cdots$ which is contained in φ. The sequence converges to ψ^*, which is a Horn core of φ.

Let $\varphi_0, \varphi_1, \ldots, \varphi_l$ be Horn CNFs, and let CORE(μ_1, μ_2) denote a Horn CNF of the canonical Horn core $can(\mu_1, \mu_1 \vee \mu_2)$. We now define the following sequence ψ_k, $k \geq 0$.

$$\begin{aligned}\psi_0 &= \varphi_0;\\ \psi_1 &= \mathrm{CORE}(\psi_0, \varphi_1);\\ \psi_2 &= \mathrm{CORE}(\psi_1, \varphi_2);\\ &\vdots\\ \psi_l &= \mathrm{CORE}(\psi_{l-1}, \varphi_l);\\ &\vdots\\ \psi_{i\cdot l+j} &= \mathrm{CORE}(\psi_{i\cdot l+j-1}, \varphi_j), \qquad i \geq 0,\ 1 \leq j \leq l\\ &\vdots\end{aligned}$$

This sequence ψ_k monotonically increases from φ_0 to

$$\psi^* = \lim_{k\to\infty} \psi_k,$$

by first increasing it in φ_1, then in φ_2 and so on. However, note that φ_l is not necessarily a Horn core of φ. The reason is that in building ψ_1 from ψ_0, say, some models of φ_1 may have been excluded which now can be added to the ψ_l such that the Horn property is still preserved. To catch this aspect, the algorithm cycles over φ_1, φ_2, …, φ_l, until no further change is possible. Observe that $\psi^* = \psi_k$ holds for some finite $k \geq 0$ since there exist only finite models. Now we have the next lemma.

Lemma 3.11 *Given Horn CNFs $\varphi_0, \varphi_1, \ldots, \varphi_l$, the above ψ^* is a Horn core of $\varphi = \vee_{i=0}^{l} \varphi_i$.*

Proof. By an inductive argument, it is easy to show that every ψ_k is a Horn CNF satisfying $\psi_k \leq \varphi$. Hence, ψ^* is a Horn CNF satisfying $\psi^* \leq \varphi$. Now assuming that ψ^* is not a Horn core of φ, we derive a contradiction.

In this case, by Lemma 4.5 (to be shown later), there exists a model $v \in \Sigma \setminus \Pi^*$ such that $\Pi^* \cup \{v\}$ is Horn, where Π^* and Σ are the theories represented by ψ^* and φ, respectively. This model v satisfies $v \in \Sigma_i$ for some i, where Σ_i is the Horn theory represented by φ_i. Let k^* be the index such that $\psi_{k^*} = \psi^*$. Since ψ^* is the limit, $\psi_{k'} = \psi^*$ holds for all $k' > k^*$. Consider the smallest index $k' > k^*$ such that $\psi_{k'} = \text{CORE}(\psi_{k'-1}, \varphi_i)$, in which the second argument is φ_i. Since $\psi_{k'-1} = \psi^*$ holds and $\Pi^* \cup \{v\}$ is Horn, this tells that ψ^* is not a Horn core of $\psi^* \vee \varphi_i$, a contradiction. □

Based on this lemma, we get the following algorithm.

Algorithm CORE*

Input: Horn CNFs $\varphi_i = \bigwedge_{j=1}^{m_i} c_{i,j}$, for $i = 0, 1, \ldots, l$.

Output: A Horn core ψ^* of $\varphi = \varphi_0 \vee \varphi_1 \vee \cdots \vee \varphi_l$.

Step 1. Set $\psi := \varphi_0$; $i := 0$; changes := 0;

Step 2. **while** changes $< l$ **do begin**
  if $i < l$ **then** $i := i + 1$ **else** $i := 1$;
  ψ_{new} := CORE(ψ, φ_i);
  if $\psi < \psi_{new}$ **then**
  begin $\psi := \bigwedge_{c \in HC(\varphi_0, \varphi_1, \ldots, \varphi_l) \text{ s.t. } \psi_{new} \leq c} c$;
    changes := 0;
  end
  else changes := changes + 1;
 end{while};

Step 3. Output the Horn CNF $\psi^* = \psi$. □

Note that, whenever $\psi_{new} = \text{CORE}(\psi, \varphi_i)$ increases from ψ during the iteration of Step 2, the algorithm replaces it by

$$\psi = \bigwedge_{c \in HC(\varphi_0, \varphi_1, \ldots, \varphi_l) \text{ s.t. } \psi_{new} \leq c} c. \tag{3.5}$$

By Lemma 3.7, this replacement does not affect the correctness of the algorithm. The algorithm halts, if no change within l consecutive calls is detected, i.e., CORE(ψ, φ_i) = CORE(ψ, φ_{i+1}) $= \ldots =$ CORE(ψ, φ_{i-1}) holds, where $i \geq 1$ and $i - 1 = l$ if $i = 1$.

Example 3.5 *Let us apply the algorithm* CORE* *to the disjunction $\varphi = \varphi_0 \vee \varphi_1 \vee \varphi_2$, where φ_0, φ_1 and φ_2 are given in Example* 3.4.

In the first iteration of Step 2*, by calling algorithm* CORE *described in Subsection* 3.2*, we have*

$$\psi_{new} = \text{CORE}(\varphi_0, \varphi_1) = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_5)(\overline{x}_2 \vee \overline{x}_3 \vee x_5)(\overline{x}_3 \vee x_5 \vee \overline{x}_7).$$

Based on

$$HC(\varphi_0, \varphi_1, \varphi_2) = \{(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_5), (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_6), (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_7), (\overline{x}_1 \vee \overline{x}_3 \vee x_5 \vee \overline{x}_7), (\overline{x}_1 \vee \overline{x}_3 \vee x_6 \vee \overline{x}_7)\},$$

we have

$$\psi_1 = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_5)(\overline{x}_1 \vee \overline{x}_3 \vee x_5 \vee \overline{x}_7).$$

In the second iteration, we have $\psi_{new} = \mathrm{CORE}(\psi_1, \varphi_2) = \psi_1$*, and hence* $\psi_2 = \psi_1$ *holds. Since* $l\,(= 2)$ *consecutive members* ψ_1 *and* ψ_2 *satisfies* $\psi_1 = \psi_2$*, the algorithm outputs* ψ_1 *and halts.* $\Box$

Theorem 3.12 *Algorithm* CORE* *outputs a Horn core* ψ *of* $\varphi = \varphi_0 \vee \varphi_1 \vee \cdots \vee \varphi_l$ *satisfying* $\varphi_0 \leq \psi \leq \varphi$*. Moreover, it runs in polynomial time if* l *is bounded by some constant.*

Proof. The correctness of the algorithm is established by Lemma 3.11. Notice that in the algorithm, each ψ_{new} = CORE(ψ, φ_i) is replaced by the increased Horn theory ψ of (3.5) for which $\psi_{new} \leq \psi \leq \varphi$ holds. The sequence these ψ also converges to ψ^*.

It thus remains to show that the algorithm is polynomial in l. Observe that each CNF ψ (except for the initial φ_0) contains only clauses from $HC(\varphi_0, \varphi_1, \ldots, \varphi_l)$. There are at most $l \cdot \Pi_{i=0}^{l} m_i$ clauses in $HC(\varphi_0, \varphi_1, \ldots, \varphi_l)$, where m_i is the number of clauses in φ_i. Clearly, for a constant l, this number is polynomial in the input size of $\varphi_0, \varphi_1, \ldots, \varphi_l$. Moreover, since $T(\psi)$ (i.e., its set of models) strictly increases in every l iterations, the set of clauses $HC(\varphi_0, \varphi_1, \ldots, \varphi_l)$ which are logically entailed by ψ monotonically decreases, if ψ changes. Thus, the number of iterations is bounded by $l \cdot |HC(\varphi_0, \varphi_1, \ldots, \varphi_l)|$, which is polynomial if l is bounded by a constant. Since a call to CORE(ψ, φ_i) is done in polynomial time by Theorem 3.8, this implies that CORE* requires polynomial time. $\Box$

Remarks. (1) To keep ψ of (3.5) small, we may further replace it by a prime CNF. Note that a prime CNF for any Horn CNF ψ is computable in quadratic time [10].
(2) We can use CORE* as a reasonable "any time algorithm" for computing an approximation of some Horn core. In fact, we can interrupt the algorithm at any step, and the contents of ψ is a Horn theory which implies φ. The approximation improves with computation time, and eventually yields a Horn core. $\Box$

4 Characteristic Models

For a Horn theory Σ, a model $v \in \Sigma$ is called *characteristic* [12], if $v \notin Cl_\wedge(\Sigma \setminus \{v\})$ holds. The set of all characteristic models of Σ, the *characteristic set of* Σ, is denoted by $C^*(\Sigma)$. Note that every Horn theory Σ has the unique characteristic set $C^*(\Sigma)$ and that $\max(\Sigma) \subseteq C^*(\Sigma)$. For example, a Horn theory $\Sigma = \{(0101), (1001), (1000), (0001), (0000)\}$ has $C^*(\Sigma) = \{(0101), (1001), (1000)\}$ and $\max(\Sigma) = \{(0101), (1001)\}$. In the rest of this section, we reconsider the problems in the previous section, assuming that Horn theories are now represented by their characteristic models.

4.1 Horn property

Like in the case of CNFs, deciding whether a disjunction of Horn theories is Horn is intractable in general.

Theorem 4.1 *Given characteristic sets $C^*(\Sigma_1)$, $C^*(\Sigma_2)$, $\ldots$, $C^*(\Sigma_l)$ of Horn theories Σ_1, Σ_2, $\ldots$, Σ_l, respectively, deciding whether $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ is Horn is* co-NP-*complete.*

Proof. The membership part of co-NP is similar to the proof of Theorem 3.1.

For the hardness part, we consider the Horn theory $\Sigma(t)$ and its characteristic set $C^*(\Sigma(t))$, defined from a term t whose positive (resp., negative) literals have indices from $P(t) \subseteq V = \{1, 2, \ldots, n\}$ (resp., $N(t) \subseteq V$):

$$\begin{aligned} \Sigma(t) &= \{x^{V \setminus A} \mid N(t) \subseteq A \subseteq V \setminus P(t)\} \\ C^*(\Sigma(t)) &= \{x^{V \setminus A} \mid N(t) \subseteq A \subseteq V \setminus P(t),\ |A \setminus N(t)| \leq 1\}. \end{aligned}$$

Clearly, $C^*(\Sigma(t))$ is constructible from t in polynomial time. Hence, the DNFs in the proof of Theorem 3.1 can be transformed in polynomial time into equivalent theories Σ_i, $i = 1, 2, \ldots, l$, and $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$, given by $C^*(\Sigma_i)$ and $C^*(\Sigma)$, respectively. Thus the argument in the proof of Theorem 3.1 proves the result. $\Box$

The next result is immediate from Corollary 3.5.

Corollary 4.2 *Unless P=NP, there is no polynomial total time algorithm which, given characteristic sets $C^*(\Sigma_1)$, $C^*(\Sigma_2)$, $\ldots$, $C^*(\Sigma_l)$ of Horn theories Σ_1, Σ_2, $\ldots$,$\Sigma_l \subseteq \{0, 1\}^n$, computes any number k (≥ 1) of Horn cores of $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$.*

The next result gives a precise semantical characterization of the Horn property of a disjunction.

Lemma 4.3 *Let $M_i \subseteq \{0, 1\}^n$, $i = 1, 2, \ldots, l$, be sets of models. Then $\Sigma = \bigcup_{i=1}^{l} Cl_\wedge(M_i)$ is Horn if and only if*

$$\bigwedge_{v \in S} v \in \Sigma \ \text{ holds for every } S \subseteq \textstyle\bigcup_{i=1}^{l} M_i \text{ such that } 1 \leq |S| \leq l. \tag{4.1}$$

Proof. Since a Horn theory is closed under intersection, the only-if part is obvious. For the if-part, we show by induction on $k \geq l$ that (4.1) implies

$$\bigwedge_{v \in S} v \in \Sigma \ \text{ holds for every } S \subseteq \textstyle\bigcup_{i=1}^{l} M_i \text{ such that } 1 \leq |S| \leq k, \tag{4.2}$$

which clearly implies that $\Sigma = \bigcup_{i=1}^{l} Cl_\wedge(M_i)$ is Horn. For the base $k = l$, the statement holds by (4.1). For the induction step, assume that (4.2) holds for some $k \geq l$, and consider the case $k + 1$.

Take a set $S \subseteq \bigcup_{i=1}^{l} M_i$ such that $|S| = k + 1$ arbitrarily, and consider the model $w \in \bigwedge_{v \in S} v$. For each $S_j \subseteq S$ such that $|S_j| = |S| - 1$, we denote $u^{(j)} = \bigwedge_{v \in S_j} v$. All $u^{(j)}$ belong to Σ by the induction hypothesis, and there are $k + 1$ such S_j. If these contain different S_{j_1} and S_{j_2} such that $u^{(j_1)} = u^{(j_2)}$, then $w = u^{(j_1)} \wedge u^{(j_2)} = u^{(j_1)}$ holds, and $w \in \Sigma$ follows from the induction hypothesis. Otherwise, there are $k + 1 \geq l + 1$ different models $u^{(1)}, u^{(2)}, \ldots, u^{(k+1)}$. Hence, by the pigeonhole principle, $u^{(1)}$ and $u^{(2)}$, say, are both contained in $Cl_\wedge(M_{i^*})$ for some i^*. Since $w = u^{(1)} \bigwedge u^{(2)} \in Cl_\wedge(M_{i^*})$ by definition, it follows $w \in \Sigma$ from the induction hypothesis. $\Box$

As an immediate consequence, we obtain the following result.

Theorem 4.4 *If l is bounded by a constant, the problem in Theorem* 4.1 *can be solved in polynomial time.* $\Box$

Theorems 4.1 and 4.4, respectively, correspond to their CNF versions Theorems 3.1 and 3.3.

4.2 Computing Horn cores

Assuming Horn CNF representations, we have presented in Section 3 a polynomial time algorithm for computing a Horn core ψ satisfying $\varphi_1 \leq \psi \leq \varphi_1 \vee \varphi_2$. For characteristic models, a similar algorithm is hard to find. This is suggested by the following theorem (proved after Lemma 4.5), which tells that recognizing a Horn core is intractable, even for the case of two Horn theories.

Lemma 4.5 *Let Σ_1 be a Horn theory and Σ_2 an arbitrary theory. Then, Σ_1 is not a Horn core of $\Sigma = \Sigma_1 \cup \Sigma_2$ if and only if there exists some model $v \in \Sigma_2 \setminus \Sigma_1$ such that $v \bigwedge w \in \Sigma_1 \cup \{v\}$ holds for all $w \in C^*(\Sigma_1)$.*

Proof. Suppose Σ_1 is not a Horn core, but Π is a Horn core such that $\Sigma_1 \subset \Pi \subseteq \Sigma_1 \cup \Sigma_2$. Let v be any minimal model in $\Pi \setminus \Sigma_1$. Then $v \bigwedge w \in \Pi$ holds for all $w \in C^*(\Sigma_1)$. By the minimality, however, this implies $v \bigwedge w \in \Sigma_1 \cup \{v\}$. Conversely, if there is a model v satisfying $v \bigwedge w \in \Sigma_1 \cup \{v\}$ for all $w \in C^*(\Sigma_1)$, then $v \wedge w \in \Sigma_1 \cup \{v\}$ holds for all $w \in \Sigma_1$, since each $w \in \Sigma_1$ can be represented by $w = \bigwedge_{u \in S} u$ for some $S \subseteq C^*(\Sigma_1)$ and $v \bigwedge u \in \Sigma_1$ for all $u \in S$. Hence, $\Sigma_1 \cup \{v\}$ is Horn by Lemma 4.3, which means that Σ_1 is not a Horn core. $\square$

Theorem 4.6 *Given characteristic sets $C^*(\Sigma_1)$ and $C^*(\Sigma_2)$ of Horn theories $\Sigma_1, \Sigma_2 \subseteq \{0,1\}^n$, respectively, deciding whether Σ_1 is a Horn core of $\Sigma = \Sigma_1 \cup \Sigma_2$ is* co-NP*-complete.*

Proof. By Lemma 4.5, Σ_1 is not a Horn core if and only if some $v \in \Sigma_2 \setminus \Sigma_1$ exists such that $v \bigwedge w \in \Sigma_1 \cup \{v\}$ holds for every $w \in C^*(\Sigma_1)$. Such a v can be guessed and verified in polynomial time; hence the problem is in co-NP.

We then prove the co-NP-hardness by a reduction from 3SAT [8]. For a given 3-CNF formula $\varphi = \bigwedge_{i=1}^{m} c_i$ on n variables $x_1, x_2, \ldots, x_n$, we now define polynomially computable sets M_1 and M_2 of models in $\{0,1\}^{2n+m+1}$, such that $M_1 = C^*(\Sigma_1)$, $M_2 = C^*(\Sigma_2)$ for Horn theories Σ_1 and Σ_2, and Σ_1 is not a Horn core of $\Sigma = \Sigma_1 \cup \Sigma_2$ if and only if φ is satisfiable.

Without loss of generality, we assume that all literals in $L = \{x_j, \overline{x}_j \mid 1 \leq j \leq n\}$ appear in φ. Obviously, this restriction on φ does not affect the NP-completeness of SAT. Define $V = V_L \cup V_C \cup V_T$, where

$$\begin{aligned} V_L &= \{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}, \\ V_C &= \{n+1, n+2, \ldots, n+m\}, \\ V_T &= \{n+m+1\}. \end{aligned}$$

Intuitively, the elements in V_L correspond to the literals in L, the elements i in V_C to the clauses c_i in φ, and $n+m+1$ in V_T is a special tag column. Now we define the instance of our problem as follows: $M_1 = M_{1,1} \cup M_{1,2} \cup M_{1,3}$, where

$$\begin{aligned} M_{1,1} &= \{x^{(V_C \setminus \{n+i\}) \cup (V_L \setminus \{q\})} \mid n+i \in V_C,\, q \in c_i\}, \\ M_{1,2} &= \{x^{V_L \setminus \{j, \overline{j}\} \cup V_T} \mid 1 \leq j \leq n\}, \\ M_{1,3} &= \{x^{V_L \setminus \{j, \overline{j}, q\} \cup V_T} \mid 1 \leq j \leq n,\, q \in V_L \setminus \{j, \overline{j}\}\}, \end{aligned}$$

and $M_2 = M_{2,1} \cup M_{2,2}$, where

$$\begin{aligned} M_{2,1} &= \{x^{(V_L \setminus \{q\}) \cup V_T} \mid q \in V_L\}, \\ M_{2,2} &= \{x^{V_L \setminus \{j, \overline{j}\}} \mid 1 \leq j \leq n\}. \end{aligned}$$

Intuitively, each model in $M_{1,1}$ corresponds to the selection of a literal q from a clause c_i. The intersection of such models after choosing at least one model for each clause c_i corresponds to a choice of one literal from each clause. The models in $M_{1,2}$ and $M_{1,3}$ serve to cover all choices in which two opposite literals are chosen for at least one j. The models in $M_{2,1}$ similarly correspond to the choice of a single literal, but the special component $n + m + 1$ is set to 1. The models in $M_{2,2}$ have a similar role as those in $M_{1,2}$.

First, it is not hard to see that $M_i = C^*(M_i)$ holds, for $i = 1, 2$, and hence M_i is indeed the characteristic set of $\Sigma_i = Cl_\wedge(M_i)$. The sets M_1 and M_2 are obviously constructible in polynomial time from φ.

We claim that Σ_1 is not a Horn core of Σ if and only if φ is satisfiable.

Let us first show the only-if-part. Suppose that Σ_1 is not a Horn core of Σ. By Lemma 4.5, there exists some $v \in \Sigma_2 \setminus \Sigma_1$ such that $v \bigwedge w \in \Sigma_1 \cup \{v\}$ holds for all $w \in M_1$. Consider the following two cases.

Case 1: The component v_{n+m+1} is 0. Then, $v = \bigwedge_{w \in S} w$ for some $S \subseteq M_2$ such that $S \cap M_{2,2} \neq \emptyset$. Hence, for some $j \in \{1, 2, \ldots, n\}$, it holds that components j and $\overline{j}$ of v are both 0. Thus, the model v can be generated in Σ_1 by taking the intersection of a model in $M_{1,2}$ and models from $M_{1,1}$ (recall that every literal occurs in some clause). This means $v \in \Sigma_1$, which is a contradiction.

Case 2: The component v_{n+m+1} is 1. Then, v must be the intersection of some models in $M_{2,1}$. Moreover, for each $j = 1, 2, \ldots, n$, at least one of v_j and $v_{\overline{j}}$ is 1; indeed, any model $v \in \Sigma_2$ with $v_{n+m+1} = 1$ and $v_j = v_{\overline{j}} = 0$ for some j can be generated by intersections of models in $M_{1,2} \cup M_{1,3}$, which would again imply $v \in \Sigma_1$, a contradiction. Now, by assumption, $v \bigwedge w \in \Sigma_1 \cup \{v\}$ holds for all $w \in M_1$. Then, for a suitable $w \in M_{1,1}$, we obtain that $u = v \bigwedge w$ coincides with v on V_L, and has all other components (including $n + m + 1$) being 0. This implies $u \in \Sigma_1$, and hence u can be generated in Σ_1 only by the intersection of at least m models from $M_{1,1}$ including $v^{(1)}, v^{(2)}, \ldots, v^{(m)}$ such that $v^{(i)}_{n+i} = 0$ holds for all i. By the definition of $M_{1,1}$, this corresponds to a choice of one literal from each clause c_i, such that no opposite literals are chosen. It then follows that φ is satisfiable.

For the if-part, suppose that φ is satisfiable. Then, there exists a choice of one literal q_i from each clause c_i, $i = 1, \ldots, m$, such that no opposite literals are chosen. We claim that the model $v = x^B$, where $B = (V_L \setminus \{q_i \mid i = 1, \ldots, m\}) \cup V_T$, satisfies $v \in \Sigma_2 \setminus \Sigma_1$ and $v \bigwedge w \in \Sigma_1$ for all $w \in \Sigma_1$. Hence, by Lemma 4.5, it follows that Σ_1 is not a Horn core of Σ.

Clearly, such a v satisfies $v \in \Sigma_2$ (i.e., by the intersection of models in $M_{2,1}$) and $v \notin \Sigma_1$. Observe that the model v' which results from v by switching component $n+m+1$ to 0 is in Σ_1 (by the intersection of models in $M_{1,1}$). Hence, if $w \in M_{1,1}$, then $v \bigwedge w = v' \bigwedge w \in \Sigma_1$. On the other hand, if $w \in M_{1,2} \cup M_{1,3}$, then $v \bigwedge w$ (whose component $n+m+1$ takes value 1) is obtainable as the intersection of models in $M_{1,2} \cup M_{1,3}$; hence, $v \bigwedge w \in \Sigma_1$ also holds. □

Thus, computing a canonical Horn core is presumably difficult in general. We next point out that the difficulty can be avoided if Σ_2 is restricted in the following sense. Call a Horn theory Σ *sparse*, if $|\Sigma| \leq p(|C^*(\Sigma)|)$ holds for some polynomial $p(\cdot)$. Then, based on the next lemma proved in [7], we have the following theorem.

Lemma 4.7 *Given the characteristic set $C^*(\Sigma)$ of a Horn theory $\Sigma \subseteq \{0,1\}^n$, the models of Σ can be enumerated with $O(n^2|C^*(\Sigma)|)$ time delay.*

Theorem 4.8 *Given characteristic sets $C^*(\Sigma_1)$ and $C^*(\Sigma_2)$ of Horn theories $\Sigma_1, \Sigma_2 \subseteq \{0,1\}^n$, respectively, computing $C^*(\Pi_1)$ for $\Pi_1 = can(\Sigma_1, \Sigma_1 \cup \Sigma_2)$ is polynomial, if Σ_2 is known to be sparse.*

Proof. We can compute $C^*(\Pi_1)$ for $\Pi_1 = can(\Sigma_1, \Sigma_1 \cup \Sigma_2)$ as follows.

(1) Using the algorithm in [7], construct Σ_2 from $C^*(\Sigma_2)$. By Lemma 4.7, this is possible in polynomial time. Set $\Pi := C^*(\Sigma_1)$. (Here $Cl_\wedge(\Pi)$ will become Π_1.)

(2) Exploiting Lemma 4.5, check whether some model $v \in \Sigma_2 \setminus Cl_\wedge(\Pi)$ exists such that $Cl_\wedge(\Pi) \cup \{v\}$ is Horn.

(3) If such a model v is found, then set $\Pi := \Pi \cup \{v\}$ and return to (2).

(4) Compute $C^*(\Pi)$ and output it as the result.

In Step (2), $Cl_\wedge(\Pi) \cup \{v\}$ is Horn if and only if $v \bigwedge w \in Cl_\wedge(\Pi) \cup \{v\}$ holds for all $w \in \Pi$"; this avoids the re-computation of $C^*(\Pi)$ in every iteration. Overall, this procedure is polynomial since Σ_2 is sparse. □

Observe that the Horn theory Σ_2 in the proof of Theorem 4.6 is not sparse, and hence the above algorithm requires exponential time.

4.3 Exponential sizes of Horn cores

The intractability of recognizing and, as a corollary along standard arguments, of computing a canonical Horn core does not immediately rule out the possibility of computing one of the Horn cores in polynomial time. However, our next result implies that a polynomial time algorithm for this task is impossible.

Theorem 4.9 *For every $n \geq 1$, there exist Horn theories $\Sigma_1, \Sigma_2 \subseteq \{0,1\}^{4n+1}$ such that $|C^*(\Sigma_1)| = |C^*(\Sigma_2)| = 2n$ but every Horn core Π of $\Sigma = \Sigma_1 \cup \Sigma_2$ has size $|C^*(\Pi)|$ not bounded by a polynomial in n.*

Proof. Given a fixed $n\,(\geq 1)$, define two sets of models $S_1, S_2 \subseteq \{0,1\}^{4n+1}$ as follows. Let $V_k = \{(k-1)*n+j \mid j = 1,2,\ldots,n\}$ for $k = 1,2,3,4$, and $V = \cup_{k=1}^{4} V_k \cup \{4n+1\} = \{1,2,\ldots,4n+1\}$. Observe that $V_1 = \{1,2,\ldots,n\}$ contains the first n components, V_2 the next n components, and so on. Then,

$$\begin{aligned} S_1 &= \{x^{V\setminus(V_3\cup\{i,3n+i\})}, x^{V\setminus(V_3\cup\{n+i,3n+i\})} \mid 1 \leq i \leq n\}, \\ S_2 &= \{x^{V\setminus(V_4\cup\{i,2n+i,4n+1\})}, x^{V\setminus(V_4\cup\{n+i,2n+i,4n+1\})} \mid 1 \leq i \leq n\}. \end{aligned}$$

Informally, each model v in $S_1 \cup S_2$ consists of 4 blocks $b_1(v), \ldots, b_4(v)$, each of which has n bits (addressed by $V_1, \ldots, V_4$), plus an extra bit v_{4n+1}. The set S_1 contains all models v such that (1) exactly $n+2$ bits are 0, (2) the i-th bit in either $b_1(v)$ or $b_2(v)$ is 0, (3) the i-th bit in $b_4(v)$ is 0, for some $i \in \{1,2,\ldots,n\}$, and (4) $b_3(v)$ is the zero vector (0,0, …,0) (for short, **0**). Similarly, S_2 contains all models v such that (1) exactly $n+3$ bits are 0, (2) the i-th bit in either $b_1(v)$ or $b_2(v)$ is 0, (3) the i-th bit of $b_3(v)$ is 0, and (4) $b_4(v)$ is **0** and $v_{4n+1} = 0$. Intuitively, each $v \in S_1$ represents a choice from two alternatives (either $b_1(v)$ or $b_2(v)$) for assigning 0 to a position i. This is similar for each $v \in S_2$.

Clearly, $|S_1| = |S_2| = 2n$, and $S_l = C^*(S_l)$ holds for $l = 1,2$, since all the models in S_l are incomparable. Hence S_l is the characteristic set of $\Sigma_l = Cl_\wedge(S_l)$. Note that $\Sigma_1 \cap \Sigma_2 = \emptyset$ holds.

Let

$$H_0 = H_{0,1} \cup H_{0,2}, \tag{4.3}$$

where

$$\begin{aligned} H_{0,1} &= \{x^{I_1 \cup I_2} \mid I_1 \subseteq V_1, I_2 \subseteq V_2, I_1 \cap \{j-n \mid j \in I_2\} = \emptyset\}, \\ H_{0,2} &= \{x^{I_1 \cup I_2 \cup \{4n+1\}} \mid I_1 \subseteq V_1, I_2 \subseteq V_2, I_1 \cap \{j-n \mid j \in I_2\} = \emptyset\}, \end{aligned}$$

i.e., the set of all models v which have (1) either $b_1(v)_i = 0$ or $b_2(v)_i = 0$ or both, for all $i = 1, 2, \ldots, n$, (2) $b_3(v) = b_4(v) = \mathbf{0}$, and (3) v_{4n+1} is set arbitrarily.

Fact 1: Every Horn core Π of $\Sigma = \Sigma_1 \cup \Sigma_2$ satisfies $\Pi \supseteq H_0$.

To prove this fact, we note that $H_0 \subseteq \Sigma$, and, for each $v \in H_0$ and $w \in \Sigma$, it holds that $v \bigwedge w \in H_0$. Let Π be a Horn core of Σ. Then $\Pi \cup H_0 \subseteq \Sigma$ holds, and moreover, by Lemma 4.3, $\Pi \cup H_0$ is Horn. Hence, the maximality of a Horn core implies $H_0 \subseteq \Pi$.

Let us first consider the canonical Horn cores $\Pi_l = can(\Sigma_l, \Sigma)$, for $l = 1, 2$. It follows from Fact 1 that $H_{0,l} \subseteq \Pi_l$ holds. Denote, for any model v,

$$\chi(v) = \{j \in \{1, 2, \ldots, n\} \mid b_1(v)_j = 0 \text{ or } b_2(v)_j = 0\},$$

i.e., $\chi(v)$ contains the positions set to 0 in either $b_1(v)$ or $b_2(v)$. Then we obtain

$$\max(H_{0,l}) = \{v \in H_{0,l} \mid \chi(v) = \{1, 2, \ldots, n\}\},$$

whose size is obviously exponential in n. We now claim that

$$\max(H_{0,l}) \quad \subseteq \quad C^*(\Pi_l) \quad \text{for } l = 1, 2, \tag{4.4}$$

which implies that $|C^*(\Pi_l)|$ is exponential in n.

We only consider the case of $l = 1$, since the case of $l = 2$ is similar. Assume that some $v \in \max(H_{0,1})$ is not contained in $C^*(\Pi_1)$. Note that $v_{4n+1} = 0$, and furthermore $w_{4n+1} = 1$ holds for every $w \in \Sigma_1$. Hence $C^*(\Pi_1)$ must contain at least one model $u \in \Sigma_2$ such that $u > v$. Since $u \in \Sigma_2$, there exists an index j such that either $b_1(u)_j = 0$ or $b_2(u)_j = 0$. Let $w \in S_1 (\subseteq \Sigma_1)$ be the vector that satisfies $b_1(w)_j = 0$ if $b_1(u)_j = 0$ (resp., $b_2(w)_j = 0$ if $b_2(u)_j = 0$). Then, for $x = u \wedge w$, we have $b_1(x) = b_1(u)$, $b_2(x) = b_2(u)$, $b_3(x) = b_4(x) = \mathbf{0}$, and $x_{4n+1} = 0$. Since $x \in \Sigma$ must hold, it follows that $x \in \Sigma_2$. Thus, $x = \bigwedge_{y \in S} y$ for some $S \subseteq S_2$. Since $b_3(x) = \mathbf{0}$, it follows that either $b_1(x)_j = b_1(u)_j = 0$ or $b_2(x)_j = b_2(u)_j = 0$ holds for all $j = 1, \ldots, n$. But this implies $u \in H_{0,1}$, which is a contradiction. Hence (4.4) holds.

As easily seen, besides Π_1 and Π_2, further Horn cores exist. Proposition 3.9 and Fact 1 imply that for any such Horn core Π, the sets $H_1 = (\Sigma_1 \cap \Pi) \setminus H_0$ and $H_2 = (\Sigma_2 \cap \Pi) \setminus H_0$ are both nonempty. Moreover, Π has the following property.

Fact 2. For all $v^{(1)} \in H_1$ and $v^{(2)} \in H_2$, it holds that $v^{(1)} \bigwedge v^{(2)} \in H_0$.

To see this, let $w = v^{(1)} \bigwedge v^{(2)}$. Then $b_3(w) = b_4(w) = \mathbf{0}$ holds, since $v^{(1)} \in \Sigma_1$ and $v^{(2)} \in \Sigma_2$. This and $w \in \Pi \subseteq \Sigma_1 \cup \Sigma_2$ imply $w \in H_0$.

By Fact 2, $w = v^{(1)} \bigwedge v^{(2)}$ satisfies either $b_1(w)_j = 0$ or $b_2(w)_j = 0$ (or both) for all j. Thus the models $v^{(1)}$ and $v^{(2)}$ satisfy $\chi(v^{(1)}) \cup \chi(v^{(2)}) = \{1, 2, \ldots, n\}$. Since this holds for all such $v^{(1)}$ and $v^{(2)}$, every set $\chi(v^{(1)})$ must include $\{1, 2, \ldots, n\} \setminus \bigcap_{v^{(2)} \in H_2} \chi(v^{(2)})$, and by symmetry every $\chi(v^{(2)})$ must include $\{1, 2, \ldots, n\} \setminus \bigcap_{v^{(1)} \in H_1} \chi(v^{(1)})$. This implies the following fact.

Fact 3. There exists a set $I \subset \{1, 2, \ldots, n\}$ of size $|I| \geq n/2$ such that either (1) $I \subseteq \chi(v^{(1)})$ holds for every $v^{(1)} \in H_1$, or (2) $I \subseteq \chi(v^{(2)})$ holds for every $v^{(2)} \in H_2$.

Assume that (1) holds; the case (2) is similar. Choose $v^{(1)} \in \max(H_1)$ arbitrarily. Then, the maximality of Horn core Π implies that every model $w \in \Sigma_1 \setminus H_0$ such that $\chi(w) = \chi(v^{(1)})$ and w coincides with $v^{(1)}$ on the remaining part (i.e., on $V_3 \cup V_4 \cup \{4n+1\}$) is contained in Π (cf. Facts 1 and 2). There exist $2^{|\chi(v^{(1)})|}$ maximal such models w, each of which chooses 0 either in block $b_1(w)$ or in $b_2(w)$, but not in both, for the positions in $\chi(v^{(1)})$. Since $I \subseteq \chi(v^{(1)})$, at least $2^{|I|}$ models from $\Sigma_1 \setminus H_0$ are contained in Π. Note that these models are maximal in Π. Since $\max(\Pi) \subseteq C^*(\Pi)$, it follows that $|C^*(\Pi)| \geq 2^{|I|} \geq 2^{n/2}$. Thus, the size of $C^*(\Pi)$ is not bounded by a polynomial in n. This proves the result. $\square$

Corollary 4.10 *There is no polynomial time algorithm for computing $C^*(\Pi)$ of any Horn core Π for $\Sigma_1 \cup \Sigma_2$, from the characteristic sets $C^*(\Sigma_1)$, $C^*(\Sigma_2)$ of Horn theories $\Sigma_1, \Sigma_2 \subseteq \{0,1\}^n$, respectively.*

5 Horn Envelope of a Disjunction of Horn Theories

In this section, we briefly discuss the Horn envelope of a disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ of Horn theories Σ_i, $i = 1, 2, \ldots, l$. For model-based representation, we can compute the Horn envelope in polynomial time. Indeed, the characteristic set of Σ is given by $C^*(M)$, where $M = \bigcup_{i=1}^{l} C^*(\Sigma_i)$. Clearly, $C^*(M)$ is computable from M in polynomial time, by removing all models v which are represented by the intersection of other models in M. Thus we have the following theorem.

Theorem 5.1 *Given characteristic sets $C^*(\Sigma_1)$, $C^*(\Sigma_2)$, $\ldots$, $C^*(\Sigma_l)$ of Horn theories Σ_1, Σ_2, $\ldots$, Σ_l, respectively, the Horn envelope of $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ can be computed in polynomial time.*

As for the CNF representation, computing the Horn envelope requires in general exponential time (and space). This is because any Horn CNF that represents the envelope of Σ may be exponential in the size of Horn CNFs $\varphi_1, \varphi_2, \ldots, \varphi_l$ of Σ_1,Σ_2,..., Σ_l, even if $l = 2$.

For example, let $\varphi_1 = x_0$ and $\varphi_2 = (\overline{x}_1 \vee \overline{x}_2 \ldots \vee \overline{x}_m) \wedge \bigwedge_{j=1}^{m}(x_j \vee \overline{y}_j)$. Then $\varphi = \varphi_1 \vee \varphi_2$ is equivalent to the CNF $(x_0 \vee \overline{x}_1 \vee \overline{x}_2 \ldots \vee \overline{x}_m) \wedge \bigwedge_{j=1}^{m}(x_0 \vee x_j \vee \overline{y}_j)$. Thus the envelope of φ can be represented by

$$\psi = \bigwedge_{z_1 \in \{x_1, y_1\}} \bigwedge_{z_2 \in \{x_2, y_2\}} \cdots \bigwedge_{z_m \in \{x_m, y_m\}} (x_0 \vee \bigvee_{j=1}^{m} \overline{z}_j).$$

This CNF has 2^m clauses. Since all clauses in ψ are prime and removing some clause in ψ does not produce the envelope, we have the next theorem.

Theorem 5.2 *There exist Horn CNFs φ_i, $i = 1, 2, \ldots, l$, such that the Horn envelope of $\varphi = \bigvee_{i=1}^{l} \varphi_i$ has size exponential in the size of φ_i, $i = 1, 2, \ldots, l$, even if $l = 2$. Hence, the Horn envelope of φ cannot be computed in polynomial time.*

6 Related Work and Conclusion

In this paper, we have considered the problems on a disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ of Horn theories Σ_i. The results for the three problems, i.e., recognition of the Horn property, computation of cores and computation of envelope, are shown in Table 1 corresponding to the two representations of CNF and characteristic sets, and also corresponding to the cases in which l is bounded by a constant and is general.

The computation of a Horn core of a propositional theory Σ has been considered by several authors, and different algorithms have been proposed for different representations, e.g. [14, 2, 1, 15]. The papers [14, 2, 1] present algorithms for theories represented by CNFs, which require exponential time in the worst case. It should be noted that these algorithms are not immediately applicable if Σ is given by the disjunction of two Horn CNF formulas φ_1 and φ_2. This is because these algorithms require a CNF formula $\varphi = \varphi_1 \vee \varphi_2$ for

Table 1: Complexity Results for Disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$

	Horn		*Core*		*Envelope*	
	CNF	C^*	CNF	C^*	CNF	C^*
l is constant	P	P	P	NPH	EXP	P
l is general	coNPC	coNPC	NPH	NPH	EXP	P

P: polynomial time, NPH: NP-hard
EXP: exponential time, coNPC: coNP-complete

the input, but, since a smallest CNF φ may be exponential in the sizes of φ_1 and φ_2, the resulting procedure requires exponential time (and space) in general.

Concerning the model based representation, our intractability result on computing a Horn core for a disjunction $\Sigma = \bigcup_{i=1}^{l} \Sigma_i$ of Horn theories Σ_i represented by their characteristic sets $C^*(\Sigma_i)$, contrasts with a positive result in [15] that all Horn cores of Σ can be enumerated with polynomial delay, if all models of Σ are given for input. Intuitively, this is explained by the fact that $C^*(\Sigma)$ can be a succinct representation of Σ, and eliciting all models from it, as needed by the algorithm in [15], is not feasible in polynomial time in the input size of $C^*(\Sigma)$.

Algorithms for computing the Horn envelope restricted to special classes of formulas are contained in [14, 2, 15, 4, 5, 16]. In particular, [14, 2, 4, 5] consider CNF formulas, while [15, 16] cover the case of theories which are represented by a disjunction of minterms (i.e., terms such that all variables occur either positively or negatively in them). All these algorithms require exponential time. We have shown that the Horn envelope of a disjunction of Horn theories $\Sigma_1, \ldots, \Sigma_l$ can be computed from their characteristic sets $C^*(\Sigma_1), \ldots, C^*(\Sigma_l)$ in polynomial time, but it cannot be computed from Horn CNFs in polynomial time, even if $l = 2$. Observe that the results in [15, 16] imply that, for the formula-based representation, a polynomial total-time computation of the Horn envelope of the disjunction of Horn theories $\Sigma_1, \ldots, \Sigma_l$ (i.e., polynomial in the size of the input and the output of any irredundant prime CNF for the envelope) is hard to find, even if each Σ_i has a single model (which is computable from a Horn CNF in polynomial time).

Some problems remain for further work. One issue is on Horn cores when l is bounded by some constant; construct an efficient algorithm for generating all Horn cores for both representations, and construct an output-efficient computation of a Horn core (under a suitable notion) for the characteristic set representation. Another issue is constructing a polynomial total algorithm for computing the Horn envelope from Horn CNFs.

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